Rotation equivalence and superrigidity

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Arctic Set theory workshop

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Background

- In the ’80s Zimmer promoted the program of understanding the actions of higher rank Lie groups and their lattices on compact manifolds (aka Zimmer’s program).
- Adams and Kechris (2000) used Zimmer’s cocycle superrigidity theorem to prove the existence of continuum many pairwise incomparable countable Borel equivalence relations, a ground breaking result in descriptive set theory.
- This work aims at expanding Adams and Kechris’ methods, and investigating those actions from the viewpoint of descriptive set theory.

Descriptive set theory focuses on definable (= Borel) objects

- Let X be a **standard Borel space**.
  E.g., $2^\omega, \omega^\omega, [0,1], \mathbb{R}^n, \mathbb{C}^n, \mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n$ with the usual σ-algebra of Borel sets.
- Let E be an equivalence relations on X.
- Assume that E is a **Borel** subset of $X \times X$.

Definition

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Define the corresponding **induced orbit equivalence relation** on X by

$$\mathcal{R}(\Gamma \curvearrowright X) \coloneqq \{(x, y) \in X \times X \mid \exists g \in \Gamma \; (g \cdot x = y)\}.$$

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Theorem (Feldman–Moore 1975)

If E is a countable Borel equivalence relation on a standard Borel space X, then there exists a countable group Γ and a Borel action of $\Gamma \curvearrowright X$ such that $E = \mathcal{R}(\Gamma \curvearrowright X)$.

Definition

Let E and F equivalence relations on the standard Borel space X and Y, respectively.

- A function $f\colon X \to Y$ is a **homomorphism** if

$$x \mathrel{E} y \implies f(x) \mathrel{E} f(y).$$

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Notation: We write $E \sim_B F$ iff $E \leq_B F$ and $F \leq_B E$.

Proposition

The following are equivalent:

1. $E \leq_B F$
2. *There is an injection from* X/E *into* Y/F *admitting Borel lifting.*

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d] & Y \arrow[d] \\ X/E \arrow[r, hook] & Y/F \end{tikzcd} \tag{1}$$

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If $E \sim_B F$, then we say that the quotient spaces X/E and Y/F have the same **Borel cardinality** $|X/E|_B = |Y/F|_B$.

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Theorem (Adams-Kechris)

$$|\mathbb{T}^m/\mathrm{GL}_m(\mathbb{Z})|_B = |\mathbb{T}^n/\mathrm{GL}_n(\mathbb{Z})|_B \iff m = n.$$

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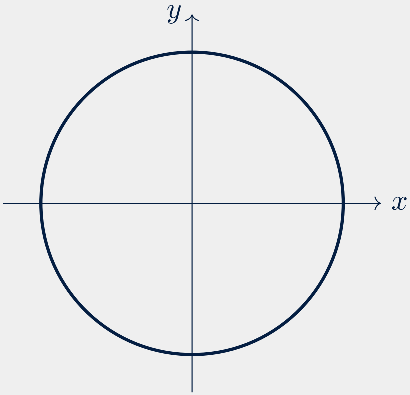
We can use **descriptive set theory** to investigate the action of countable groups on manifolds and find new instances of rigidity.

«There is geometry in the humming of the strings, there is music in the spacing of the spheres.»

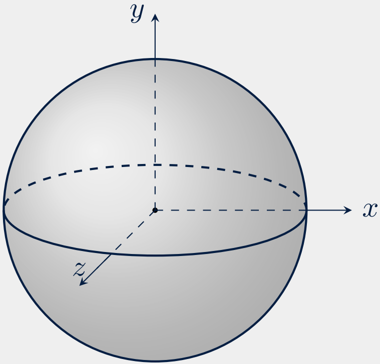
(Pythagoras)

Definition

Let $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ be the **sphere** $\mathbb{R}^n$.



$\mathbb{S}^1$ is the unit circle



$\mathbb{S}^2$ is the unit sphere

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Definition

We call $\mathcal{R}_n := \mathcal{R}(\mathrm{SO}_n(\mathbb{Q}) \curvearrowright \mathbb{S}^{n-1})$ the **(rational) rotation equivalence** in dimension n.

