Critical Cardinals 3: The Race to the Critical Point Theory

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Definition

We say that a cardinal κ is a $critical\ cardinal$ if it is the critical point of an elementary embedding $j\colon V_{\kappa+1} \to M$, where M is some transitive set.

Theorem

Let κ be a critical cardinal. Then,

1. *κ is regular and a strong limit. Equivalently, $V_\kappa \models \mathsf{ZF}_2$.*
2. *κ is measurable and carries a normal measure.*
3. *κ satisfies all the "usual" reflected properties (e.g. Mahlo, weakly critical).*

It is consistent with ZF, however, that a measurable cardinal carries no normal measures, or that it is not a limit cardinal, or that it is not reflecting all the "reasonable" properties.

Theorem

The following are equivalent in ZFC*:*

1. κ *is critical.*
2. κ *is the critical point of an embedding* $j\colon V \to M$.
3. κ *is the critical point of an embedding* $j\colon V \to M$ *where* $M^\kappa \subseteq M$.
4. κ *is the critical point of an ultrapower embedding.*
5. κ *is measurable.*

In ZF no two of these are equivalent.*

(Most of the proofs are mostly written.)

(It is also not clear whether 1 and 2 are equivalent or not with the current tools.)

Let $\mathbb{P}$ be a forcing notion. The action of $\pi \in \operatorname{Aut}(\mathbb{P})$ on $\mathbb{P}$-names is defined recursively:

$$\pi\dot{x} = \{\langle \pi p, \pi\dot{y}\rangle \mid \langle p, \dot{y}\rangle \in \dot{x}\}.$$

The Symmetry Lemma states that $p \Vdash \varphi(\dot{x}) \iff \pi p \Vdash \varphi(\pi\dot{x})$.

We say that $\langle \mathbb{P}, \mathscr{G}, \mathscr{F}\rangle$ is a *symmetric system* if $\mathbb{P}$ is a forcing notion, $\mathscr{G}$ is a group of automorphisms of $\mathbb{P}$, and $\mathscr{F}$ is a normal filter of subgroups on $\mathscr{G}$.

1. Say that $\dot{x}$ is $\mathscr{F}$*-symmetric* if $\operatorname{sym}_{\mathscr{G}}(\dot{x}) = \{\pi \in \mathscr{G} \mid \pi\dot{x} = \dot{x}\} \in \mathscr{F}$.
2. If $\dot{x}$ is $\mathscr{F}$-symmetric, and this property holds hereditarily for names appearing in $\dot{x}$, then it is *hereditarily* $\mathscr{F}$*-symmetric*.
3. The class of hereditarily $\mathscr{F}$-symmetric names is denoted by $\mathsf{HS}_{\mathscr{F}}$.
4. If $G \subseteq \mathbb{P}$ is a V-generic filter, then $\mathsf{HS}_{\mathscr{F}}^G = \{\dot{x}^G \mid \dot{x} \in \mathsf{HS}_{\mathscr{F}}\}$ is a transitive model of ZF extending V. We call such model a *symmetric extension*.
5. The symmetric forcing relation $\Vdash^{\mathsf{HS}}$ is defined by relativisation and behaves as expected.

Suppose that $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$ is a symmetric system. We can get a discount at the assumptions store, and get the following for free.

1. $\mathscr{F}$ is a normal filter *base*.
2. For every $p \in \mathbb{P}$, there is some $K \in \mathscr{F}$ such that for all $\pi \in K$, $\pi p = p$.

Example

1. $\mathbb{P}$ is $\mathrm{Add}(\lambda, \lambda)$ for a regular cardinal λ.
2. $\mathscr{G}$ is the group of permutations of λ acting on $\mathbb{P}$, $\pi p(\pi\alpha, \beta) = p(\alpha, \beta)$.
3. $\mathscr{F}$ is generated by $\mathrm{fix}(E)$ for $E \in [\lambda]^{<\lambda}$, $\mathrm{fix}(E) = \{\pi \in \mathscr{G} \mid \pi \upharpoonright E = \mathrm{id}\}$.

If $\dot{x}_\alpha$ is the canonical name for the αth Cohen subset, $\pi\dot{x}_\alpha = \dot{x}_{\pi\alpha}$. The symmetric extension contains the set $\{x_\alpha \mid \alpha < \lambda\}$, but no well-orderable subset (in the symmetric extension) has size λ.
Less trivially, $\mathsf{DC}_{<\lambda}$ holds in the symmetric extension.

