Compactness vs. hugeness at ω_2

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Overview

Question

Is the tree property at ω_2 consistent with either the existence of a saturated ideal on ω_2 or $(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$?

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Answers so far

- Not if $2^\omega \leq \omega_2$.
- Not if the generic hugeness embedding comes from a Kunen-style lift of a ground model hugeness embedding.
- With Neeman's pure side conditions, we can get the tree property with a weaker hugeness property ("weak Chang's conjecture").

Definitions

A normal ideal I on κ is called *weakly presaturated* if, whenever $G \subseteq \mathcal{P}(\kappa)/I$ is generic, then $\mathrm{Ult}(V, G)$ is well-founded up to $(\kappa^+)^V + 1$ and $j_G(\kappa) = (\kappa^+)^V$.

Some abbreviations:

1. $\mathrm{sat}(\kappa)$: There is a saturated ideal on κ, i.e. a normal ideal I such that $\mathcal{P}(\kappa)/I$ has the κ^+-c.c.
2. $\mathrm{wps}(\kappa, S)$: There is a weakly presatuated normal ideal I on κ such that $\kappa \setminus S \in I$.
3. $\mathrm{CC}(\kappa)$: $\{X \subseteq \kappa^+ : X \cap \kappa \in \kappa \text{ and } \mathrm{ot}(X) = \kappa\}$ is stationary.
4. $\mathrm{wCC}(\kappa, S)$: For every stucture $\mathfrak{A}$ on H_{κ^+} in a countable language, there is $\alpha \in S$ such that the set $\{\mathrm{ot}(M \cap \kappa^+) : M \prec \mathfrak{A} \wedge M \cap \kappa = \alpha\}$ is unbounded in κ.

Fact: If $\kappa = \mu^+$ and $S^\kappa_\mu = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \mathrm{cf}(\mu)\}$, then

$$(\mathrm{CC}(\kappa) \vee \mathrm{sat}(\kappa)) \implies \mathrm{wps}(\kappa, S^\kappa_\mu) \implies \mathrm{wCC}(\kappa, S^\kappa_\mu)$$

Negative results

The following is a kind of diagnosis of why all the attempts of myself and Sean Cox to combine compactness and hugeness at ω_2 failed:

Theorem (Cox-E.)

Suppose $j : V \to M$ is an elementary embedding with critical point κ definable from parameters in V. Suppose $\mathbb{P} * \dot{\mathbb{Q}}$ is a two-step iteration such that:

1. *M is $|\mathbb{P}|$-closed, and $|\mathbb{P}| < j(\kappa)$.*
2. *$\mathbb{P} * \dot{\mathbb{Q}}$ collapses all ordinals in the open interval $(\kappa, j(\kappa))$.*
3. *Whenever $G * H$ is $\mathbb{P} * \dot{\mathbb{Q}}$-generic over V, then in some outer model, j can be lifted to $j' : V[G * H] \to M[G' * H']$, such that $\mathcal{P}_\kappa(\mathrm{Ord})^{V[G*H]} \subseteq M[G']$.*

Then $\mathbb{P}$ forces that $\kappa = \mu^+$ for some $\mu < \kappa$, and $\square^*_\mu$ holds.

Negative results

Theorem (Cox-E.)

Suppose κ is a regular cardinal, $2^{<\kappa} \leq \kappa^+$, and there is a weakly presaturated ideal on κ^+ concentrating on $\mathrm{cof}(\kappa)$. Then $\square^*_\kappa$ holds.

Very vaugely, the argument somehow imitates the usual proof of $\square^*_{\omega_1}$ from CH, replacing $\omega_1^\omega = \omega_1$ with the fact that $([\omega_2]^\omega)^V$ has size ω_1 in a generic ultrapower, working on the j-side to build a weak square sequence of length $j(\omega_2) = \omega_3^V$, and then reflecting.

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The assumption that the ideal concentrates on the highest cofinality is important:

Theorem (Woodin-Sargsyan)

It is consistent relative to a Woodin limit of Woodins that $2^\omega = \omega_2$, $\mathrm{TP}(\omega_2)$, and $\mathrm{wps}(\omega_2, S^{\omega_2}_\omega)$. (Moreover BMM holds.)

Equivalences in terms of elementary submodels

Lemma

TFAE:

1. $\mathrm{wps}(\kappa, S)$.
2. *There is* $\theta > \kappa$ *and a stationary* $T \subseteq \{M \in [H_\theta]^{<\kappa} : M \cap \kappa \in S\}$ *such that the following holds: For all* $f : \kappa \to \kappa$ *and all* $A \subseteq S$ *such that* $A^* = \{M \in T : M \cap \kappa \in A\}$ *is stationary, there are stationary-many* $M \in A^*$ *such that* $\mathrm{ot}(M \cap \kappa^+) > f(M \cap \kappa)$.

