Pseudo-Prikry sequences

(Joint and ongoing work with Spencer Unger)

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I: Historical background

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$\mathbb{P}_U$ is known as *Prikry forcing* (with respect to U). There is now a large class of variations on Prikry forcing, known collectively as *Prikry-type forcings*, which add diagonalizing sequences to a large cardinal κ, to a set of the form $\mathcal{P}_\kappa(\lambda)$, or to a sequence of such objects.

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Theorem (Džamonja-Shelah, [3])

Suppose that:

1. *V is an inner model of W;*
2. *κ is an inaccessible cardinal in V and a singular cardinal of cofinality θ in W;*
3. $(\kappa^+)^W = (\kappa^+)^V$;
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Then, in W, there is a sequence $\langle \gamma_i \mid i < \theta \rangle$ of ordinals such that, for all $\alpha < \kappa^+$ and all sufficiently large $i < \theta$, $\gamma_i \in C_\alpha$.

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A similar theorem is proven by Gitik [4], and it is extended by Magidor and Sinapova [5], who also prove the following generalization.

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Theorem (Magidor-Sinapova, [5])

Suppose that $n < \omega$ and:

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2. *κ is a regular cardinal in V and, for all $m \leq n$, $(\kappa^{+m})^V$ has countable cofinality in W;*
3. *$(\kappa^+)^W = (\kappa^{+n+1})^V$;*
4. *$\langle D_\alpha \mid \alpha < \kappa^{+n+1} \rangle \in V$ is a sequence of clubs in $\mathcal{P}_\kappa(\kappa^{+n})$.*

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Then, in W, there is a sequence $\langle x_i \mid i < \omega \rangle$ of elements of $(\mathcal{P}_\kappa(\kappa^{+n}))^V$ such that, for all $\alpha < \kappa^{+n+1}$ and all sufficiently large $i < \omega$, $x_i \in D_\alpha$.

Theorem (Cummings-Schimmerling in the context of Prikry forcing, [2])

Suppose that V is an inner model of W, κ is inaccessible in V and a singular cardinal of countable cofinality in W, and $(\kappa^+)^W = (\kappa^+)^V$.

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Theorem (Brodsky-Rinot, [1])

Suppose that λ is a regular, uncountable cardinal, $2^\lambda = \lambda^+$, and $\mathbb{P}$ is a λ^+-c.c. forcing notion of size $\leq \lambda^+$. Suppose moreover that, in $V^{\mathbb{P}}$, λ is a singular ordinal and $|\lambda| > \mathrm{cf}(\lambda)$.

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Then there is a λ^+-Souslin tree in $V^{\mathbb{P}}$.

II: Fat trees and pseudo-Prikry sequences

Definition

Suppose κ is a regular, uncountable cardinal, $n < \omega$, and, for all $m \leq n$, $\lambda_m \geq \kappa$ is a regular cardinal. Then

$$T \subseteq \bigcup_{k \leq n+1} \prod_{m < k} \kappa_m$$

is a *fat tree* of type $(\kappa, \langle \lambda_0, \ldots, \lambda_n \rangle)$ if:

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2. for all $\sigma \in T$ such that $k := \mathrm{lh}(\sigma) \leq n$, $\mathrm{succ}_T(\sigma) := \{\alpha \mid \sigma^\frown \langle \alpha \rangle \in T\}$ is $(< \kappa)$-club in κ_k.

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Lemma

If $\mathcal{C}$ is a club in $\mathcal{P}_\kappa(\kappa^{+n})$, then there is a fat tree of type $(\kappa, \langle \kappa^{+n}, \kappa^{+n-1}, \ldots, \kappa \rangle)$ such that, for every maximal $\sigma \in T$, there is $x \in \mathcal{C}$ such that, for all $m \leq n$, $\sup(x \cap \kappa^{+m}) = \sigma(n-m)$.

Outside guessing of fat trees

Theorem

Suppose that:

1. V *is an inner model of* W*;*
2. *in* V, $\kappa < \lambda$ *are cardinals, with* κ *regular;*
3. *in* W, $\theta < \theta^{+2} < |\kappa|$, θ *is a regular cardinal, and there is a* $\subseteq$*-increasing sequence* $\langle x_i \mid i < \theta \rangle$ *from* $(\mathcal{P}_\kappa(\lambda))^V$ *such that* $\bigcup_{i<\theta} x_i = \lambda$*;*
4. $(\lambda^+)^V$ *remains a cardinal in* W*;*
5. $n < \omega$ *and, in* V, $\langle \lambda_i \mid i \leq n \rangle$ *is a sequence of regular cardinals from* $[\kappa, \lambda]$ *and* $\langle T(\alpha) \mid \alpha < \lambda^+ \rangle$ *is a sequence of fat trees of type* $(\kappa, \langle \lambda_0, \ldots, \lambda_n \rangle)$*.*

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Then, in W*, there is a sequence* $\langle \sigma_i \mid i < \theta \rangle$ *such that, for all* $\alpha < \lambda^+$ *and all sufficiently large* $i < \theta$, σ_i *is a maximal element of* $T(\alpha)$*.*

Proof sketch ($n = 0$)

Our sequence of fat trees is just a sequence $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ of clubs in λ_0.

