On a playfully defined family of models

Ben De Bondt

IMJ-PRG Université de Paris

Arctic set theory workshop
February 16 2022

The Cofund MathInParis PhD project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 754362.

This talk reports on ongoing joint work with my thesis advisor **Boban Veličković**.

Preliminaries

Consider the following situation:

$\lambda < \theta$ are regular cardinals and $M \prec H_\theta$ is countable with $H_\lambda \in M$.

Then for every $x \in H_\lambda$, define

$$\mathtt{Hull}(M, x) = \{f(x) : f \in M \text{ a function with } \mathrm{dom}(f) = H_\lambda\}.$$

Thus

$$M \cup \{x\} \subseteq \mathtt{Hull}(M, x) \prec H_\theta$$

and $\mathtt{Hull}(M, x)$ is minimal with this property.

Let NS_{ω_1} be the ideal of nonstationary subsets of ω_1,
let $\mathsf{NS}^+_{\omega_1} = P(\omega_1) \setminus \mathsf{NS}_{\omega_1}$, we also refer to the poset $(\mathsf{NS}^+_{\omega_1}, \subseteq)$ as $\mathsf{NS}^+_{\omega_1}$.

The selfgenericity game G_M^{sg}

Let $|P(P(\omega_1))| < \lambda < \theta$ be regular cardinals.
We associate to every countable $M \prec H_\theta$ with $H_\lambda \in M$ a game G_M^{sg}.

IMJ-PRG
Institut de Mathématiques
de
Jussieu-Paris Rive Gauche

The selfgenericity game G^{sg}_M

Let $|P(P(\omega_1))| < \lambda < \theta$ be regular cardinals.
We associate to every countable $M \prec H_\theta$ with $H_\lambda \in M$ a game G^{sg}_M.

There are two players $P1$ ("the Challenger") and $P2$ ("the Constructor"), the game has length ω. As the game G^{sg}_M is played, a sequence $(M_n)_{n<\omega}$ will be constructed, with $M_0 = M$.

The selfgenericity game G^{sg}_M

Let $|P(P(\omega_1))| < \lambda < \theta$ be regular cardinals.
We associate to every countable $M \prec H_\theta$ with $H_\lambda \in M$ a game G^{sg}_M.

There are two players $P1$ ("the Challenger") and $P2$ ("the Constructor"), the game has length ω. As the game G^{sg}_M is played, a sequence $(M_n)_{n<\omega}$ will be constructed, with $M_0 = M$.

In round n of the game,

- $P1$ plays some maximal antichain $\mathcal{A}_n \subseteq \mathsf{NS}^+_{\omega_1}$ with $\mathcal{A}_n \in M_n$,
- $P2$ plays some $S_n \in \mathcal{A}_n$ such that $\omega_1 \cap M_n \in S_n$,
- define $M_{n+1} = \mathtt{Hull}(M_n, S_n)$.

$$
\begin{array}{c|cccccc}
 & M_0 & \overset{+S_0}{\rightsquigarrow} & M_1 & \overset{+S_1}{\rightsquigarrow} & M_2 & \dots \\
 & \Cup & & \Cup & & \Cup & \\
P1 & \mathcal{A}_0 & & \mathcal{A}_1 & & \mathcal{A}_2 & \dots \\
\hline
P2 & & S_0 & & S_1 & & \dots
\end{array}
$$

Institut de Mathématiques de Jussieu-Paris Rive Gauche

The selfgenericity game G^{sg}_M

$$
\begin{array}{c|cccccc}
 & M_0 & \overset{+S_0}{\rightsquigarrow} & M_1 & \overset{+S_1}{\rightsquigarrow} & M_2 & \dots \\
 & \Cup & & \Cup & & \Cup & \\
P1 & \mathcal{A}_0 & & \mathcal{A}_1 & & \mathcal{A}_2 & \dots \\
\hline
P2 & & S_0 & & S_1 & & \dots
\end{array}
$$

Winning condition:

- A player who disobeys the rules loses,
- else $P2$ wins iff $\forall n\ M_n \cap \omega_1 = M \cap \omega_1$.

The selfgenericity game G^{sg}_M

Definition

$M \prec H_\theta$ is called **selfgeneric** if for every maximal antichain $\mathcal{A} \subseteq \mathsf{NS}^+_{\omega_1}$, with $\mathcal{A} \in M$, there exists $S \in \mathcal{A} \cap M$ such that $\omega_1 \cap M \in S$.

