On a playfully defined family of models

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This talk reports on ongoing joint work with my thesis advisor **Boban Veličković**.



Preliminaries

Consider the following situation:

 $\lambda < \theta$ are regular cardinals and $M \prec H_{\theta}$ is countable with $H_{\lambda} \in M$. Then for every $x \in H_{\lambda}$, define

 $\operatorname{Hull}(M, x) = \{f(x) : f \in M \text{ a function with } \operatorname{dom}(f) = H_{\lambda}\}.$

Thus

$$M \cup \{x\} \subseteq \operatorname{Hull}(M, x) \prec H_{\theta}$$

and Hull(M, x) is minimal with this property.

Let NS_{ω_1} be the ideal of nonstationary subsets of ω_1 , let $NS_{\omega_1}^+ = P(\omega_1) \setminus NS_{\omega_1}$, we also refer to the poset $(NS_{\omega_1}^+, \subseteq)$ as $NS_{\omega_1}^+$.

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Let $|P(P(\omega_1))| < \lambda < \theta$ be regular cardinals. We associate to every countable $M \prec H_{\theta}$ with $H_{\lambda} \in M$ a game G_{M}^{sg} .



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There are two players P1 ("the Challenger") and P2 ("the Constructor"), the game has length ω . As the game G_M^{sg} is played, a sequence $(M_n)_{n<\omega}$ will be constructed, with $M_0 = M$.

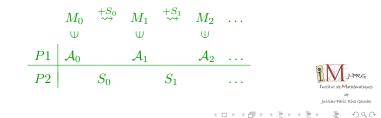


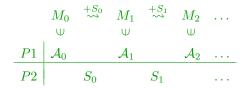
Let $|P(P(\omega_1))| < \lambda < \theta$ be regular cardinals. We associate to every countable $M \prec H_{\theta}$ with $H_{\lambda} \in M$ a game G_{M}^{sg} .

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In round n of the game,

- P1 plays some maximal antichain $\mathcal{A}_n \subseteq \mathsf{NS}^+_{\omega_1}$ with $\mathcal{A}_n \in M_n$,
- P2 plays some $S_n \in \mathcal{A}_n$ such that $\omega_1 \cap M_n \in S_n$,
- define $M_{n+1} = \operatorname{Hull}(M_n, S_n)$.





Winning condition:

- A player who disobeys the rules loses,
- else P2 wins iff $\forall n \ M_n \cap \omega_1 = M \cap \omega_1$.



Definition

 $M \prec H_{\theta}$ is called **selfgeneric** if for every maximal antichain $\mathcal{A} \subseteq \mathsf{NS}^+_{\omega_1}$, with $\mathcal{A} \in M$, there exists $S \in \mathcal{A} \cap M$ such that $\omega_1 \cap M \in S$.



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Observation

P2 has a winning strategy for the game G_M^{sg} iff there exists some countable selfgeneric $\widetilde{M} \prec H_{\theta}$ such that $M \subseteq \widetilde{M}$ and $M \cap \omega_1 = \widetilde{M} \cap \omega_1$.



The strong selfgenericity game G_M^{ssg}

We consider a second game, called the strong selfgenericity game G_M^{ssg} . In round n, the player P1 now has two options:

- either P1 makes a move $\mathcal{A}_n \in M_n$ as in the game G_M^{sg} ,
 - the player P2 then has to answer with some S_n just as in the game G_M^{sg} ,
 - define $M_{n+1} = \operatorname{Hull}(M_n, S_n)$.



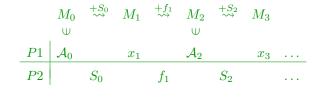
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 - the player P2 then has to answer with some S_n just as in the game G^{sg}_M,
 define M_{n+1} = Hull(M_n, S_n).
- or P1 decides to play some $x_n \in H_\lambda$ (x_n not necessarily $\in M_n$),
 - P2 then has to play some function $f_n : \omega_1 \to H_\lambda$ with $f_n(\omega_1 \cap M_n) = x_n$,
 - define $M_{n+1} = \operatorname{Hull}(M_n, f_n)$.

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The strong selfgenericity game G_M^{ssg}



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Observation

Both the games G_M^{sg} and G_M^{ssg} are closed for P2, and hence determined.

Definition

Given $M \prec H_{\theta}$ countable with $H_{\lambda} \in M$,

M is called **weakly victorious** iff there exists a winning strategy for P2 in the game G_M^{sg} .

M is called **victorious** iff there exists a winning strategy for P2 in the game G_M^{ssg} .



By the earlier observation, the following are equivalent:

- NS_{ω_1} is precipitous,
- there exist projective stationary many selfgeneric $M \prec H_{\theta}$,
- there exist projective stationary many weakly victorious $M \prec H_{\theta}$.

Question

Is also the existence of (many) **victorious** models $M \prec H_{\theta}$ consistent from large cardinals?



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Question

Is also the existence of (many) victorious models $M \prec H_{\theta}$ consistent from large cardinals?

Answer

Yes!



In fact,

Proposition

The following are equivalent:

- NS_{ω_1} is precipitous,
- there exist projective stationary many countable weakly victorious $M \prec H_{\theta}$,
- there exist projective stationary many countable victorious $M \prec H_{\theta}$.