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E_0 is hyperfinite because $E_0 = \bigcup_{m \in \mathbb{N}} F_m$ where

$$x \, F_m \, y \iff \forall n \geq m(x(n) = y(n)).$$

Fact

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Theorem (Gao-Jackson 2015)

If Γ *is abelian, then* $\mathcal{R}(\Gamma \curvearrowright X)$ *is hyperfinite.*

An obvious consequence is that $\mathcal{R}_2 \leq_B = E_0$.

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Proposition

If the action $\Gamma \curvearrowright X$ is continuous, then TFAE:

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Proposition

Let G be a countable group acting on a Polish space X continuously. If $\mathcal{R}(\Gamma \curvearrowright X)$ is generically ergodic and every orbit is meager (e.g., when X is perfect), then $\mathcal{R}(\Gamma \curvearrowright X)$ is not smooth.

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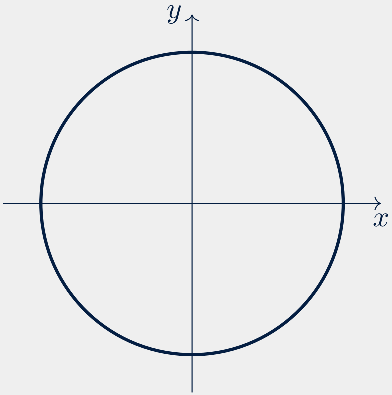

Consider $M = \frac{1}{5}\begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \in \mathrm{SO}_n(\mathbb{Q})$.

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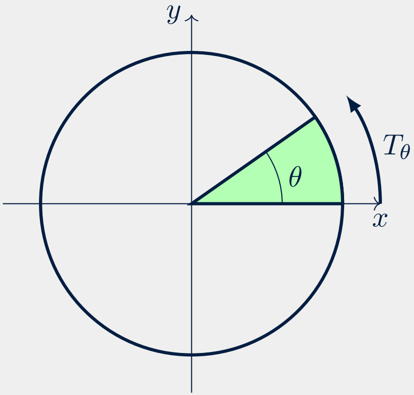

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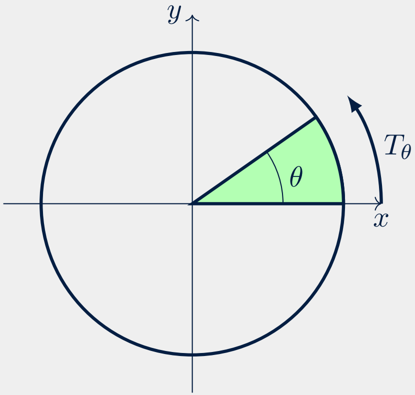

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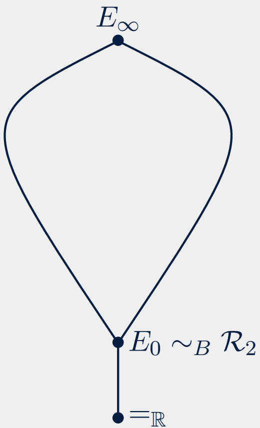
Each orbit $\{T_\theta^n(x) : n \in \mathbb{Z}\}$ is dense in $\mathbb{S}^1$.



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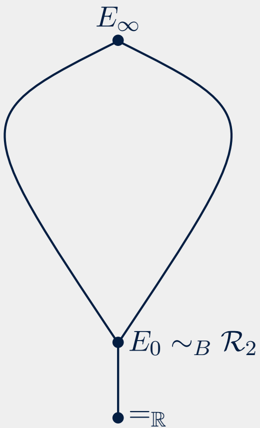
A CBER F is universal if $E \leq_B F$ for every CBER E.



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Theorem (Dougherty–Jackson–Kechris 1994)

The induced equivalence relation E_∞ induced by the shift action $\mathbb{F}_2 \curvearrowright 2^{\mathbb{F}_2}$ is universal.

Proving that some CBER is in between E_0 and E_∞ is gener(ic)ally hard.

Theorem (Hjorth–Kechris 1996)

Let E be a countable Borel equivalence relation on a Polish space X. Then there is a comeager invariant Borel set $C \subseteq X$ such that $E \restriction C$ is hyperfinite.

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Warning

We borrow tools from ergodic theory, and measure group theory.