The selfgenericity game G_M^{sg}

Definition

$M \prec H_\theta$ is called **selfgeneric** if for every maximal antichain $\mathcal{A} \subseteq \mathsf{NS}_{\omega_1}^+$, with $\mathcal{A} \in M$, there exists $S \in \mathcal{A} \cap M$ such that $\omega_1 \cap M \in S$.

Observation

$P2$ has a winning strategy for the game G_M^{sg} iff there exists some countable selfgeneric $\widetilde{M} \prec H_\theta$ such that $M \subseteq \widetilde{M}$ and $M \cap \omega_1 = \widetilde{M} \cap \omega_1$.

The strong selfgenericity game G_M^{ssg}

We consider a second game, called the **strong selfgenericity game** G_M^{ssg}. In round n, the player $P1$ now has two options:

- either $P1$ makes a move $\mathcal{A}_n \in M_n$ as in the game G_M^{sg},
  - the player $P2$ then has to answer with some S_n just as in the game G_M^{sg},
  - define $M_{n+1} = \mathtt{Hull}(M_n, S_n)$.

The strong selfgenericity game G_M^{ssg}

We consider a second game, called the **strong selfgenericity game** G_M^{ssg}. In round n, the player $P1$ now has two options:

- either $P1$ makes a move $\mathcal{A}_n \in M_n$ as in the game G_M^{sg},
  - the player $P2$ then has to answer with some S_n just as in the game G_M^{sg},
  - define $M_{n+1} = \mathtt{Hull}(M_n, S_n)$.
- or $P1$ decides to play some $x_n \in H_\lambda$ (x_n not necessarily $\in M_n$),
  - $P2$ then has to play some function $f_n : \omega_1 \to H_\lambda$ with $f_n(\omega_1 \cap M_n) = x_n$,
  - define $M_{n+1} = \mathtt{Hull}(M_n, f_n)$.

$$M_0 \overset{+S_0}{\leadsto} M_1 \overset{+f_1}{\leadsto} M_2 \overset{+S_2}{\leadsto} M_3$$

$$M_0 \ni \mathcal{A}_0, \qquad M_2 \ni \mathcal{A}_2$$

$P1$	$\mathcal{A}_0$		x_1		$\mathcal{A}_2$		x_3	$\dots$
$P2$		S_0		f_1		S_2		$\dots$

The strong selfgenericity game G_M^{ssg}

$$
\begin{array}{ccccccc}
M_0 & \overset{+S_0}{\rightsquigarrow} & M_1 & \overset{+f_1}{\rightsquigarrow} & M_2 & \overset{+S_2}{\rightsquigarrow} & M_3 \\
\Cup & & & & \Cup & & \\
\end{array}
$$

$$
\begin{array}{c|cccccccc}
P1 & \mathcal{A}_0 & & x_1 & & \mathcal{A}_2 & & x_3 & \dots \\
\hline
P2 & & S_0 & & f_1 & & S_2 & & \dots
\end{array}
$$

Winning condition:

- A player who disobeys the rules loses,
- else $P2$ wins iff $\forall n\ M_n \cap \omega_1 = M \cap \omega_1$.

IMJ-PRG
Institut de Mathématiques
de
Jussieu-Paris Rive Gauche

Victorious models

Observation

Both the games G^{sg}_M and G^{ssg}_M are closed for $P2$, and hence determined.

Definition

Given $M \prec H_\theta$ countable with $H_\lambda \in M$,

M is called **weakly victorious** iff there exists a winning strategy for $P2$ in the game G^{sg}_M.

M is called **victorious** iff there exists a winning strategy for $P2$ in the game G^{ssg}_M.

Victorious models

By the earlier observation, the following are equivalent:

- NS_{ω_1} is precipitous,
- there exist projective stationary many selfgeneric $M \prec H_\theta$,
- there exist projective stationary many weakly victorious $M \prec H_\theta$.

Question

Is also the existence of (many) **victorious** models $M \prec H_\theta$ consistent from large cardinals?

Victorious models

By the earlier observation, the following are equivalent:

- NS_{ω_1} is precipitous,
- there exist projective stationary many selfgeneric $M \prec H_\theta$,
- there exist projective stationary many weakly victorious $M \prec H_\theta$.

Question

Is also the existence of (many) **victorious** models $M \prec H_\theta$ consistent from large cardinals?

