LOGIC COLLOQUIUM 78
M. Boffa, D. van Dalen, K. McAloon (eds.)

ABSTRACT LOGIC AND SET THEORY. I. DEFINABILITY

Jouko Väänänen
Department of mathematics
University of Helsinki
Hallituskatu 15
SF-00100 Helsinki 10
Finland

Definability in an abstract logic is compared with definability in set theory. This leads to set theoretical characterizations of implicit definability, Löwenheim-numbers and Hanf-numbers of various abstract logics. A new logic, sort logic, is introduced as the ultimate limit of abstract logics definable in set theory.

§ 0. Introduction

The aim of this paper is to bring together, in a coherent framework, both old and new results about unbounded abstract logics (a logic is unbounded if it is able to characterize the notion of well-ordering). Typical problems that can be asked about any logic are:

(1) Which model classes are implicitly (with extra predicates and sorts) definable?
(2) Which classes of cardinals are spectra?
(3) What is the Löwenheim-number?
(4) What is the Hanf-number?

In the case of unbounded logics these problems are particularly relevant as such logics fail to be axiomatizable and mostly lack workable model theory. An attempt to shed light on (1)-(4) is the main purpose of this paper.

Out method is to build, right from the beginning, a close connection between abstract logic and set theory.

The basic notion of the whole paper is that of symbiosis. We say that an abstract logic L^* and a predicate P of set theory are symbiotic if, roughly speaking, the family of $\Delta(L^*)$-definable model classes coincides with the family of model classes which are $\Delta_1(P)$. For example, second order logic L^{II} is symbiotic with the power-set operation, or, what amounts to the same,

$$\Delta(L^{II}) = \{K \mid \text{the model class } K \text{ is } \Delta_2\}.$$

In Chapter 2 we give a new proof of the following result (essentially due to

Oikkonen [10]): If L^* and P are symbiotic, then

$$\Delta_n(L^*) = \{K \mid \text{the model class } K \text{ is } \Delta_n(P)\}.$$

As a corollary we get for $n > 1$:

$$\Delta_n(L_{\omega\omega}) = \{K \mid \text{the model class } K \text{ is } \cdot\Delta_n\}.$$

Consideration of the logics $\Delta_n(L_{\omega\omega})$ leads very naturally to what we call sort logic. To grasp the idea of sort logic, let us consider a typical many-sorted structure

$$\mathbf{M} = \langle M_1,\ldots,M_n;R_1,\ldots,R_m;a_1,\ldots,a_k\rangle.$$

$\mathbf{M}$ consists of three kinds of objects: universes M_i, relations R_i and individuals a_i. To quantify over the individuals we have first order logic; to quantify over relations we have second order logic; but to quantify over universes (i.e. sorts) we need a new logic. Accordingly, let sort logic L^S be the many-sorted logic which allows quantification over individuals, relations and sorts. It is clearly impossible to define the semantics of sort logic in set theory, but it can be done, for example, in MKM (Morse-Kelley-Mostowski) theory of classes.

It follows readily from the above analysis of $\Delta_n(L_{\omega\omega})$ that

$$L^S = \{K \mid \text{the model class } K \text{ is definable in set theory}\}$$

(stated in [8] p. 174).

The rest of Chapter 2 is devoted to an analysis of the non-syntactic nature of the Δ-operation. We show, for example, that the set of L^{II}-sentences which give rise to $\Delta(L^{II})$-definitions, is Π_3- but not Σ_3-definable in set theory. This result reflects the difficultness of finding a simple syntax for $\Delta(L^{II})$.

Chapter 3 is concerned with a restricted Δ-operation, Δ^1_1, which does not allow the use of new sorts (or universes). This operation is clearly related to L^{II} as we may think of L^{II} as $\Delta^1_{(\omega)}(L_{\omega\omega})$. The key notion of this chapter is that of a flat formula of set theory. We obtain the following characterization of generalized second order logic: If L^* and P satisfy a strengthend symbiosis assumption, then

$$\Delta^1_{(\omega)}(L^*) = \{K \mid \text{the model class } K \text{ is defined by a flat formula of the language } \{\varepsilon,P\}\}.$$

In particular

$$L^{II} = \{K \mid \text{the model class } K \text{ is defined by a flat formula of set theory}\}.$$

These results are proved in a level-by-level form.

