Independence Logic and Quantum Physics

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Arctic Set Theory Workshop 2022

Team Semantics

- Team semantics: Hodges 1997
- Dependence logic: Väänänen 2007
- Independence logic: Grädel–Väänänen 2013

Given a structure $\mathfrak{A}$ and a finite set V of variables, a team of $\mathfrak{A}$ is a set X of assignments $s\colon V \to A$.

	x	y	z
s_0	0	0	0
s_1	0	0	1
s_2	0	1	0

- $\mathfrak{A} \models_X =(\vec{x}, \vec{y})$ if "$\vec{x}$ functionally determines $\vec{y}$", i.e.

$$\forall s, s' \in X(s(\vec{x}) = s'(\vec{x}) \implies s(\vec{y}) = s'(\vec{y})).$$

- $\mathfrak{A} \models_X \vec{x} \perp_{\vec{y}} \vec{z}$ if "$\vec{x}$ is independent of $\vec{z}$ when $\vec{y}$ is fixed"

$$\forall s, s' \in X\big(s(\vec{y}) = s'(\vec{y}) \implies \exists s'' \in X\big(s''(\vec{x}\vec{y}) = s(\vec{x}\vec{y}) \wedge s''(\vec{z}) = s'(\vec{z})\big)\big).$$

- $=(\vec{x}, \vec{y}) \equiv \vec{y} \perp_{\vec{x}} \vec{y}$.

Let X be the following team:

	x	y	z	w
s_0	0	0	0	1
s_1	0	1	1	0
s_2	1	0	0	1
s_3	1	1	2	1
s_4	2	0	0	1
s_5	2	1	0	1

Then

- X satisfies $=(z, w)$ but not $=(x, y)$.
- X satisfies $x \perp y$ but not $y \perp z$.

Syntax of independence logic:

$$\varphi ::= \alpha \mid \neg\alpha \mid \vec{x} \perp_{\vec{y}} \vec{z} \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x \varphi \mid \forall x \varphi,$$

where α is first-order atomic

- $\mathfrak{A} \models_X \varphi$ if $\mathfrak{A} \models_s \varphi$ for all $s \in X$, whenever φ is a first-order atomic or negated atomic formula.
- $\mathfrak{A} \models_X \varphi \wedge \psi$ if $\mathfrak{A} \models_X \varphi$ and $\mathfrak{A} \models_X \psi$.
- $\mathfrak{A} \models_X \varphi \vee \psi$ if $\mathfrak{A} \models_Y \varphi$ and $\mathfrak{A} \models_Z \psi$ for some $Y, Z \subseteq X$ such that $X = Y \cup Z$.
- $\mathfrak{A} \models_X \exists x \varphi$ if $\mathfrak{A} \models_{X[F/x]} \varphi$ for some function $F\colon X \to \mathcal{P}(A) \setminus \{\emptyset\}$, where $X[F/x] = \{s(a/x) \mid s \in X, a \in F(s)\}$.
- $\mathfrak{A} \models_X \forall x \varphi$ if $\mathfrak{A} \models_{X[A/x]} \varphi$, where $X[A/x] = \{s(a/x) \mid s \in X, a \in A\}$.

Sort Logic

We consider many-sorted structures and include in our syntax the quantifiers of sort-logic [Väänänen 2014].

- $\mathfrak{A} \models_X \tilde{\exists} x \varphi \iff \mathfrak{B} \models_X \exists x \varphi$ for some expansion $\mathfrak{B}$ of $\mathfrak{A}$ by the sort of x
- $\mathfrak{A} \models_X \tilde{\forall} x \varphi \iff \mathfrak{B} \models_X \forall x \varphi$ for all expansions $\mathfrak{B}$ of $\mathfrak{A}$ by the sort of x

Hidden-Variable Models of Quantum Mechanics

- Could the non-deterministic nature of quantum mechanics be explained by including “hidden” variables in the models?
- Brandenburger & Yanofsky: a purely probabilistic framework
- Abramsky: a relational (possibilistic) framework

Empirical & Hidden-Variable Teams

We consider variables of three sorts:

- $V_\mathrm{m} = \{x_0, \ldots, x_{n-1}\}$ ("measurement variables"),
- $V_\mathrm{o} = \{y_0, \ldots, y_{n-1}\}$ ("outcome variables"), and
- $V_\mathrm{h} = \{z_0, \ldots, z_{l-1}\}$ ("hidden variables").

