Generalized descriptive set theory under I0

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Objective

The study of definable subsets of non-separable spaces with singular uncountable weight.
Or, doing generalized descriptive set theory with singular cardinals instead of regular ones.

Inspiration (Kechris)

"Descriptive set theory is the study of definable sets in Polish spaces", and of their regularity properties.

Classical case

Polish spaces: separable completely metrizable spaces, e.g. the *Cantor space* ${}^{\omega}2$ and the *Baire space* ${}^{\omega}\omega$.
Definable subsets: *Borel* sets, *analytic* sets, *projective* sets...
Regularity properties: Perfect set property (PSP), Baire property, Lebesgue measurability...

Classical results

- ${}^{\omega}2$ is the unique compact perfect zero-dimensional Polish space
- Every zero-dimensional Polish space is homeomorphic to a closed space of ${}^{\omega}\omega$ and a G_δ subset of ${}^{\omega}2$, therefore results on the Cantor space spread to all zero-dimensional Polish spaces
- Every Polish space is continuous image of a closed subset of ${}^{\omega}\omega$
- Lusin Separation Theorem and Souslin Theorem (i.e., Borel = bi-analytic)
- Every analytic set satisfies PSP, Baire property and Lebesgue measurability
- Silver Dichotomy, i.e., PSP for co-analytic equivalence relations

By the first and second points, all the other points are true in any zero-dimensional Polish space, and "partially" true in every Polish space.

We are going now to analyze past approaches to generalize this setting.
There is a branch of research in set theory called "Generalized descriptive set theory". It consists of replacing ω with κ everywhere, where κ is regular and most of the time $\kappa^{<\kappa} = \kappa$. It has remarkable connections with other areas of set theory and model theory.

GDST

Generalized Cantor and Baire spaces: ${}^{\kappa}2$ and ${}^{\kappa}\kappa$, endowed with the *bounded topology*, i.e., the topology generated by the sets $N_s = \{x \in {}^{\kappa}2 : s \sqsupseteq \kappa\}$ with $s \in {}^{<\kappa}2$ or ${}^{<\kappa}\kappa$ respectively.
Definable subsets: κ^+-*Borel* sets = sets in the κ^+-algebra generated by open sets; κ-*analytic* sets = continuous images of closed subsets of ${}^{\kappa}\kappa$
Regularity properties: κ-PSP for a set A = either $|A| \leq \kappa$ or ${}^{\kappa}2$ topologically embeds into A; κ-Baire property (sometimes)...

What happens to the "nice" properties that we had on the classical case? A lot is lost, most of the nontrivial results are either false or independent of ZFC:

GDST results

Complete metrizability: If $\kappa > \omega$ and regular, then ${}^{\kappa}2$ is not completely metrizable. Therefore we cannot talk of "κ-Polish spaces".
Lusin separation and Souslin theorems: False when $\kappa > \omega$.
PSP: κ-PSP for closed/κ^+-Borel/analytic sets is independent of ZFC.
Silver Dichotomy: κ-Silver Dichotomy is independent of ZFC (very false in L for κ inaccessible).
Bonus: ${}^{\kappa}\kappa \not\approx {}^{\kappa}2$ only if κ is weakly compact.

The culprit here seems to be the fact that κ is regular. What if κ is singular? There is already some bibliography on that...

A.H.Stone, *Non-separable Borel sets*, 1962

Baire spaces: $\Pi_{n\in\omega} T_n$, where each T_n is discrete. In particular, the space $B(\lambda) = {}^{\omega}\lambda$ and, if $\mathrm{cof}(\lambda) = \omega$, the space $C(\lambda) = \Pi_{n\in\omega}\lambda_n$, where λ_n's are cofinal in λ.
Definable subsets: *Borel* sets (σ-algebra generated by open sets); λ-analytic sets = continuous image of $B(\lambda)$
Regularity properties: λ-PSP for a set A = either $|A| \leq \lambda$ or ${}^{\lambda}2$ topologically embeds into A.

