ITP at successors of singular cardinals

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joint work with various friends

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A key point: like TP, the strong tree property and ITP can also hold at successor cardinals.

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An immediate difficulty: no elementary embedding with such critical point.

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5. We use that $j \restriction \kappa = \text{id}$ and that levels of T have size less than κ.

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At successors of singulars, this does not quite work.

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The obstacle to an easy ITP-adaptation: the required branch must pass though some prefixed nodes stationary often.

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  - b is **an ineffable branch** if $\{a \mid x_a = b \cap a\}$ is stationary.

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The main obstacle with implementing Magidor-Shelah's strategy: the measures on say κ_0 and κ_{n+1} do not cohere.

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- do a careful interplay between the various normal measures.
- Finally: pick $\bar{\alpha} < \mu$ above which all of the latter split. Show that one of them must be ineffable by partitioning a stationary subset of μ into κ_{n} many sets.

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Need branch preservation lemmas.

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- Here each κ_n becomes $\aleph_{n+3}$ and μ becomes $\aleph_{\omega+1}$.

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Key point: at all times have to consider all possible branches thought all possible names for lists.

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So if κ is singular strong limit and SCH holds at κ, then the tree property fails at κ^{++}.

Some major challenges: tension between

- failure of SCH at κ (an instance of non-compactness) vs.
- TP/ITP at κ^+ (an instance of compactness).

Solovay: SCH holds above a strong compact cardinal.
Q: Does ITP_κ imply SCH above κ? Or the strong tree property?

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In particular, SCH fails at $\lambda^{+\omega}$.

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ITP_λ in the generic extension:

- Since λ is supercompact in V, V $\models \mathsf{ITP}_\lambda$.
- To show that this is still the case in the generic extension,
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- pull back the branch to the right model.

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The construction:use Neeman's model i.e.force with a version of the diagonal Gitik-Sharon forcing after adding subsets of κ.

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The construction:use Neeman's model i.e.force with a version of the diagonal Gitik-Sharon forcing after adding subsets of κ.
A key feature: cannot lift over the Prikry. So, we must work with names of list.

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