Answer

Yes!

Victorious models

In fact,

Proposition

The following are equivalent:

- NS_{ω_1} *is precipitous,*
- *there exist projective stationary many countable weakly victorious* $M \prec H_\theta$,
- *there exist projective stationary many countable victorious* $M \prec H_\theta$.

Victorious models

In fact,

Proposition

The following are equivalent:

- NS_{ω_1} *is precipitous,*
- *there exist projective stationary many countable weakly victorious* $M \prec H_\theta$,
- *there exist projective stationary many countable victorious* $M \prec H_\theta$.

Proof.

If time permits. □

Application: forcing special generic iterations

Definition

Let $\overline{M}_0$ be a countable transitive model of the theory

$$\mathsf{ZFC}^- + \text{“}P(P(\omega_1)) \text{ exists”}.$$

A generic iteration of $\overline{M}_0$ of length $\omega_1 + 1$ is a sequence

$$\mathrm{It} = (\overline{M}_\beta, g_\alpha, j_{\alpha\beta} : \alpha < \beta \leq \omega_1)$$

such that

- for every $\alpha < \omega_1$, $\overline{M}_{\alpha+1}$ is the transitivised generic ultrapower of $\overline{M}_\alpha$ w.r.t. the $(\overline{M}_\alpha, (\mathsf{NS}^+_{\omega_1})^{M_\alpha})$-generic filter g_α and $j_{\alpha\alpha+1}$ is the induced ultrapower embedding,
- for every limit ordinal $\gamma \leq \omega_1$, $(\overline{M}_\gamma, j_{\alpha\gamma} : \alpha < \gamma)$ is the transitivised direct limit of the system $(\overline{M}_\beta, g_\alpha, j_{\alpha\beta} : \alpha < \beta < \gamma)$.

Application: forcing special generic iterations

Definition

Let $\overline{M}_0$ be a countable transitive model of the theory

$$\mathsf{ZFC}^- + \text{“}P(P(\omega_1)) \text{ exists”}.$$

A generic iteration of $\overline{M}_0$ of length $\omega_1 + 1$ is a sequence

$$\text{It} = (\overline{M}_\beta, g_\alpha, j_{\alpha\beta} : \alpha < \beta \leq \omega_1)$$

such that

- for every $\alpha < \omega_1$, $\overline{M}_{\alpha+1}$ is the transitivised generic ultrapower of $\overline{M}_\alpha$ w.r.t. the $(\overline{M}_\alpha, (\mathsf{NS}^+_{\omega_1})^{\overline{M}_\alpha})$-generic filter g_α and $j_{\alpha\alpha+1}$ is the induced ultrapower embedding,
- for every limit ordinal $\gamma \leq \omega_1$, $(\overline{M}_\gamma, j_{\alpha\gamma} : \alpha < \gamma)$ is the transitivised direct limit of the system $(\overline{M}_\beta, g_\alpha, j_{\alpha\beta} : \alpha < \beta < \gamma)$.

Definition

$\overline{M}$ is called iterable if $\overline{M} \models$ “NS_{ω_1} is precipitous” and for every iteration It of $\overline{M}$ of limit length $\delta \leq \omega_1$ the direct limit of It is wellfounded.

Application: forcing special generic iterations

Observation

Working in a stationary set preserving forcing extension of V, note that the following two data describe the same kind of object:

- a generic iteration $\mathrm{It} = (\overline{M}_\beta, g_\alpha, j_{\alpha\beta} : \alpha < \beta \leq \omega_1)$ of a countable transitive model $\overline{M}_0$ of length $\omega_1 + 1$, with $\overline{M}_{\omega_1} = H^V_\lambda$,
- a continuous sequence $(M_\beta)_{\beta \leq \omega_1}$ such that $M_{\omega_1} = H^V_\lambda$ and such that for every $\beta < \omega_1$, $M_\beta \prec H^V_\lambda$ is countable and selfgeneric and $M_{\beta+1} = \mathtt{Hull}(M_\beta, M_\beta \cap \omega_1)$.

Institut de Mathématiques de Jussieu-Paris Rive Gauche

Application: forcing special generic iterations

Working under the assumption that NS_{ω_1} is precipitous, we will use the projective stationary set of victorious models to show that under this assumption there exists a stationary set preserving forcing that adds a generic iteration $\mathrm{It} = (\overline{M}_\beta, g_\alpha, j_{\alpha\beta} : \alpha < \beta \leq \omega_1)$ of a countable transitive model $\overline{M}_0$, with $\overline{M}_{\omega_1} = H_\lambda^V$.