Woodin, *Suitable extender models II*, 2012

Baire space: $V_{\lambda+1}$, where λ satisfies $\mathrm{I0}(\lambda)$, with the topology where the open sets are $O_{a,\alpha} = \{x \subseteq V_\lambda : x \cap V_\alpha = a\}$, with $\alpha < \lambda$ and $a \subseteq V_\alpha$.
Definable subsets: very complicated, the simplest are in $L_1(V_{\lambda+1})$, therefore λ-projective
Regularity properties: different definitions of PSP (the details later).

Also Džamonja-Väänänen suggested that maybe singular cardinals could give a better picture: they studied a bit of generalized descriptive set theory with κ singular of cofinality ω, mainly in connection with model theory (models of ω-chain logic).

We wanted to give some order to this variety of approaches, and define a single framework where they all live, and that is close to the "classical" approach.

Baire and Cantor spaces

Fix λ uncountable cardinal of cofinality ω, and λ_n cofinal sequence in it. So, we have four different approaches:

$$^{\lambda}2 \quad B(\lambda) = {}^{\omega}\lambda \quad C(\lambda) = \Pi_{n\in\omega}\lambda_n \quad V_{\lambda+1}$$

Proposition (Džamonja-Väänänen, D.-Motto Ros)

The following spaces are homeomorphic:

- $^{\lambda}2$
- $\Pi_{n\in\omega}{}^{\lambda_n}2$
- $\Pi_{n\in\omega}2^{\lambda_n}$ where 2^{λ_n} is discrete
- $^{\omega}(2^{<\lambda})$ where $2^{<\lambda}$ is discrete

It is therefore immediate to see that when λ is strong limit, then all the spaces above are homeomorphic!
On the other hand, $^{\lambda}2 \not\approx {}^{\lambda}\lambda$, as $^{\lambda}\lambda$ has density $\lambda^{<\lambda} > \lambda$;

Universality properties

Definition

A space is *uniformly zero-dimensional* if for any $U \neq \emptyset$ open, every $\epsilon > 0$, every $i \in \omega$, U can be partitioned into $\geq \lambda_i$-many clopen sets with diameter $< \epsilon$.

Andrea Medini noticed that this implies ultraparacompactness, or $dim = 0$.

Theorem (A.H.Stone)

Up to homeomorphism, ${}^{\lambda}2$ is the unique uniformly zero-dimensional λ-Polish space, therefore results on the generalized Cantor space spread to all uniformly zero-dimensional λ-Polish spaces.

Proposition (D.-Motto Ros)

Every λ-Polish space is continuous image of a closed subset of ${}^{\omega}\lambda$, therefore results on the generalized Cantor space partially spread to all λ-Polish spaces.

Definable subsets

On ${}^{\lambda}2$ we consider λ^+-Borel sets, as in GDST. It can be proven that these sets can be stratified in a hierarchy with exactly λ^+-many levels. Also, since λ is singular, λ^+-Borel $=$ λ-Borel.

As for the analytic sets...

Classical case

In the classical case, tfae:

- A is a continuous image of a Polish space;
- $A = \emptyset$ or A is a continuous image of ${}^\omega\omega$;
- A is a continuous image of a closed set $F \subseteq {}^\omega\omega$;
- A is the continuous/Borel image of a Borel subset of ${}^\omega\omega$ or ${}^\omega 2$;
- A is the projection of a closed subset of $X \times {}^\omega\omega$;
- A is the projection of a Borel subset of $X \times Y$, where Y is ${}^\omega\omega$ or ${}^\omega 2$.

New case

If λ is a singular cardinal of cofinality ω, tfae:

- A is a continuous image of a λ-Polish space;
- $A = \emptyset$ or A is a continuous image of ${}^{\omega}\lambda$;
- A is a continuous image of a closed set $F \subseteq {}^{\omega}\lambda$;
- A is the continuous/Borel image of a Borel subset of ${}^{\omega}\lambda$ or ${}^{\lambda}2$;
- A is the projection of a closed subset of $X \times {}^{\omega}\lambda$;
- A is the projection of a Borel subset of $X \times Y$, where Y is ${}^{\omega}\lambda$ or ${}^{\lambda}2$.