Theorem (Silver's criterion)

$j\colon V \to M$ lifts to $j\colon V[G] \to M[H]$ if and only if we can set $j(G) = H$.

Theorem (Usuba)

W is a symmetric extension of $V \models \mathrm{ZFC}$ if and only if $W = V(x)$ for some $x \in W$.

Conjecture (Wishful, and slightly inaccurate criterion)

j lifts to symmetric extensions $W = V(x) \to M(y) = N$ if and only if we can set $j(x) = y$.

The problem is amenability. We are interested in more than the case where such lift exists *somewhere*, out there, we want it to be amenable to $V(x)$.

Inaccurately Stated Theorem (Hayut–K.)

Suppose $j\colon V\to M$ is an elementary embedding and $\langle\mathbb{P},\mathscr{G},\mathscr{F}\rangle$ is a symmetric system. Then j lifts to the symmetric extension if the following conditions hold:

1. $j(\langle\mathbb{P},\mathscr{G},\mathscr{F}\rangle)=\langle\mathbb{P},\mathscr{G},\mathscr{F}\rangle*\langle\dot{\mathbb{Q}},\dot{\mathscr{H}},\dot{\mathscr{K}}\rangle$,
2. *there is $\dot{H}\in\mathrm{HS}_{\mathscr{F}}$ which is "sufficiently M-generic" for $\dot{\mathbb{Q}}$,*
3. *the name for $\dot{H}$ is forced to be "sufficiently stable" under the action of $\dot{\mathscr{H}}$,*
4. *and $j``\mathscr{F}$ is a basis for $j(\mathscr{F})$.and $\forall K\in j(\mathscr{F})\,\exists K_0\in\mathscr{F},j(K_0)\subseteq K$.*

Theorem (Hayut–K.)

We can replace the fourth condition by

$$\forall K\in j(\mathscr{F})\,\exists K_0\in\mathscr{F},j``K_0\subseteq K$$

The new change is significant, since it opens up the door for a lot of the interesting cases where $j(\mathbb{P})=\mathbb{P}\times\mathbb{Q}$ and we already have $H\in V$.

Theorem (K.–Yuan)

If $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle \in V_\kappa$, then any ultrapower embedding is lifted to an ultrapower embedding.

Theorem (Spector)

Let $j \colon V \to V^\kappa/U$ be the ultrapower embedding. The following are equivalent:

1. *j is elementary.*
2. *V^κ/U is extensional.*
3. *V^κ/U has a unique empty set.*
4. *U-AC$_\kappa$. Given any $\{A_\alpha \mid \alpha < \kappa\}$, there is $X \in U$ such that $\prod_{\alpha \in X} A_\alpha \neq \varnothing$.*

Corollary

U-AC$_\kappa$ does not imply AC$_\omega$.

Proof of the ultrapower lifting.

Suppose that $\dot{F} \in \mathsf{HS}_{\mathscr{F}}$ is a name for a function and $\Vdash^{\mathsf{HS}} \operatorname{dom} \dot{F} = \check{\kappa}$ and for every $\alpha < \kappa$, $\dot{F}(\check{\alpha}) \neq \varnothing$. For each $\alpha < \kappa$ consider

$$D_\alpha = \{p \in \mathbb{P} \mid p \text{ has a witness that } \dot{F}(\check{\alpha}) \neq \varnothing\}.$$

There is some $X_0 \in U$ such that $\alpha \in X_0$ implies $D_\alpha = D$. For all $\alpha \in X_0$, and a fixed $p \in D$, choose $\dot{x}_\alpha$ such that $p \Vdash \dot{x}_\alpha \in \dot{F}(\check{\alpha})$.

There is some $X_{1,p} \in U$ such that for all $\alpha \in X_{1,p}$, $\operatorname{sym}(\dot{x}_\alpha) = K$, which we can assume satisfy that for all $\pi \in K$, $\pi p = p$.

Then $\dot{f}_p = \{\langle p, \langle \check{\alpha}, \dot{x}_\alpha \rangle^\bullet \rangle \mid \alpha \in X_{1,p}\}$ is a name in $\mathsf{HS}_{\mathscr{F}}$, and easily

$$p \Vdash^{\mathsf{HS}} \dot{f} \in \prod_{\alpha \in X_{1,p}} \dot{F}(\alpha).$$

Therefore U-AC_κ holds in the symmetric extension, so by Spector's theorem, the ultrapower embedding is lifted to an ultrapower embedding. $\square$

Theorem (K.–Yuan)

If $j\colon V\to M$ and $M^\lambda\subseteq M$, where $\lambda\geqslant\kappa$, then for any symmetric system in V_κ, the lifting of $j\colon W\to N$ satisfies that $N^x\subseteq N$ whenever $V\models|\dot x|\leqslant\lambda$, for some name for x. Similarly, if $V_\alpha\subseteq M$, then $W_\alpha\subseteq N$.