The witnessing ideal is just the projection of $\mathrm{NS} \restriction T$ *to* κ.

Lemma

TFAE:

1. $\mathrm{wCC}(\kappa, S)$.
2. *For all* $f : \kappa \to \kappa$*, there are stationary-many* $M \in [H_{\kappa^+}]^{<\kappa}$ *such that* $M \cap \kappa \in S$ *and* $\mathrm{ot}(M \cap \kappa^+) > f(M \cap \kappa)$.

Neeman forcing

Neeman's two-type side conditions forcing depends on the following parameters:

- Some fixed transitive model K satisfying enough ZFC.
- A collection $\mathcal{S}$ of small $M \prec K$.
- A collection $\mathcal{T}$ of transitive $M \prec K$.
- A cardinal κ.

There are some requirements on $\mathcal{S}, \mathcal{T}$ and their interrelation. The conditions are $\in$-chains of models from $\mathcal{S} \cup \mathcal{T}$ length $<\kappa$, closed under pairwise intersection of the models, ordered by $p \leq q$ when $p \supseteq q$.

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Typical example: $K = \langle V_\theta, \in, ... \rangle$ for θ inaccessible. $\mathcal{S}$ = all countable $M \prec K$, $\mathcal{T}$ = all countably closed $V_\alpha \prec K$, $\kappa = \omega$.

The forcing is strongly proper for $\mathcal{S} \cup \mathcal{T}$. In the above case, it preserves ω_1, collapses θ to become ω_2, and preserves the tree property at θ if θ was weakly compact.

Proposition

Suppose θ is measurable. There is a structure $\mathfrak{A}$ on V_θ such that, taking $\mathcal{S}, \mathcal{T}$ like above and using the finite conditions, Neeman's forcing gets a weakly presaturated ideal on ω_1.

Proof sketch: Let $\mathcal{U}$ be a normal measure on θ, let $\lhd$ be a wellorder on a sufficiently large H_λ, and let $\mathfrak{A}$ be the restriction of $\mathfrak{B} = \langle H_\lambda, \in, \mathcal{U}, \lhd \rangle$, to V_θ.

By Neeman's work, and $\mathcal{S}_G := \{M : M \in \mathcal{S} \wedge (\exists p \in G) M \in p \in G\}$ is forced to be stationary. Let I be the projection of $\mathrm{NS} \restriction \mathcal{S}_G$.

Let $\dot{A}$ be a name for an I-positive set and let p be an arbitrary condition. Let $\dot{F}$ be a name for a function $V_\theta^{<\omega} \to V_\theta$. Let $\dot{f}$ be a name for a function $\omega_1 \to \omega_1$.

Since it is forced that the inverse of the projection is stationary, there is $q \leq p$ and $M \in \mathcal{S}$ such that $\mathrm{Hull}^{\mathfrak{B}}(M) = M$, $\dot{F} \in \mathrm{Hull}^{\mathfrak{B}}(M)$ and $q \Vdash M \cap \omega_1 \in \dot{A}$. We may assume $M \in q$.

Now let $q' \leq q$ decide $\dot{f}(M \cap \omega_1) = \xi$.

By a well-known argument, if $\alpha \in \bigcap(\mathcal{U} \cap \mathrm{Hull}^{\mathfrak{B}}(M))$, then $M_1 = \mathrm{Hull}^{\mathfrak{B}}(M \cup \{\alpha\})$ has the property that $M_1 \cap V_\alpha = M$.

Repeat this until we get $M_\xi \prec \mathfrak{A}$ such that $\mathrm{ot}(M_\xi \cap \theta) \geq \xi$, and such that for some $\alpha \in \bigcap(\mathcal{U} \cap \mathrm{Hull}^{\mathfrak{B}}(M))$, $M_\xi \cap V_\alpha = M$, and $q' \in V_\alpha$.

Now put $q'' = q' \cup \{V_\alpha, M_\xi\}$. This is a condition below p forcing that $M_\xi \in \mathcal{S}_G$, M_ξ is closed under $\dot{F}^G$, $M_\xi \cap \omega_1 \in \dot{A}$, and $\mathrm{ot}(M_\xi \cap \kappa) > \dot{f}(M_\xi \cap \omega_1)$.

So q'' forces that there are stationary-many models projecting to A with the desired ordertype property, which implies weak presaturation. $\square$

We would like to generalize this to ω_2. But in the simplest generaliztion of Neeman's forcing we use countable sequences of models which are all countably closed. How can we end-extend models arbitrarily high and arrive another countably closed small model?

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Suppose κ is inaccessible. For $\lambda > \kappa$, a κ-Magidor model is an $M \prec V_\lambda$ such that $M \cap \kappa \in \kappa$ and $\mathrm{trcl}(M) = V_\alpha$ for some $\alpha < \kappa$. Magidor proved that κ is supercomapact iff for every $\lambda > \kappa$, the set of κ-Magidor $M \prec V_\lambda$ is stationary.