Again, this is not true if λ is regular.

Proposition (D.-Motto Ros)

- The collection of λ-analytic subsets of any λ-Polish space contains all open and closed sets, and is closed under λ-unions and λ-intersections. In particular, λ-Borel sets are λ-analytic.
- There are λ-analytic subsets of ${}^{\lambda}2$ that are not λ-Borel.

Generalized Luzin separation theorem (D.-Motto Ros)

If A, B are disjoint λ-analytic subsets of a λ-Polish space, then A can be separated from B by a λ-Borel set.

Generalized Souslin theorem (D.-Motto Ros)

A subset of a λ-Polish space is λ-bianalytic iff it is λ-Borel.

Perfect set property

Definition

A subset A of a topological space X has the λ-PSP if either $|A| \leq \lambda$ or else ${}^{\lambda}2$ topologically embeds into A.

A.H.Stone

Every λ-analytic subset of a uniformly zero-dimensional λ-Polish space has the λ-PSP.

Silver dichotomy

Theorem (D.-Shi)

Let λ be a strong limit cardinal of cofinality ω. Suppose that λ is limit of measurable cardinals. Let E be a coanalytic equivalence relation on ${}^{\omega}\lambda$. Then exactly one of the following holds:

- E has at most λ many classes:
- there is a continuous injection $\varphi : {}^{\omega}\lambda \to {}^{\omega}\lambda$ such that for distinct $x, y \in {}^{\omega}\lambda$ $\neg\varphi(x) E \varphi(y)$.

Corollary

Let λ be a strong limit cardinal of cofinality ω. Suppose that λ is limit of measurable cardinals. Let E be a coanalytic equivalence relation on a uniformly zero-dimensional λ-Polish space. Then exactly one of the following holds:

- E has at most λ many classes:
- there is a continuous injection $\varphi : {}^{\omega}\lambda \to {}^{\omega}\lambda$ such that for distinct $x, y \in {}^{\omega}\lambda$ $\neg\varphi(x) E \varphi(y)$.

This is the first case where the proof is not only almost cut-and-paste from the classical case, but needs some further tools. The starting point is the G_0-dichotomy: Ben Miller's proof works also on this setting.
But the Baire category argument fails. Instead of that, we have some argument that relies on the properness of the diagonal Prikry forcing..

List of things that do not generalize well in this setting:

- λ-analytic sets are not exactly those that are the continuous image of ${}^{\lambda}\lambda$: in fact, it is possible that all the λ-projective sets are the continuous image of ${}^{\lambda}\lambda$
- it is not clear how to define the λ-meager sets: either the countable union of nowhere dense sets, but then they are really small (it is not clear even if Borel sets have the Baire property), or the λ-union of nowhere dense sets, but then the whole space is meager.

A bit further...

One of the intriguing consequences of the Axiom of Determinacy is that under AD the regularities properties hold for *all* the subsets of ${}^{\omega}2$.

A pivotal property that relates large cardinals and determinacy is being κ-weakly homogeneously Suslin. Informally, a subset A of ${}^{\omega}2$ is κ-weakly homogeneously Suslin if there is a tree-structure of κ-complete ultrafilters that can reconstruct A by looking at the well-founded towers of ultrafilters.

Theorem (Woodin)

If there are ω Woodin cardinals and a measurable above, then every subset of $\mathbb{R}$ in $L(\mathbb{R})$ is κ-weakly homogeneously Suslin for some κ.

Theorem

All the κ-weakly homogeneously Suslin subsets of ${}^{\omega}2$ have the PSP, the Baire property and are Lebesgue measurable.

This structure is almost mimicked in the I0 case.

Definition

I0(λ): There exists an elementary embedding $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with critical point less than λ.