X is an *empirical team* if $\mathrm{dom}(X) = V_\mathrm{m} \cup V_\mathrm{o}$.

X is a *hidden-variable team* if $\mathrm{dom}(X) = V_\mathrm{m} \cup V_\mathrm{o} \cup V_\mathrm{h}$.

Empirical & Hidden-Variable Teams (cont.)

x_0	y_0	$\dots$	x_{n-1}	y_{n-1}	z_0	$\dots$	z_{l-1}
a_0^0	b_0^0	$\dots$	a_{n-1}^0	b_{n-1}^0	γ_0^0	$\dots$	γ_{l-1}^0
a_0^2	b_0^1	$\dots$	a_{n-1}^2	b_{n-1}^1	γ_0^1	$\dots$	γ_{l-1}^2
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
a_0^{m-1}	b_0^{m-1}	$\dots$	a_{n-1}^{m-1}	b_{n-1}^{m-1}	γ_0^{m-1}	$\dots$	γ_{l-1}^{m-1}

Empirical & Hidden-Variable Teams (cont.)

A hidden-variable team Y *realizes* an empirical team X if for all assignments s we have

$$s \in Y \iff s \upharpoonright (V_{\mathrm{m}} \cup V_{\mathrm{o}}) \in X.$$

If $\varphi(\vec{x}, \vec{y}, \vec{z})$ is a formula of independence logic, then

$\tilde{\exists} z_0 \exists z_1 \ldots \exists z_{l-1} \varphi$ defines the class of empirical teams that are realized by some hidden-variable team satisfying φ.

Properties of Empirical Teams

Weak Determinism: "the outcomes of the measurements are completely determined"

$$=(\vec{x}, \vec{y})$$

No-Signalling: "the choice of measurement by one party cannot be signalled to the other parties".

$$\bigwedge_{i<n} \{x_j \mid j \neq i\} \perp_{x_i} y_i$$

Properties of Hidden-Variable Teams

Strong Determinism: "the outcome of each individual measurement is completely determined by that measurement (and the hidden variable) alone"

$$\bigwedge_{i<n} =(x_i \vec{z}, y_i)$$

z-Independence: "the value of the hidden variable is independent of the choice of measurements"

$$\vec{z} \perp \vec{x}$$

Parameter Independence: a hidden-variable version of no-signalling

$$\bigwedge_{i<n} \{x_j \mid j \neq i\} \perp_{x_i \vec{z}} y_i$$

Relationships between the Properties

Strong determinism implies parameter independence

$$=\!(x_i\vec{z}, y_i) \vdash \{x_j \mid j \neq i\} \perp_{x_i\vec{z}} y_i$$

Empirical vs. Hidden-Variable Teams

An empirical team supports no-signalling iff it can be realized by a hidden-variable team supporting z-independence and parameter independence.

In other words, the following formulas are equivalent.

1. $\bigwedge_{i<n}\{x_j \mid j \neq i\} \perp_{x_i} y_i$,
2. $\tilde{\exists} z_0 \exists z_1 \dots \exists z_{l-1} \left(\vec{z} \perp \vec{x} \wedge \bigwedge_{i<n}\{x_j \mid j \neq i\} \perp_{x_i \vec{z}} y_i\right)$.

No-Go Theorems

There is an empirical team that cannot be realized by any hidden-variable team supporting single-valuedness and outcome independence.

Theorem

$\tilde{\exists} z_0 \exists z_1 \ldots \exists z_{l-1} \left(=\!(\vec{z}) \wedge \bigwedge_{i<n} y_i \perp_{\vec{x}\vec{z}} \{ y_j \mid j \neq i \} \right)$ *is not valid.*

Proof.

As demonstrated, for instance, by the following team.

	x_0	x_1	y_0	y_1
s	0	1	0	1
s'	0	1	1	0

We call the above the *EPR team*. □

No-Go Theorems (cont.)

There is an empirical team that cannot be realized by a hidden-variable team supporting z-independence and locality.

Theorem

$$\tilde{\exists} z_0 \exists z_1 \dots \exists z_{l-1} \left(\vec{z} \perp \vec{x} \wedge \bigwedge_{i<n} \left(\left(\{x_j \mid j \neq i\} \perp_{x_i \vec{z}} y_i \right) \wedge \left(y_i \perp_{\vec{x}\vec{z}} \{y_j \mid j \neq i\} \right) \right) \right)$$

is not valid.

Proof.