Ideally, we would like the following: There are inaccessible $\kappa < \lambda$ and a stationary $\mathcal{S}$ of κ-Magidor $M \prec V_\lambda$ such that: For every $M \in \mathcal{S}$, $\alpha < \kappa$, and $\beta < \lambda$, there is $N \in \mathcal{S}$ (key!) such that $N \cap V_\beta = M$ and $\mathrm{ot}(N \cap \lambda) > \alpha$.

This would allow us to mimic the argument for a weakly presaturated ideal on general κ. Surprisingly, this turns out to be impossible.

Let κ be supercompact and let $\lambda > \kappa$ be inaccessible. Let $\mathbb{P}$ be the Neeman forcing to make $\kappa = \omega_2$ with finite conditions. Let $G \subseteq \mathbb{P}$ be generic. In $V[G]$, let $\mathbb{Q}$ be the Neeman forcing with

$$\mathcal{S} = \{M[G] : M \text{ is } \kappa\text{-Magidor in } V \text{ and } \mathrm{cf}(\sup(M \cap \lambda)) \geq M \cap \kappa\}$$

$$\mathcal{T} = \{V_\alpha \prec V_\lambda : \mathrm{cf}(\alpha) \geq \kappa\}$$

In $V[G]$, let $\mathbb{Q}$ be the forcing with countable sequences of these $\mathcal{S}, \mathcal{T}$.

Theorem (Neeman)

$\mathbb{Q}$ is countably distributive and forces (preserves) the tree property on κ (which becomes ω_2).

If the wished-for scenario were consistent, then we would get a model violating my theorem with Sean Cox.

Assume κ is almost-huge with target λ. This means there is an elementary $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) = \lambda$, and $M^{<\lambda} \subseteq M$.

Lemma

Let $\mathfrak{A}$ be a structure on V_λ in a countable language. Let $\mathcal{E}$ be the set of all κ-Magidor $M \prec \mathfrak{A}$ such that $\mathrm{cf}(\sup(M \cap \lambda)) \geq M \cap \kappa$ and for all $\alpha < \kappa$ and $\beta < \lambda$, there is $N \prec \mathfrak{A}$ and $\gamma > \beta$ such that:

- *N is κ-Magidor,*
- $\mathrm{ot}(N \cap \lambda) > \alpha$,
- $\gamma \in N$,
- $\mathrm{cf}(\gamma) \geq \kappa$,
- $V_\gamma \prec V_\lambda$,
- $N \cap V_\gamma = M$.
- $\mathrm{cf}(\sup(N \cap \lambda)) \geq M \cap \kappa$.

Then $\mathcal{E}$ is stationary.

wCC with the tree property

Let κ be almost-huge with target λ. Let $\mathbb{P}$ be Neeman's finite conditions forcing to make $\kappa = \omega_2$, and let $\mathbb{Q}$ be the further forcing as above.

By Neeman's Theorem, the tree property holds at ω_2 after $\mathbb{P} * \dot{\mathbb{Q}}$. Let us see that $\mathrm{wCC}(\omega_2, S^{\omega_2}_{\omega_1})$ also holds.

Let $q_0 \in \mathbb{Q}$ be arbitrary. Let $\dot{F}$ be a name for a fintary function on V_λ, and let $\dot{f}$ be a name for a function $\kappa \to \kappa$. Let $\mathfrak{A}$ incorporate $\dot{F}$. Let M be a κ-Magidor model such that $q_0 \in M$ and $M \in \mathcal{E}$.

Let $q_1 \leq q_0$ decide $\dot{f}(M \cap \kappa) = \xi$. Let $N \prec \mathfrak{A}$ and γ be such that:

- N is κ-Magidor.
- $q_1 \in V_\gamma \in N$.
- $\mathrm{cf}(\gamma) \geq \kappa$.
- $N \cap V_\gamma = M$.
- $\mathrm{ot}(N \cap \lambda) > \xi$.

wCC with the tree property

Let $q_2 = q_1 \cup \{V_\gamma, N\}$.

By preservation of cofinailities, $N \cap \kappa$ has uncountable cofinality in $V[G,H]$. It is forced to be closed under $\dot{F}$ by q_2. So by the equivalent charaterication of wCC, $\mathrm{wCC}(\kappa, S^\kappa_{\omega_1})$ holds in $V[G,H]$.

This shows that $2^\omega = \omega_2 + \mathrm{TP}(\omega_2)$ is consistent with $\mathrm{wCC}(\omega_2, S^{\omega_2}_{\omega_1})$, in contrast with $\mathrm{wps}(\omega_2, S^{\omega_2}_{\omega_1})$!

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Thanks for your attention!