Automorphism Groups of Homogeneous Structures

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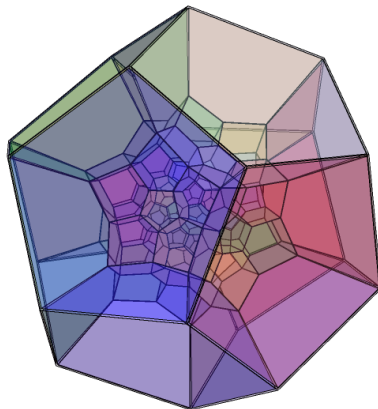
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A "CLASSICAL ENIGMA": RECONSTRUCTING FROM SYMMETRY.

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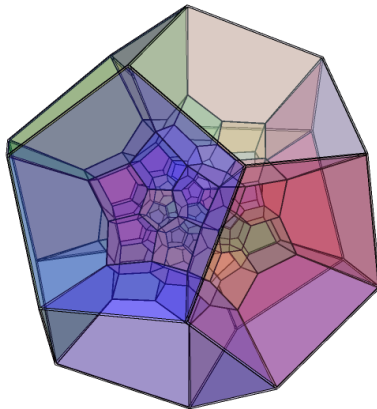$Aut(M)$.

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Tell me what is M!

Reconstructing models   The Small Index Property   SIP beyond first order   Examples: quasiminimal classes, the Zilber field, j-invariants

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Reconstructing models?

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- we follow ONE line of reconstruction, different from (but related to) the work of M. Rubin!

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  field with finite extensions of degree > 2 and prime to $char(K)$.

These are versions of the same kind of problem - but we will not
concentrate on these today. They may however be amenable to
model theoretic treatment.

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- ▶ If M is $\aleph_0$-categorical, any open subgroup of $Aut(M)$ is a stabilizer $Aut_\alpha(M)$ for some imaginary α. Also $Aut(M) \curvearrowright \{H \leq Aut(M) \mid H \text{ open}\}$ (conjugation).

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- ▸ The action $Aut(M) \curvearrowright$ is (almost) $\approx$ to $Aut(M) \curvearrowright M^{eq}$.

So, we have recovered the action of $Aut(M)$ on M^{eq} from the topology of $Aut(M)$... so, if M, N are countable $\aleph_0$-categorical structures, TFAE:

- ▸ There is a bicontinuous isomorphism from $Aut(M)$ onto $Aut(N)$

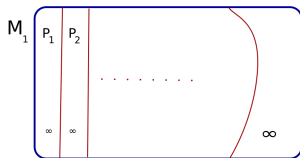- ▸ M and N are bi-interpretable.

Some (non-)Examples - Why is saturation needed?

Let M_1 be the countable
saturated model of P_i $(i < \omega)$
disjoint infinite predicates and
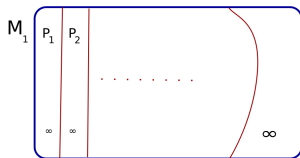
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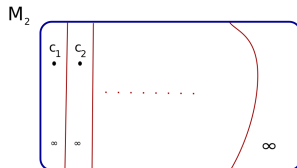Let M_1 be the countable saturated model of P_i ($i < \omega$) disjoint infinite predicates and
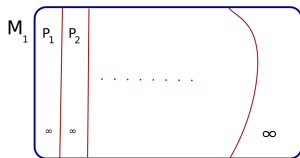

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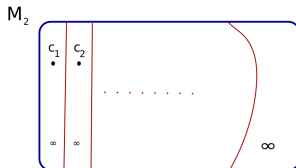

$$\text{yet } Aut(M_1) \approx Aut(M_2)$$

SIP - THE LINK BETWEEN ALGEBRA AND TOPOLOGY
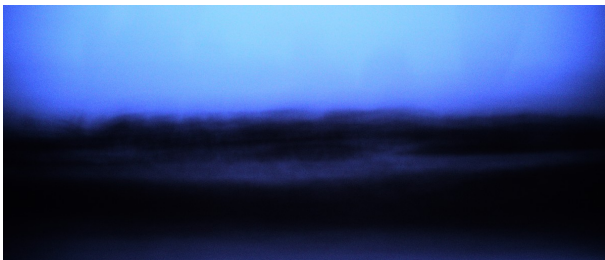
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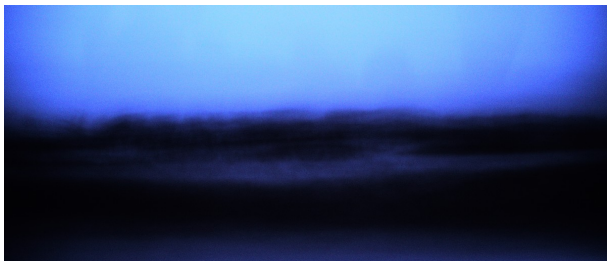
Now, to the main property of the group $Aut(M)$ that enables us to capture its topology...

The Small Index Property (countable version)

Definition (Small Index Property - SIP)

Let M be a countable structure. M has the small index property if for any subgroup H of $Aut(M)$ of index less than $2^{\aleph_0}$, there exists a finite set $A \subset M$ such that $Aut_A(M) \subset H$.