I0(λ) implies that λ is a strong limit cardinal of cofinality ω. It is an incredible large cardinal, at the very top of the hierarchy: for example it is much stronger than I3(λ), that in turn implies that λ is limit of cardinals that are n-huge for any n.

It induces on $L(V_{\lambda+1})$ a structure that is similar to $L(\mathbb{R})$ under AD. For example, $L(V_{\lambda+1}) \not\models AC$, and λ^+ is measurable in $L(V_{\lambda+1})$.

Woodin developed an analogue of κ-weakly homogeneously Suslin also under I0: $U(j)$-representability.

Theorem (Cramer-Woodin)

Suppose that j witnesses $\mathrm{I0}(\lambda)$. Then every subset of $V_{\lambda+1}$ in $L_{\lambda^+}(V_{\lambda+1})$ is $U(j)$-representable.

In still unpublished work, Cramer proved that if j witnesses $\mathrm{I0}(\lambda)$ then all subsets of $V_{\lambda+1}$ in $L(V_{\lambda+1})$ are $U(j)$-representable.

Woodin, *Suitable extender models II*, 2012

Suppose that j witnesses $\mathrm{I0}(\lambda)$. Then every set that is $U(j)$-representable has the PSP.

Shi, *Axiom I_0 and higher degree theory*, 2015

Suppose that j witnesses $\mathrm{I0}(\lambda)$, and that every set in $L(V_{\lambda+1})$ is $U(j)$-representable. Then every set has the λ-PSP (space embedded: $C(\lambda)$).

Cramer, *Inverse limit reflection and the structure of $L(V_{\lambda+1})$*, 2015

Suppose that j witnesses $\mathrm{I0}(\lambda)$. Then every set has the λ-PSP (space embedded: $B(\lambda)$).

In both the *I*0 and AD cases there is a similar double argument:

- Under some condition, proving that every set has a certain structure;
- Proving that all the sets with a certain structure have the desired regularity property.

We are looking to generalize the second statement, again defining a unique backdrop that works for any space.

Definition

A family $\mathbb{U}$ of ultrafilters is *orderly* iff there exists a set K such that for all $U \in \mathbb{U}$ there is $n \in \omega$ such that ${}^nK \in U$. Such n is called the *level* of U.

A *tower* of ultrafilters in such a $\mathbb{U}$ is a sequence $(U_n)_{n\in\omega}$ such that for all $m < n < \omega$:

- $U_n \in \mathbb{U}$ has level n;
- U_n projects to U_m;

A tower of ultrafilters $(U_n)_{n\in\omega}$ is *well-founded* iff for every sequence $(A_n)_{n\in\omega}$ with $A_n \in U_n$ there is $z \in {}^{\omega}K$ such that $z \restriction n \in A_n$ for any $n \in \omega$.

Definition

Let $\kappa \geq \lambda$ be a cardinal, and $\mathbb{U}$ be an orderly family of κ-complete ultrafilters. A $(\mathbb{U}, \kappa)$-representation for $Z \subseteq {}^{\omega}\lambda$ is a function $\pi : \bigcup_{i \in \omega} {}^{i}\lambda \times {}^{i}\lambda \to \mathbb{U}$ such that:

- if $s, t \in {}^{i}\lambda$, then $\pi(s, t)$ has level i;
- for any $(s, t) \in {}^{n}\lambda$, if $(s', t') \sqsupseteq (s, t)$ then $\pi(s', t')$ projects to $\pi(s, t)$;
- $x \in Z$ iff there is a $y \in {}^{\omega}\lambda$ such that $(\pi(x \restriction i, y \restriction i))_{i \in \omega}$ is well-founded.

If $\lambda = \omega$ and $A \subseteq \mathbb{R}$ is κ-weakly homogeneously Suslin, then A is $(\mathbb{U}, \kappa)$-representable for some $\mathbb{U}$.
Consider the homeomorphism between $V_{\lambda+1}$ and ${}^{\omega}\lambda$. Then the image of a $U(j)$-representable set is $(\mathbb{U}, \kappa)$-representable for some $\mathbb{U}, \kappa$, and viceversa.