IMJ-PRG
Institut de Mathématiques
de
Jussieu-Paris Rive Gauche

The forcing $\mathbb{P}(\lambda, \mu)$

Fix two regular cardinals $|P(P(\omega_1))| < \lambda << \mu$.

Definition

A set $B \in H_\mu$ is called "a building block" if it is a triple $B = (M, z, \Sigma)$, with:

- $M \prec H_\mu$ a victorious model,
- Σ a winning strategy for the second player in G^{ssg}_M,
- z a finite play of the game G^{ssg}_M in which $P2$ follows Σ.

The forcing $\mathbb{P}(\lambda, \mu)$

Fix two regular cardinals $|P(P(\omega_1))| < \lambda << \mu$.

Definition

A set $B \in H_\mu$ is called "a building block" if it is a triple $B = (M, z, \Sigma)$, with:

- $M \prec H_\mu$ a victorious model,
- Σ a winning strategy for the second player in G^{ssg}_M,
- z a finite play of the game G^{ssg}_M in which $P2$ follows Σ.

Notation

- If z is a finite play of the game G^{ssg}_M, write $P2(z)$ for the set of moves that have been made by $P2$.
- If B is a building block, then

$$\mathrm{Ext}(B) := \mathtt{Hull}(M, P2(z)) \prec H_\theta.$$

The forcing $\mathbb{P}(\lambda, \mu)$

Define two orders on the set of building blocks.
Let B_1, B_2 be two building blocks, then:

- $B_1 < B_2 \iff B_1 \in \mathrm{Ext}(B_2)$,
- $B_1 \sqsubseteq_{\text{game}} B_2 \iff B_1 = (M, z_1, \Sigma)$ and $B_2 = (M, z_2, \Sigma)$ and z_2 extends z_1.

The forcing $\mathbb{P}(\lambda, \mu)$

Define two orders on the set of building blocks.
Let B_1, B_2 be two building blocks, then:

- $B_1 < B_2 \iff B_1 \in \mathrm{Ext}(B_2)$,
- $B_1 \sqsubseteq_{\text{game}} B_2 \iff B_1 = (M, z_1, \Sigma)$ and $B_2 = (M, z_2, \Sigma)$ and z_2 extends z_1.

Definition (The forcing $\mathbb{P}(\lambda, \mu)$)

p is a $\mathbb{P}(\lambda, \mu)$-condition iff

- p is a finite set of building blocks,
- for every two building blocks $B_1, B_2 \in p$,

$$B_1 > B_2 \text{ or } B_1 = B_2 \text{ or } B_1 < B_2.$$

The order on $\mathbb{P}(\lambda, \mu)$: if $p, q \in \mathbb{P}(\lambda, \mu)$, then

$$q \leq p \iff (\forall B \in p)(\exists B' \in q) B \sqsubseteq_{\text{game}} B'.$$

We will conclude by sketching the behaviour of $\mathbb{P}(\lambda, \mu)$ on the flipchart.

Application: increasing $\underset{\sim}{\delta}^1_2$

The ordinal $\underset{\sim}{\delta}^1_2$ is the supremum of all lengths of $\underset{\sim}{\Delta}^1_2$-prewellorderings of $\mathbb{R}$. If the universe is closed under sharps for reals, then $\underset{\sim}{\delta}^1_2 = u_2$, where u_2 is the minimal ordinal $> \omega_1$ which is x-indiscernible for every real x.

Woodin: if $P(\omega_1)^\sharp$ exists and NS_{ω_1} is saturated, then $\underset{\sim}{\delta}^1_2 = \omega_2$.

Given λ, does there exist a forcing that forces $\underset{\sim}{\delta}^1_2$ to be at least λ?

Under NS_{ω_1} precipitous together with existence of sharps:

- Ketchersid, R., Larson, P. and Zapletal, J. (2007) *Increasing δ^1_2 and Namba-Style Forcing*, The Journal of Symbolic Logic **72**, p. 1372-1378.
- Claverie, B. and Schindler, R. (2009) *Increasing u_2 by a Stationary Set Preserving Forcing*, The Journal of Symbolic Logic **74**, p. 187-200.
- the forcing $\mathbb{P}(\lambda, \mu)$.

Thank you!