In fact,

Proposition

The following are equivalent:

- NS_{ω_1} is precipitous,
- there exist projective stationary many countable weakly victorious $M \prec H_{\theta}$,
- there exist projective stationary many countable victorious $M \prec H_{\theta}$.

Proof.

If time permits.



Definition

Let \overline{M}_0 be a countable transitive model of the theory

 $\mathsf{ZFC}^- + "P(P(\omega_1))$ exists".

A generic iteration of \overline{M}_0 of length $\omega_1 + 1$ is a sequence

 $\mathbf{It} = (\overline{M}_{\beta}, g_{\alpha}, j_{\alpha\beta} : \alpha < \beta \le \omega_1)$

such that

- for every $\alpha < \omega_1$, $\overline{M}_{\alpha+1}$ is the transitivised generic ultrapower of \overline{M}_{α} w.r.t. the $(\overline{M}_{\alpha}, (NS^+_{\omega_1})^{M_{\alpha}})$ -generic filter g_{α} and $j_{\alpha\alpha+1}$ is the induced ultrapower embedding,
- for every limit ordinal γ ≤ ω₁, (M
 γ, j{αγ} : α < γ) is the transitivised direct limit of the system (M
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Definition

 \overline{M} is called iterable if $\overline{M} \models "NS_{\omega_1}$ is precipitous" and for every iteration It of \overline{M} of limit length $\delta \leq \omega_1$ the direct limit of It is wellfounded.

Observation

Working in a stationary set preserving forcing extension of V, note that the following two data describe the same kind of object:

- a generic iteration It = (M
 _β, g_α, j_{αβ} : α < β ≤ ω₁) of a countable transitive model M
 ₀ of length ω₁ + 1, with M
 _{ω₁} = H^V_λ,
- a continuous sequence $(M_{\beta})_{\beta \leq \omega_1}$ such that $M_{\omega_1} = H^V_{\lambda}$ and such that for every $\beta < \omega_1, M_{\beta} \prec H^V_{\lambda}$ is countable and selfgeneric and $M_{\beta+1} = \operatorname{Hull}(M_{\beta}, M_{\beta} \cap \omega_1).$



Working under the assumption that NS_{ω_1} is precipitous, we will use the projective stationary set of victorious models to show that under this assumption there exists a stationary set preserving forcing that adds a generic iteration It = $(\overline{M}_{\beta}, g_{\alpha}, j_{\alpha\beta} : \alpha < \beta \leq \omega_1)$ of a countable transitive model \overline{M}_0 , with $\overline{M}_{\omega_1} = H_{\lambda}^V$.



Fix two regular cardinals $|P(P(\omega_1))| < \lambda \ll \mu$.

Definition

A set $B \in H_{\mu}$ is called "a building block" if it is a triple $B = (M, z, \Sigma)$, with:

- $M \prec H_{\mu}$ a victorious model,
- Σ a winning strategy for the second player in G_M^{ssg} ,
- z a finite play of the game G_M^{ssg} in which P2 follows Σ .

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Notation

- If z is a finite play of the game G_M^{ssg} , write P2(z) for the set of moves that have been made by P2.
- If B is a building block, then

 $\operatorname{Ext}(B) := \operatorname{Hull}(M, P2(z)) \prec H_{\theta}.$

Define two orders on the set of building blocks. Let B_1 , B_2 be two building blocks, then:

- $B_1 < B_2 \iff B_1 \in \operatorname{Ext}(B_2)$,
- $B_1 \sqsubseteq_{\text{game}} B_2 \iff B_1 = (M, z_1, \Sigma) \text{ and } B_2 = (M, z_2, \Sigma)$ and z_2 extends z_1 .

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Definition (The forcing $\mathbb{P}(\lambda, \mu)$)

- p is a $\mathbb{P}(\lambda,\mu)\text{-condition}$ iff
 - p is a finite set of building blocks,
 - for every two building blocks $B_1, B_2 \in p$,

 $B_1 > B_2$ or $B_1 = B_2$ or $B_1 < B_2$.

The order on $\mathbb{P}(\lambda, \mu)$: if $p, q \in \mathbb{P}(\lambda, \mu)$, then

 $q \leq p \iff (\forall B \in p) (\exists B' \in q) B \sqsubseteq_{\text{game}} B'.$

We will conclude by sketching the behaviour of $\mathbb{P}(\lambda,\mu)$ on the flipchart.



Application: increasing δ_2^1

The ordinal δ_2^1 is the supremum of all lengths of Δ_2^1 -prewellorderings of \mathbb{R} . If the universe is closed under sharps for reals, then $\delta_2^1 = u_2$, where u_2 is the minimal ordinal $> \omega_1$ which is x-indiscernible for every real x.

Woodin: if $P(\omega_1)^{\sharp}$ exists and NS_{ω_1} is saturated, then $\delta_2^1 = \omega_2$.

Given λ , does there exist a forcing that forces δ_2^1 to be at least λ ?

Under NS_{ω_1} precipitous together with existence of sharps:

- Ketchersid, R., Larson, P. and Zapletal, J. (2007) *Increasing* δ¹₂ and *Namba-Style Forcing*, The Journal of Symbolic Logic **72**, p. 1372-1378.
- Claverie, B. and Schindler, R. (2009) *Increasing* u_2 by a Stationary Set *Preserving Forcing*, The Journal of Symbolic Logic **74**, p. 187-200.
- the forcing $\mathbb{P}(\lambda, \mu)$.

Thank you!