Fact

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For $\mathrm{SO}_n(\mathbb{Q}) \curvearrowright \mathbb{S}^{n-1}$ we have

$$\mu_n\left(\operatorname{Fr}_{\mathrm{SO}_n(\mathbb{Q})} \mathbb{S}^{n-1}\right) = 1.$$

Main Theorem (C. 22+)

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For $\{m, n\} \neq \{3, 4\}$,

$$|\mathbb{S}^{m-1} / \operatorname{SO}_m(\mathbb{Q})|_B = |\mathbb{S}^{n-1} / \operatorname{SO}_n(\mathbb{Q})|_B \iff m = n.$$

Why $n \geq 5$?

Theorem (Margulis 1980)

Let $n \geq 5$. If p is prime and $p \equiv 1 \pmod 4$, then $\mathrm{SO}_n(\mathbb{Z}[\frac{1}{p}])$ is a dense subgroup of $\mathrm{SO}_n(\mathbb{R})$ with property (T).

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Theorem (Zimmer 1984)

For $n = 3, 4$, there is no infinite subgroup of $\mathrm{SO}_n(\mathbb{R})$ with property (T).

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is a Borel reduction from $\mathcal{R}_n$ to $\mathcal{R}_m$, then $A = f^{-1}(\mathrm{Fr}\,\mathbb{S}^m)$ is a Borel $\mathrm{SO}_n(\mathbb{Q})$-invariant set.It follows that $\mu_n(A) = 1$ or $\mu_n(A) = 0$ by ergodicity.

Define $\mathcal{R}_m^* := \mathcal{R}(\mathrm{SO}_n(\mathbb{Q}) \curvearrowright \mathrm{Fr}_{\mathrm{SO}_n(\mathbb{Q})}\,\mathbb{S}^{n-1})$, the **restriction of $\mathcal{R}_n$ to the free part**.

Theorem (C. ’21+)

For all $n \geq 5$ and $2 \leq m < n$, the relation $\mathcal{R}_n$ is not Borel reducible to $\mathcal{R}_m^*$.

Definition

Let $\Gamma \curvearrowright X$ and Δ be a group. A Borel function $\alpha \colon \Gamma \times X \to \Delta$ is called a **(strict) cocycle** if for all $g, h \in \Gamma$,

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$\alpha \colon \Gamma \times X \to \Delta$ is the **cocycle associated to** f.

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That is, there is a Borel map $B\colon X \to \Lambda$ such that

$$\delta(g) = B(g \cdot x)\alpha(g, x)B(x)^{-1} \qquad m_G - \text{a.e.}(x).$$

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- The associated cocycle lift to a cocycle $\alpha\colon \widetilde{\Gamma} \times \widetilde{\mathrm{SO}_n(\mathbb{R})} \to \mathrm{SO}_m(\mathbb{Q})$ where $\widetilde{\Gamma} = p^{-1}(\mathrm{SO}_n(\mathbb{Z}[\frac{1}{5}]))$ and p is the covering map.

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- Using ergodic theory and Margulis superrigidity theorem for S-arithmetic groups we can exclude both.

Theorem (C. ’22+)

Suppose that $3 \leq m < n$ and that $f : \mathbb{S}^{n-1} \to \mathbb{S}^{m-1}$ is a weak Borel reduction from $\mathcal{R}_n$ to $\mathcal{R}_m$. Then, there is a Borel $\mathrm{SO}_n(\mathbb{Q})$-invariant $Y \subseteq X$ with $\mu(Y) = 1$ such that $f(Y)$ is contained in the free part.

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Whenever $m \geq 4$ the main ingredient is the following lemma.

Lemma

Let $n > 5$ and let p be a prime number such that $p \equiv 1 \mod 4$. Let X be a standard Borel $\mathrm{SO}_n\left(\mathbb{Z}\left[\frac{1}{p}\right]\right)$-space with an invariant ergodic probability measure. Suppose that G is an algebraic $\mathbb{Q}$-group such that $\dim G < \frac{n(n-1)}{2}$ and that $H \leq G(\mathbb{Q})$. Then for every Borel cocycle $\alpha\colon \mathrm{SO}_n\left(\mathbb{Z}\left[\frac{1}{p}\right]\right) \times X \to H$, there exists an equivalent cocycle β such that $\beta\left(\mathrm{SO}_n\left(\mathbb{Z}\left[\frac{1}{p}\right]\right) \times X\right)$ is contained in a finite subgroup of H.

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This proof avoids the machinery of Zimmer's superrigidity cocycle theorem.

THANKS