Basic facts on countable SIP

SIP allows us to recover the topological structure of $Aut(M)$ from its pure group structure:

Open neighborhoods of 1 in pointwise convergence topology =

Subgroups containing pointwise stabilizers $Aut_A(M)$ for some finite A.

BASIC FACTS ON COUNTABLE SIP

SIP allows us to recover the topological structure of $Aut(M)$ from its pure group structure:

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- ▶ SIP holds for random graph, infinite set, DLO, vector spaces over finite fields, generic relational structures, $\aleph_0$-categorical $\aleph_0$-stable structures, etc.
- ▶ It fails e.g. for $M \models ACF_0$ with ∞ transc. degree.

Reconstructing models   **The Small Index Property**   SIP beyond first order   Examples: quasiminimal classes, the Zilber field, j-invariants

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Galois group (of a theory)

The Galois group of a model M,

$$Gal(M) := Aut(M)/Autf(M),$$

is invariant across saturated models of a theory.
Possible failures of SIP are encoded in this quotient.

To the uncountable / the non-elementary

SIP FOR UNCOUNTABLE STRUCTURES

We now switch focus to the uncountable, first order, case.
Fix $\lambda = \lambda^{<\lambda}$ an uncountable cardinal, and fix M a saturated model of cardinality λ.

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We now use the topology $\mathcal{T}^\lambda$ on $Aut(M)$, whose basic open sets around 1_M are stabilizers of subsets of size $< \lambda$ - as before $Aut_A(M)$ but now $A \subset M$ with $|A| < \lambda$.

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$Aut(M)$ with this topology is of course no longer a Polish space. The techniques from Descriptive Set Theory that have been used for the countable case need to be replaced (Friedman, Hyttinen and Kulikov have a start of Descriptive Set Theory for some uncountable cardinalities, however).

Lascar-Shelah's Theorem

Theorem (Lascar-Shelah: Uncountable saturated models have the SIP)

Let M be saturated, of cardinality $\lambda = \lambda^{<\lambda}$ and let G be a subgroup of $Aut(M)$ such that $[Aut(M) : G] < 2^\lambda$. Then there exists $A \subset M$ with $|A| < \lambda$ such that $Aut_A(M) \subset G$.

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The proof consists of building directly (assuming that G does not contain any open set $Aut_A(M)$ around the identity) a **binary tree** of height λ of automorphisms of M in such a way that every two of them are not conjugate. This is enough but requires two crucial notions: **generic** and **existentially closed (sequences of) automorphisms**. These are obtained by assuming that G is not open.

Beyond First Order

Although results on the reconstruction problem, so far have been stated and <u>proved</u> for saturated models in first order theories, the scope of the matter can go far beyond:

▶ Abstract Elementary Classes with well-behaved closure notions, and the particular case:

Beyond First Order

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- ▶ Abstract Elementary Classes with well-behaved closure notions, and the particular case:
- ▶ Quasiminimal (qm excellent) Classes.

The setting: Strong amalgamation classes

A setting for homogeneity: let $(\mathcal{K}, \prec_{\mathcal{K}})$ be an AEC, with $LS(\mathcal{K}) \leq \lambda$, $|M| = \kappa > \lambda$, $\kappa^{<\kappa} = \kappa$.

Let $\mathcal{K}^{<}(M) := \{N : N \preceq_K M, |N| < \kappa\}$ and fix $M \in \mathcal{K}$ homogeneous.

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Let $\mathcal{K}^{<}(M) := \{N : N \prec_K M, |N| < \kappa\}$ and fix $M \in \mathcal{K}$ homogeneous.

The topology τ^{cl}: base of open neighborhoods given by sets of the form $Aut_X(M)$ where $X \in \mathcal{C}$, where $\mathcal{C} := \left\{ \mathrm{cl}^M(A) : A \subseteq M \text{ such that } |A| < \kappa \right\}$ and the "closure operator" is $cl^M(A) := \bigcap_{A \subset N \prec_{\mathcal{K}} M} A$.

This class of cl^M-closed sets has enough structure for the proof of SIP.

THE MAIN RESULT: SIP FOR HOMOGENEOUS AEC.

Theorem (SIP for $(Aut(M), \mathcal{T}^{cl})$ - Ghadernezhad, V.)

Let M be a homogeneous model in an AEC $(\mathcal{K}, \prec_{\mathcal{K}})$, with $|M| = \lambda = \lambda^{<\lambda} > LS(\mathcal{K})$, such that $\mathcal{K}^{<\lambda}$ is a strong amalgamation class. Let $G \leq Aut(M)$ with $[Aut(M) : G] \leq \lambda$ (this is, G has small index in $Aut(M)$). Then there exists $X \in \mathcal{C}$ such that $Aut_X(M) \leq G$ (i.e., G is open in $(Aut(M), \mathcal{T}^{cl})$).