Definition

A $(\mathbb{U},\kappa)$-representation π for a set $Z \subseteq {}^{\omega}\lambda$ has the *tower condition* if there exists $F : ran\pi \to \bigcup \mathbb{U}$ such that:

- $F(U) \in U$ for all $U \in ran\pi$
- for every $x, y \in {}^{\omega}\lambda$, the tower of ultrafilters $(\pi(x \restriction i, y \restriction i))_{i\in\omega}$ is well-founded iff there is $z \in {}^{\omega}K$ such that $z \restriction i \in F(\pi(x \restriction i, y \restriction i))$ for all $i \in \omega$.

If κ is much larger than λ (e.g., $\lambda = \omega$ and κ measurable), then the tower condition is for free.

Scott Cramer proved that under I0 every representation has a tower condition.

Theorem (D.-Motto Ros)

Let λ be strong limit with $\mathrm{cof}(\lambda) = \omega$ and let $\kappa \geq \lambda$ be a cardinal. If $Z \subseteq {}^{\omega}\lambda$ admits a $(\mathbb{U}, \kappa)$-representation with the tower condition, then Z has the λ-PSP.

Corollary

Assume $\mathrm{I0}(\lambda)$, as witnessed by j. If $A \in \mathcal{P}(V_{\lambda+1}) \cap L(V_{\lambda+1})$ is $U(j)$-representable, then A has the λ-PSP.

Corollary

Assume $\mathrm{I0}(\lambda)$. All λ-projective subsets of any uniformly zero-dimensional λ-Polish space have the λ-PSP.

Corollary

Assume $\mathrm{I0}(\lambda)$, as witnessed by a proper j with $\mathrm{crt}(j) = \kappa$. Let $\mathbb{P}$ be the Prikry forcing on κ with respect to the measure generated by j. Then there exists a $\mathbb{P}$-generic extension $V[G]$ of V in which all κ-projective subsets of any uniformly zero-dimensional κ-Polish space have the κ-PSP.

A look into the future

This new approach opens a lot problems. All the "classical" results are up for grabs.

The following is a *personal* selection of possible developments.

Open problem

Is it necessary for the Silver dichotomy for λ to be limit of measurable cardinals?

Open problem

How to define meager, comeager, Baire property?

Open problem

Is there a model where all the subsets of ${}^{\lambda}2$ are $(\mathbb{U}, \kappa)$-representable, different from $L(\mathbb{R})$ under large cardinals or $L(V_{\lambda+1})$ under I0? Maybe $L(\mathcal{P}(\aleph_\omega))$ under generic I0? Or $L(V_{\lambda+1})$ when λ is limit of Berkeley cardinals that are limits of extendible cardinals?

Theorem (AD+V=$L(\mathbb{R})$, Hjorth, *A dichotomy for the definable universe*, 1995)

Let E be an equivalence relation on ${}^{\omega}2$. Then exactly one of the following holds:

- the classes of E are well-ordered;
- there is a continuous injection $\varphi : {}^{\omega}2 \to {}^{\omega}2$ such that for distinct $x, y \in {}^{\omega}2$ $\neg\varphi(x) E \varphi(y)$.

Open problem

Same thing, under I0?

Theorem (D.-Shi)

Suppose I0(λ), as witness by j, and let $(\lambda_n)_{n\in\omega}$ be the critical sequence of j. Suppose that all subsets of $V_{\lambda+1}$ are $U(j)$-representable. Let $E \in L(V_{\lambda+1})$ be an equivalence relation such that if $x, y \in {}^{\omega}\lambda$ differs only in one coordinate, then $\neg xEy$, then there is a continuous injection $\Pi_{n\in\omega}\lambda_n \to \Pi_{n\in\omega}\lambda_n$ such that for distinct $x, y \in \Pi_{n\in\omega}\lambda_n$ $\neg\varphi(x) E \varphi(y)$.

Thanks for watching.