Proof.

Note that $j(\mathrm{HS}_{\mathscr{F}})=\mathrm{HS}_{\mathscr{F}}\cap M$ and that $j(\mathbb{P})=\mathbb{P}$. So any name of size λ is already in M. This is similar for proving that $W_\alpha\subseteq N$. □

Corollary

If κ is supercompact and $\langle\mathbb{P},\mathscr{G},\mathscr{F}\rangle\in V_\kappa$, then κ remains supercompact in the symmetric extension.

Theorem (K.–Yuan)

It is consistent that there is a critical cardinal, but no elementary embedding is an ultrapower embedding.

Proof.

Let κ be a measurable cardinal. Consider $\mathbb{P}$ the Easton support product of $\langle \mathbb{Q}_\alpha, \mathscr{G}_\alpha, \mathscr{F}_\alpha \rangle$, where $\mathbb{Q}_\alpha = \mathrm{Add}(\alpha^+, \alpha^+)$; $\mathscr{G}_\alpha$ is the group of permutations of α^+ acting on the forcing, and $\mathscr{F}_\alpha$ is generated by fixing pointwise α coordinates. The products of the groups and filters are also taken pointwise with Easton support.

Then $j(\mathbb{P}) = \mathbb{P} \times \mathbb{R}$, where $\mathbb{R}$ is has an M-generic filter in V. We also note that if $K \in j(\mathscr{F})$, then K_α is nontrivial only on a bounded number of coordinates below any regular cardinal α (in M). Set $K_0 = K \upharpoonright \kappa$, then $K_0 \in \mathscr{F}$ and $j``K_0 \subseteq K$. So by the slightly improved lifting theorem, j lifts.

On the other hand, set A_α as the set of Cohen subsets added by $\mathbb{Q}_\alpha$. The function $F(\alpha) = A_\alpha$ admits no partial inverse from any subset of size κ. □

Theorem (K.–Yuan)

It is consistent that κ is a critical cardinal, but no elementary embedding has a countably closed target.

Proof.

Repeat the same construction as before, but with $\mathscr{F}_\alpha$ being generated by finite supports. Since we have an M-generic for $\mathbb{R}$ in V, the Dedekind-finite sets added (in N) for $\alpha > \kappa$ can be enumerated in W. In particular, N cannot be countably closed. $\square$

This can be modified to any $\lambda < \kappa$ and to preserve $\mathsf{DC}_{<\lambda}$ if need be. Another option is to replace the support of the product of $\mathscr{F}_\alpha$s by λ-support.

Observation

If $j(\mathbb{P}) = \mathbb{P} * \dot{\mathbb{Q}}$, and $H \in W$ is M-generic for $\mathbb{Q}$, then N cannot be "more closed" then the closure of H or $j(\mathscr{F})$.

Definition

κ is *weakly critical* if for every $A \subseteq V_\kappa$ there is an elementary embedding between transitive sets $j\colon X \to M$ with $\operatorname{crit}(j) = \kappa$ such that $\kappa, A, V_\kappa \in X \cap M$.

Proposition

κ is weakly critical if and only if for every $A \subseteq V_\kappa$ there is a transitive set M which is an elementary end-extension of V_κ such that $A \in M$.

Proposition

If κ is critical, then it reflects being weakly critical.

Theorem (Hayut–K.)

It is consistent relative to a measurable cardinal that the least weakly critical cardinal is the least measurable cardinal.

Proof.

Let κ be a measurable cardinal. Consider the Easton support product of $\mathbb{Q}_\alpha$ for $\alpha < \kappa$ inaccessible, which adds a non-reflecting stationary subset to S^α_ω. This partial order is homogeneous, so we can take the product of the automorphism groups acting pointwise on each $\mathbb{Q}_\alpha$, with $\mathscr{F}$ being the filter generated by $\prod_{\alpha<\kappa} H_\alpha$, where on a tail of α, $H_\alpha = \mathrm{Aut}(\mathbb{Q}_\alpha)$.