In Chapter 4 we extend the analysis of the set theoretic nature of model theoretic definability to spectra and Löwenheim-numbers $\mathcal{l}(L^*)$. We characterize the spectra of symbiotic logics and prove for L^*, symbiotic with P,

$$\mathcal{l}(\Delta_n(L_A^*)) = \sup\{\alpha \mid \alpha \text{ is } \Pi_n(P)\text{-definable with parameters in } A\}$$

$$\mathcal{l}(\Delta_n(L_A)) = \sup\{\alpha \mid \alpha \text{ is } \Delta_n\text{-definable with parameters in } A\} \ (n > 1).$$

A similar analysis of Hanf-numbers $h(L^*)$ is carried out in Chapter 5. The non-preservation of Hanf-numbers under Δ necessiates the introduction of a bounded Δ-operation Δ^B, and respective set theoretical notions $\sum_1^B$, Π_1^B and Δ_1^B. The main result says: If L^* and P are symbiotic in a sufficiently bounded way, then

$$h(L_A^*) = \sup\{\alpha \mid \alpha \text{ is } \textstyle\sum_1^B(P)\text{-definable with parameters in } A\}$$

and for $n > 1$,

$$h(\Delta_n(L_A)) = \sup\{\alpha \mid \alpha \text{ is } \textstyle\sum_n(P)\text{-definable with parameters in } A\}.$$

In the rest of Chapter 5 we consider the numbers

$$\mathcal{l}_n = \sup\{\alpha \mid \alpha \text{ is } \Pi_n\text{-definable}\}$$

$$h_n = \sup\{\alpha \mid \alpha \text{ is } \textstyle\sum_n\text{-definable}\}.$$

Note that $\mathcal{l}_n = \mathcal{l}(\Delta_n(L_{\omega\omega}))$ and $h_n = h(\Delta(L_{\omega\omega}))$ (for $n > 1$). It turns out that for $n > 1$,

$$\mathcal{l}_n = \sup\{\alpha \mid \alpha \text{ is } \Delta_n\text{-definable}\}$$

and

$$\mathcal{l}_n < h_n = \mathcal{l}_{n+1}.$$

In particular, we get

$$\mathcal{l}(L^S) = h(L^S) = \sup\{\alpha \mid \alpha \text{ is definable in set theory}\}.$$

This paper is based on Chapter 2 of the author's Ph.D. thesis and the author wishes to use this opportunity to express his gratitude to his supervisor P.H.G. Aczel for the help he provided during the preparation of the thesis. This work was financially supported by Osk. Huttunen Foundation.

§ 1. Preliminaries

We give at first a rough sketch of the preliminaries, which should be enough for a casual reader familiar with [2] and [8]. More detailed preliminaries then follow.

Our abstract logics are defined roughly as in [2]. If Q is a generalized quantifier LQ is like $L_{\omega\omega}[Q]$ in [8]. I is the Härtig-quantifier, W the well-ordering quantifier and Q_H the Henkin-quantifier. L^{II} is second order logic. All logics are understood to be many-sorted. The logic which is obtained from L^{II} by adding quantification over sorts is called sort logic and denoted L^s. If L^* is an abstract logic, $\sum(L^*)$ is the family of PC-classes of L^* in the sense of [8]. $\Pi(L^*)$ consists of the complements of PC-classes of L^*. $\sum_n(L^*)$ and $\Pi_n(L^*)$ are obtained by iterations of the $\sum$- and Π-operations. $\Delta_n(L^*)$ refers to the intersection of $\sum_n(L^*)$ and $\Pi_n(L^*)$. The families $\sum^1_n(L^*)$, $\Pi^1_n(L^*)$ and $\Delta^1_n(L^*)$ are defined similarly but the PC-definitions are not allowed to introduce new sorts. This ends the sketch.

1.1. Abstract logics

For many-sorted logic we refer to [5]. Types are sets of sorts, relation-symbols and constant-symbols. If L is a type, the class of all structures of type L is denoted $Str(L)$. If $M \in Str(L)$ and K is a type such that $K \subset L$, then $M|_K$ denotes the reduct of M to K. If $x \in L$, then x^M denotes the interpretation of x in M. $|M|$ denotes the union of the universes of M.

A quasilogic is a pair $L^* = \langle Stc^*, \models^* \rangle$ such that

(L1) If $\varphi \in L^*$ (that is $Stc^*(L,\varphi)$), then L is a type and φ is a set called an L^*-sentence,

(L2) If $M \models^* \varphi$ (that is $\models^* (M,\varphi)$), then there is a type L such that $M \in Str(L)$ and $\varphi \in L^*$,

(L3) If $M \models^* \varphi$ and $M \cong N$, then $N \models^* \varphi$.