As is demonstrated, for instance, by the following team.

	x_0	x_1	x_2	y_0	y_1	y_2
s_0	0	0	0	0	0	1
s_1	0	0	0	0	1	0
s_2	0	0	0	1	0	0
s_3	0	0	0	1	1	1
s_4	0	1	1	0	0	0
s_5	0	1	1	0	1	1
s_6	1	0	1	1	0	1
s_7	1	1	0	1	1	0

This is an example of a *GHZ team*. □

Quantum-Mechanical Teams

Definition

A probabilistic empirical team $\mathbb{X}$ is *quantum-mechanical* if it represents the probability distribution of measurement outcomes in a finite-dimensional quantum system.

Define a new atomic formula QR such that an ordinary team X satisfies QR if X is the possibilistic collapse of a quantum-mechanical team.

Definition

Let M and O be sets of n-tuples, and denote $M_i = \{a_i \mid \vec{a} \in M\}$ and $O_i = \{b_i \mid \vec{b} \in O\}$. A quantum system of type (M, O) is a tuple $(\mathcal{H}, (A_i^{a,b})_{a \in M_i, b \in O_i, i<n}, \rho)$, where

- $\mathcal{H}$ is the tensor product $\bigotimes_{i<n} \mathcal{H}_i$ of finite-dimensional Hilbert spaces $\mathcal{H}_i$, $i < n$,
- for all $i < n$ and $a \in M_i$, $\{A_i^{a,b} \mid b \in O_i\}$ is a positive operator-valued measure on $\mathcal{H}_i$, and
- ρ is a density operator on $\mathcal{H}$, i.e. $\rho = \sum_{j<k} p_j \left|\psi_j\right\rangle \left\langle\psi_j\right|$, where $\left|\psi_j\right\rangle$ is a unit vector of $\mathcal{H}$ and $p_j \in [0,1]$ for all $j < k$ and $\sum_{j<k} p_j = 1$.

For each measurement $\vec{a} \in M$, we define the probability distribution $p_{\vec{a}}$ of outcomes by setting $p_{\vec{a}}(\vec{b}) := \mathrm{Tr}(A^{\vec{a},\vec{b}}\rho)$, where $A^{\vec{a},\vec{b}}$ denotes the operator $\bigotimes_{i<n} A_i^{a_i,b_i}$.

Definition

Let $\mathbb{X}$ be a probabilistic team with variable domain $V_{\mathrm{m}} \cup V_{\mathrm{o}}$ and denote $M = \{s(\vec{x}) \mid s \in \operatorname{supp} \mathbb{X}\}$ and $O = \{s(\vec{y}) \mid s \in \operatorname{supp} \mathbb{X}\}$. We say that $\mathbb{X}$ is *quantum-mechanical* if there exists a quantum system

$$(\mathcal{H}, (A_i^{a,b})_{a \in M_i, b \in O_i, i<n}, \rho)$$

of type (M, O) such that for all assignments s, we have $\mathbb{X}(s) = p_{s(\vec{x})}(s(\vec{y}))/\,|M|$. We call a quantum-mechanical team $\mathbb{X}$ a *quantum realization* of an empirical team X if X is the possibilistic collapse of $\mathbb{X}$.

Quantum-Mechanical Teams (cont.)

Proposition

1. *The EPR team is a collapse of a quantum-mechanical team, hence*

$$\mathsf{QR} \not\models \exists \vec{z} \left(=\!(\vec{z}) \wedge \bigwedge_{i<n} y_i \perp\!\!\!\perp_{\vec{x}\vec{z}} \{y_j \mid j \neq i\} \right) .$$

2. *A GHZ team is a collapse of a quantum-mechanical team, hence*

$$\mathsf{QR} \not\models \exists \vec{z} \left(\vec{z} \perp\!\!\!\perp \vec{x} \wedge \bigwedge_{i<n} \{x_j \mid j \neq i\} \perp\!\!\!\perp_{x_i \vec{z}} y_i \wedge \bigwedge_{i<n} y_i \perp\!\!\!\perp_{\vec{x}\vec{z}} \{y_j \mid j \neq i\} \right).$$

Proposition

The set $\{X \mid X \models \mathrm{QR}\}$ *is undecidable but recursively enumerable.*

Proof idea: There is a many-one reduction from two-player one-round non-local games that have a perfect quantum strategy to teams that have a quantum realization.

Determining whether a non-local game has a perfect quantum strategy is undecidable. [Slofstra 2019]

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