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3. for all $\alpha, \beta < \lambda^+$, there is $\gamma < \lambda^+$ such that $f_\beta^{C_\alpha} < f_\gamma$.

Proof sketch (cont.)

Move now to W, where we have $\langle x_i \mid i < \theta \rangle$. Define a sequence $\vec{g} = \langle g_\beta \mid \beta < \lambda^+ \rangle$ from θ to λ_0 by letting $g_\beta(i) = f_\beta(x_i)$. Note that:

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2. for all $\gamma \in S^{\lambda^+}_{>\theta}$, there is a club D_γ in γ such that, for all $\beta \in D_\gamma$, $g_\beta < g_\gamma$;
3. $\theta^{+3} < \lambda^+$.

Therefore, $\vec{g}$ has an *exact upper bound*, i.e. a $<^*$-upper bound h such that, for every $h' <^* h$, there is $\beta < \lambda^+$ such that $h' <^* g_\beta$. Moreover, we may assume $\mathrm{cf}(h)(i) > \theta$ for all $i < \theta$, so $h : \theta \to \lambda_0$.

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If not, then there is $\alpha < \lambda^+$ and an unbounded $A \subseteq \theta$ such that, for all $i \in A$, $\gamma_i \notin C_\alpha$.

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$h' < h$, so there is $\beta < \lambda^+$ such that $h' <^* g_\beta$. But then there is $\gamma < \lambda^+$ such that $f_\beta^{C_\alpha} < f_\gamma$.

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$$\max(C_\alpha \cap h(i)) < g_\beta(i) < h(i) < \min(C_\alpha \setminus g_\beta(i)) < g_\gamma(i).$$

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$$\max(C_\alpha \cap h(i)) < g_\beta(i) < h(i) < \min(C_\alpha \setminus g_\beta(i)) < g_\gamma(i).$$

In particular, h is *not* a $<^*$-upper bound for $\vec{g}$. Contradiction!

III: Diagonal sequences

Diagonal clubs

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2. If $\kappa \leq \mu_0$ is a regular cardinal, then a *diagonal club in* $\mathcal{P}_\kappa(\vec{\mu})$ is a sequence $\langle D_i \mid i < \theta \rangle$ such that, for all $i < \theta$, D_i is club in $\mathcal{P}_\kappa(\mu_i)$.

Diagonal ordinal sequences

Theorem

Suppose that:

1. V *is an inner model of* W*;*
2. *in* V*,* μ *is a singular cardinal of cofinality* θ*;*
3. *there is* $\kappa < \mu$ *such that every* V*-regular cardinal in* $[\kappa, \mu)$ *has cofinality* θ *in* W*;*
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4. *in W, $(\mu^+)^V$ remains a cardinal and $\theta^{+2} < |\mu|$.*

Then there are:

- *an increasing sequence of regular cardinals $\vec{\mu} = \langle \mu_i \mid i < \theta \rangle \in V$, cofinal in μ;*

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such that, for every $\langle C_i \mid i < \theta \rangle \in V$ *that is a diagonal club in* $\vec{\mu}$, *for all sufficiently large* $i < \theta$, $g(i) \in C_i$.

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3. *in* V, $\vec{\mu} = \langle \mu_i \mid i < \theta \rangle$ *is an increasing sequence of regular cardinals, cofinal in* μ, *with* $\kappa \leq \mu_0$*;*
4. *in* W, *there is a* $\subseteq$*-increasing sequence* $\langle x_i \mid i < \theta \rangle$ *from* $(\mathcal{P}_\kappa(\mu))^V$ *such that* $\bigcup_{i<\theta} x_i = \mu$*;*
5. *in* W, $(\mu^+)^V$ *remains a cardinal and* $\mu \geq 2^\theta$*;*
6. *in* V, $\langle \vec{D}(\alpha) \mid \alpha < \mu^+ \rangle$ *is a sequence of diagonal clubs in* $\mathcal{P}_\kappa(\vec{\mu})$.

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1. V *is an inner model of* W*;*
2. *in* V, $\mathrm{cf}(\mu) = \theta < \kappa = \mathrm{cf}(\kappa) < \mu$ *are cardinals, with* μ *strong limit;*
3. *in* V, $\vec{\mu} = \langle \mu_i \mid i < \theta \rangle$ *is an increasing sequence of regular cardinals, cofinal in* μ*, with* $\kappa \leq \mu_0$*;*
4. *in* W*, there is a* $\subseteq$*-increasing sequence* $\langle x_i \mid i < \theta \rangle$ *from* $(\mathcal{P}_\kappa(\mu))^V$ *such that* $\bigcup_{i<\theta} x_i = \mu$*;*
5. *in* W, $(\mu^+)^V$ *remains a cardinal and* $\mu \geq 2^\theta$*;*
6. *in* V, $\langle \vec{D}(\alpha) \mid \alpha < \mu^+ \rangle$ *is a sequence of diagonal clubs in* $\mathcal{P}_\kappa(\vec{\mu})$*.*

Then, in W*, there is* $\langle y_i \mid i < \theta \rangle$ *such that, for all* $\alpha < \mu^+$ *and all sufficiently large* $i < \theta$, $y_i \in D(\alpha)_i$*.*

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Thank you!