This definition is somewhat weaker than the definition of a system of logics in [2], and substantially weaker than the definition of a logic in [8].

The quasilogic $L_{\infty\omega}$ is defined as usual. If $Q^1,\ldots,Q^n$ are generalized quantifiers, we let $L_{\infty\omega}(Q^1,\ldots,Q^n)$ denote the quasilogic which is obtained from $L_{\infty\omega}$ by addition of the new quantifiers $Q^1\ldots Q^n$. Second order infinitary logic, which is denoted by $L^{II}_{\infty\omega}$, is obtained from $L_{\infty\omega}$ by addition of quantification over (finitary) relations. The following generalized quantifiers play a special role in this paper:

Härtig-quantifier: $IxyA(x)B(y) \leftrightarrow card(A) = card(B)$,

Well-ordering-quantifier: $WxyA(x,y) \leftrightarrow$ A well-orders its domain,

Regularity-quantifier: $RxyA(x,y) \leftrightarrow$ A orders its domain in the type of a regular cardinal,

Henkin-quantifier: $Q_H xyuvA(x,y,u,v) \leftrightarrow \forall f \forall g \exists x \exists y A(x,y,f(x),g(y))$.

Note that our Henkin-quantifier is the dual of the original one.

If L^* is a quasilogic we let L^*_A be the quasilogic the sentences of which are those $\varphi \in L^*$ for which $L \in A$ and $\varphi \in A$, and the semantics of which follows that of L^*. For example, $(L_{\infty\omega})_{H(\kappa)}$ will be $L_{\kappa\omega}$ if the syntax of $L_{\infty\omega}$ is defined in the usual set theoretical way (see e.g. [3]). We denote L^*_{HF} by $L^*_{\omega\omega}$ and in general $L^*_{H(\kappa)}$ by $L^*_{\kappa\omega}$. $L_{\omega\omega}(Q^1,\ldots,Q^n)$ and $L^{II}_{\omega\omega}$ are shortend to $L(Q^1,\ldots,Q^n)$ and L^{II}. As usual, L_A denotes $(L_{\infty\omega})_A$. For $\omega < \lambda \in A$, we use $L_{A\lambda}$ to denote $(L_{\infty\lambda})_A$. Similarly L_{AG} denotes $(L_{\infty G})_A$. If $\kappa < \lambda$, then $L_{\kappa\lambda}$ does not make much sense, but we redefine it as $L_{\kappa\kappa}$ added with the weak second order quantifiers $\exists X(|X| \leq \alpha \wedge \ldots)$ for $\alpha < \lambda$. The obvious set theoretic definition gives $L_{\kappa\lambda} \subset H(\kappa + |\alpha|)$ whenever $\lambda = \aleph_\alpha$.

A class of structures of the same type is called a <u>model class</u> if it is closed under isomorphisms. If $K \subseteq Str(L)$ is a model class, then the model class $Str(L) - K$ is denoted by $\overline{K}$. A model class K is L^*-definable if there are L and $\varphi \in L^*$ such that $K = Mod(\varphi) = \{M \in Str(L) | M \models^* \varphi\}$. We say, as usual, that a quasilogic L^* is a sublogic of another quasilogic L^+, $L^* \leq L^+$, if every L^*-definable model class is L^+-definable. L^* and L^+ are equivalent, $L^* \sim L^+$, if they are soblogics of each other.

An <u>abstract logic</u> is a quasilogic L^* such that

(L4) If L and L' are types such that $L \subseteq L'$, then $L^* \subseteq L'^*$ and for $\varphi \in L^*$, $M \in Str(L')$,

$$M \models^* \varphi \text{ if and only if } M|_L \models^* \varphi.$$

(L5) For every rudimentary set A, type $L \in A$ and $\varphi, \psi \in L^*_A$, there are $\varphi \wedge \psi$ and $\varphi \vee \psi$ in L^*_A such that $Mod(\varphi\wedge\psi) = Mod(\varphi) \cap Mod(\psi)$ and $Mod(\varphi\wedge\psi) = Mod(\varphi) \cap Mod(\psi)$.

(L6) For every rudimentary set A, types $L, L' \in A$ and $\varphi \in L'^*$ there are $\exists c\varphi$ and $\forall c\varphi$ in L^*_A such that if $L' - L$ consists of the constant-symbol c, then

$$Mod(\exists c\varphi) = \{M | \exists N \in Str(L')(N|_L = M \;\&\; N \models^* \varphi)\}$$

$$Mod(\forall c\varphi) = \{M | \forall N \in Str(L')(N|_L = M \rightarrow N \models^* \varphi)\}.$$

(L7) $L_{\omega\omega} \leq L^*_{\omega\omega}$.