Given any name $\dot{A} \in \mathsf{HS}$ for a subset of W_κ, find in V a transitive set M which is closed under $<\kappa$-sequences, and $j\colon V_\kappa \to M$ with the relevant predicates. Since M is sufficiently closed, and $j(\mathbb{P}) = \mathbb{P} \times \mathbb{R}$, with $\mathbb{R}$ being κ-strategically closed, we can find an M-generic filter for $\mathbb{R}$. This ensures that j lifts with the interpretation of $\dot{A}$ as our predicate, as wanted.

Finally, it is easy to see that κ is the least weakly critical. To see that κ remains measurable, note that by homogeneity, any set of ordinals is added in a bounded part of the product. However, by simple cardinality arguments, any measure on κ in V has a unique extension in any bounded part of the product, and it is easy to see that the union of these is a measure on κ. $\square$

Where do we go next?

Question

If κ is critical and it admits an ultrapower embedding, does it admits a $normal$ ultrapower embedding?

"Proof".

Let U be a measure such that $j\colon V \to V^\kappa/U$ is elementary. Define $D = \{A \subseteq \kappa \mid \kappa \in j(A)\}$, which is a normal measure on κ.

We want to define an embedding $i\colon V^\kappa/D \to V^\kappa/U$ by $[f]_D \mapsto j(f)(\kappa)$. In the ZFC case, we prove this embedding is elementary by showing it commutes with the other two ultrapower embeddings. But here we are trying to show the ultrapower embedding j_D is elementary by using the elementarity of i. Which we have yet proved in this case.

It is enough to show that if $j(f)(\kappa) \neq \varnothing$, then there is some g such that $j(g)(\kappa) \in j(f)(\kappa)$. If we can show that, then $\kappa \in j(\{\alpha < \kappa \mid g(\alpha) \in f(\alpha)\})$, in which case $V^\kappa/D \models [g]_D \in_D [f]_D$. But there is no reason for this g to exist.

So maybe the answer is negative?

Question

Suppose κ is critical. Does that mean there is an embedding from the whole universe?

Suppose that κ was a measurable, and pick some regular $\lambda > \kappa$. Then $\mathrm{Add}(\lambda, \lambda)$ does not add any sets of rank κ. Consider the symmetric extension defined by permutations of λ acting on the indices of the Cohen subsets, with $[\lambda]^{<\kappa}$ as the supports generating the filter of subgroups.

In this extension κ certainly remains critical, since $V_{\kappa+1}$ is not changed. But it is not clear if the embedding can be lifted to the whole symmetric extension.

Let X denote the set of Cohen generics for λ, then X does not have subsets of size κ. But $j(X)$ must have subsets of size κ. This is not enough to conclude that j cannot be lifted, but it is a good indication that this is not at all obvious.

Question

How can we improve the lifting theorem even more?

It seems unlikely that the improvements can be done with condition 4.

However, it seems that the correct approach is to require "enough pieces" of an M-generic filter for $j(\mathbb{P})$ to exist.

This will also alleviate the limitations with requiring $j(\mathbb{P}) = \mathbb{P} * \dot{\mathbb{Q}}$.

Can we also have a better understanding of how much closure is preserved?

Question

Can we extend the lifting to iterations of symmetric extensions?

We can iterate symmetric extensions. While it is likely that we can present an iteration of symmetric extensions as a single symmetric extension, it is not immediately clear how.

Question

Does the Bristol model construction lift elementary embedding?

The Bristol model is an intermediate model of ZF between V and $V[c]$ not of the form $V(x)$, where c is a Cohen real. It exists provided that V satisfies GCH and $\square^*_\lambda$ on every singular cardinal λ. The construction preserves measures, but it is not clear that it preserves embedding.

Theorem (K.–Schilhan)

Bristol models satisfy DC.

If we can prove that $\mathsf{AC}_{\mathsf{WO}}$ is actually true in Bristol models, then U-AC_κ will follow, and therefore the embeddings will lift.

Question

Can we get a non-trivial failure of choice with supercompactness?

Recall that Woodin proved that if δ is supercompact, then there is a forcing $\mathbb{P}$ such that $\mathbb{1} \Vdash_{\mathbb{P}} \check{\delta} = \omega_1 \wedge \mathsf{DC}$. We say that a failure of choice is trivial, if $\mathbb{1} \Vdash_{\mathbb{P}} \mathsf{AC}$.

We saw that due to the Levy–Solovay phenomenon, small symmetric extensions violate choice and preserve supercompactness. But these failures are trivial, in the sense presented here.

For a non-trivial failure of choice, we would need to perform a class-size symmetric extension that will also lift many different embeddings and preserve their closure.

Thank you
For
Your attention!

(And apologies for whatever I screwed up.)