Getting many non-conjugates

<u>Proof</u> (rough sketch): suppose G has small index in $Aut(M)$ but is
not open (does not contain any basic $Aut_X(M)$ for $X \in \mathcal{C}$).
We have enough tools (generic sequences and strong amalgamation
bases) to build a Lascar-Shelah tree to reach a contradiction (2^λ
many branches giving automorphisms of M g_σ for $\sigma \in 2^\lambda$ such that
if $\sigma \neq \tau \in 2^\lambda$ then $g_\sigma^{-1} \circ g_\tau \notin G$).
Of course, the possibility of getting these 2^λ-many automorphisms
requires using the non-openness of G to get the construction going.

LASCAR-SHELAH TREE FOR OUR SITUATION

A λ-Lascar-Shelah tree for M and $G \leq Aut(M)$ is a binary tree of height λ with, for each $s \in 2^{<\lambda}$, a model $M_s \in \mathcal{K}^<(M)$, $g_s \in Aut(M_s)$, $h_s, k_s \in Aut_{M_s}(M)$ such that

- $h_{s,0} \in G$ and $h_{s,1} \notin G$ for all $s \in \mathcal{S}$;

- $k_{s,0} = k_{s,1}$ for all $s \in \mathcal{S}$;

- for $s \in \mathcal{S}$ and all $t \in \mathcal{S}$ such that $t \leqslant s : h_t [M_s] = M_s$ (i.e. $h_t \in Aut_{\{M_s\}}(M)$) and ...;

- for $s \in \mathcal{S}$ and all $t \in \mathcal{S}$ such that $t \leqslant s : g_s \cdot (h_t \restriction M_s) \cdot g_s^{-1} = k_t \restriction M_s$;

- for $s \in \mathcal{S}$ and $\beta < length\,(s)$: $a_s \in M_s$;

- for all s, the families $\{h_t : t \leqslant s, t \in \mathcal{S}\}$ and $\{k_t : t \leqslant s, t \in \mathcal{S}\}$ are elements of $\mathcal{F}$ (i.e. they are generic).

Generic sequences and Strong amalgamation bases

The main technical tools in the construction of a LS tree are

- ▶ Guaranteeing <u>generic</u> sequences of automorphisms
  ($g \in Aut(M)$ is generic if
  $\forall N \in \mathcal{K}^<(M)$ such that $g \upharpoonright N \in Aut(N)$
  $\forall N_1 \succ_{\mathcal{K}} N, N_1 \in \mathcal{K}^<(M)$
  $\forall h \supset g \upharpoonright N, h \in Aut(N_1)$
  $\exists \alpha \in Aut_N(M)$ such that $g \supset \alpha \circ h \circ \alpha^{-1}$),

- ▶ showing they are unique up to conjugation,

- ▶ getting a generic sequence $\mathcal{F} = (g_i : i \in I)$ such that
    1. the set $\{i \in I : g_i \upharpoonright M_0 = h \text{ and } g_i \notin G\}$ has cardinality κ for all
       $M_0 \in \mathcal{K}^<(M)$ and $h \in Aut(M_0)$;
    2. the set $\{i \in I : g_i \in G\}$ has cardinality κ.

Quasiminimal pregeometry classes

In a language L, a <u>quasiminimal pregeometry</u> class $\mathcal{Q}$ is a class of pairs $\langle H, \mathrm{cl}_H \rangle$ where H is an L-structure, cl_H is a pregeometry operator on H such that the following conditions hold:

1. Closed under isomorphisms,
2. For each $\langle H, \mathrm{cl}_H \rangle \in \mathcal{Q}$, the closure of any finite set is countable.
3. If $\langle H, \mathrm{cl}_H \rangle \in \mathcal{Q}$ and $X \subseteq H$, then $\langle \mathrm{cl}_H (X), \mathrm{cl}_H \restriction \mathrm{cl}_H (X) \rangle \in \mathcal{Q}$.
4. If $\langle H, \mathrm{cl}_H \rangle, \langle H', \mathrm{cl}_{H'} \rangle \in \mathcal{Q}$, $X \subseteq H$, $y \in H$ and $f : H \to H'$ is a partial embedding defined on $X \cup \{y\}$, then $y \in \mathrm{cl}_H (X)$ if and only if $f(y) \in \mathrm{cl}_{H'} (f(X))$.
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These can all be generated by ONE canonical structure.

VERY RECENT UPDATES

- ► In November 2016 - about a week ago, Sébastien Vasey has posted a paper on the ArXiV proving that quasiminimal pregeometries do not <u>require</u> the exchange axiom of pregeometries. This makes it in principle easier to prove that classes are quasiminimal!

- ► Vasey has also suggested that our theorem applies to wider classes (excellent classes, and even wider: certain "non-forking frames"). This is work in progress now.

Example: quasiminimal classes, "Zilber field"

- $\mathcal{Q}$ quasiminimal pregeometry class. $M \in \mathcal{Q}$ of size $\aleph_1$, $\mathcal{C} = \{cl(A) \mid A \subset M, A \text{ small}\}$ then $\mathcal{C}$ has the free aut-independence amalgamation property. (Based on Haykazyan's paper on qm classes.)

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- The "Zilber field" has SIP.
- The j-invariant has the SIP.

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Thanks, thanks, to you all (and especially to J and J)