It is obvious that $L_{\infty\omega}$, $L_{\infty\omega}(Q^1,\ldots,Q^n)$, $L^{II}_{\infty\omega}$, and their fragments are abstract logics. If the analogue of (L5) and (L6) holds for the negation, we call L^* a Boolean logic.

When we are only interested in the definable model classes of an abstract logic L^*, we sometimes write

$$L^* = \{K \mid \text{the model class } K \text{ is}\ldots\}$$

meaning that an arbitrary model class K is L^*-definable if and only if K is... .

1.2. Sort logic

The class of formulae of infinitary sort logic $L^s_{\infty\omega}$ is obtained if the following formation rule is added to the recursive definition of $L^{II}_{\infty\omega}$-formulae:

> If φ is a formula and s is a sort, then $\exists s\varphi$ and $\forall s\varphi$ are formulae.

To define the semantics of $L^s_{\infty\omega}$ we have to work in the MKM theory of classes or in any other theory in which satisfaction for formulae of set theory is definable. If L is a type, s a sort, $s \notin L$ and $L' = L \cup \{s\}$, then for any $M \in \mathrm{Str}(L)$ we define

$$M \models \exists s\varphi \text{ if and only if } \exists N \in \mathrm{Str}(L')(N\restriction_L = M \;\&\; N \models \varphi)$$

$$M \models \forall s\varphi \text{ if and only if } \forall N \in \mathrm{Str}(L')(N\restriction_L = M \to N \models \varphi).$$

This defines $L^s_{\infty\omega}$ as an abstract logic. We denote L^s_{HF} by L^s. It appears that sort logic has not been singled out as a logic before, although it has been studied in a semantical form in [10].

Let $\sum_n(L_{\infty\omega})$, $n \geq 1$, be the sublogic of $L^s_{\infty\omega}$ the formulae of which have the form

$$\exists s_1 \forall s_2 \ldots \exists(\forall) s_n \varphi$$

where $s_1,\ldots,s_n$ are sorts and $\varphi \in L^{II}_{\infty\omega}$. Let $\Pi_n(L_{\infty\omega})$ be the sublogic of $L^s_{\infty\omega}$ consisting of formulae of the form

$$\forall s_1 \exists s_2 \ldots \forall(\exists) s_n \varphi$$

where $s_1,\ldots,s_n$ are sorts and $\varphi \in L^{II}_{\infty\omega}$. For each $n < \omega$, the abstract logics $\sum_n(L_{\infty\omega})$ and $\Pi_n(L_{\infty\omega})$ are definable in ZF. Note that $\sum_n(L_{\infty\omega})$ and $\Pi_n(L_{\infty\omega})$ are closed under second order quantifiers.

Let $\Delta_n(L_{\infty\omega})$ be the sublogic of $\sum_n(L_{\infty\omega})$ the formulae of which are equivalent to $\Pi_n(L_{\infty\omega})$-formula. $\Delta_n(L_{\infty\omega})$ is an abstract logic but it does not seem to

have such a simple syntax as $\sum_n(L_{\infty\omega})$ and $\Pi_n(L_{\infty\omega})$. In fact the class of $\Delta_n(L_{\infty\omega})$-formulae reflects to a certain extent the properties of the underlying model of set theory and changes when the model is changed. Note that $\Delta_1(L_{\infty\omega})$ is just $\Delta(L_{\infty\omega})$ in the sense of [8]. More generally, $\Delta(\sum_n(L_{\infty\omega}) \cup \Pi_n(L_{\infty\omega})) \sim \Delta_{n+1}(L_{\infty\omega})$. The fragments $\sum_n(L_A)$, $\Pi_n(L_A)$ and $\Delta_n(L_A)$ are defined similarly.

1.3. Extension-operations

We review the definition of the Δ-operation from [8] because the definition naturally leads to both more general and more restricted operations.

If $\varphi \in L^*$, $L' \subset L$ and $M \in \mathrm{Str}(L')$, let

$$E(M,\varphi) = \{N \in \mathrm{Str}(L) \mid N\big|_{L'} = M \ \&\ N \models^* \varphi\}.$$

A model class K of type L' is $\sum$-*defined* by φ if $L - L'$ is finite and

$$K = \{M \in \mathrm{Str}(L') \mid E(M,\varphi) \neq \emptyset\}.$$

K is Π-*defined* by φ is $L - L'$ is finite and

$$K = \{M \in \mathrm{Str}(L') \mid \forall N \in \mathrm{Str}(L)(N\big|_{L'} = M \to N \models^* \varphi)\}.$$

K is $\sum(L^*)$-*definable* ($\Pi(L^*)$-*definable*) if it is $\sum$-defined (Π-defined) by some $\varphi \in L^*$. Finally, K is $\Delta(L^*)$-*definable* if it is both $\sum(L^*)$- and $\Pi(L^*)$-definable. $\Delta(L^*)$ gives rise to the semantics of an abstract logic, but to find a syntax for that logic seems as difficult as finding a syntax for $\Delta_n(L_{\infty\omega})$. Note however, that in special cases $\Delta(L^*)$ has a beautiful syntax (see e.g. [8] §4). To be specific let us agree that $\Delta(L^*)$ is the abstract logic the sentences of which are 4-tuples $\langle\varphi,L,\psi,L'\rangle$ where $\varphi \in L^*$, $\psi \in L'^*$ and φ $\sum$-defines the same model class as ψ Π-defines. One of our results will imply that $\Delta(L^*)$ hardly has a less artificial syntax. The families $\sum(L^*)$ and $\Pi(L^*)$ can also be made into abstract logics if, for example, a model class which is $\sum$-defined by $\varphi \in L^*$ is associated an artificial sentence $\langle\varphi,L\rangle$. Note that this syntax for $\sum(L^*)$ depends only on the syntax of L^* and not on the underlying set theory. We say that L^* is *unbounded* if W is $\Delta(L^*)$-definable.

By induction on $n < \omega$ we define $\sum_{n+1}(L^*) = \sum(\Pi_n(L^*))$, $\Pi_{n+1}(L^*) = \Pi(\sum_n(L^*))$ and $\Delta_{n+1}(L^*) = \Delta(\sum_n(L^*) \cup \Pi_n(L^*))$. Now we have two interpretations for $\sum_n(L_{\infty\omega})$, either as a $\sum_n$-extension of $L_{\infty\omega}$ or as a fragment of $L^S_{\infty\omega}$, but it is obvious that the two interpretations are essentially equivalent. All standard logics are sublogics of $\Delta_3(L_{\infty\omega})$ and therefore the operations Δ_n, $n > 2$, are relatively uninteresting, apart from their relation to sort logic.

If the above definition of $\sum(L^*)$ and $\Pi(L^*)$ is modified by requiring that

L has no new sorts over and above those of L', the essentially weaker notions of $\Pi^1_1(L^*)$- and $\Sigma^1_1(L^*)$-definability are obtained. Let $\Delta^1_1(L^*)$ be defined as $\Delta(L^*)$ above. More generally we define $\Sigma^1_{n+1}(L^*) = \Sigma^1_1(\Pi^1_n(L^*))$, $\Pi^1_{n+1}(L^*) = \Pi^1_1(\Sigma^1_n(L^*))$ and $\Delta^1_{n+1}(L^*) = \Delta^1_1(\Sigma^1_n(L^*) \cup \Pi^1_n(L^*))$.

1.4. Set theory

Our set theoretical notation follows mostly that of [4]. However, we write R_α for the α'th level of the ramified hierarchy. $Cd(x)$ is the predicate "x is a cardinal number (initial ordinal)", $Rg(x)$ is the predicate "x is a regular cardinal" and $Pw(x,y)$ is the predicate $x = P(y)$, where P is the power-set operation. $Card(x)$ is the least ordinal which has the same power as x. $HC(x) = \max(card(TC(x)), \aleph_0)$. We sometimes use $\aleph_n$ as a predicate, meaning the predicate "$x \in \aleph_n$", of course. $P_\kappa(y)$ is the set $\{x \subseteq y \mid |x| < \kappa\}$ and $Pw_\kappa(x,y)$ is the predicate "$y = P_\kappa(x)$". The sets of $\Sigma_n(P)$- and $\Pi_n(P)$-formulae are defined as usual. A predicate is $\Sigma_n(P)$ ($\Pi_n(P)$) w.p.i. (= with parameters in) A if it is definable with a $\Sigma_n(P)$ ($\Pi_n(P)$)-formula w.p.i. A. A predicate is $\Delta_n(P)$ w.p.i. A if it is both $\Sigma_n(P)$ and $\Pi_n(P)$ w.p.i. A. An ordinal α is $\Sigma_n(P)$ ($\Pi_n(P), \Delta_n(P)$)-definable w.p.i. A if the predicate "$x \in \alpha$" is.

§ 2. The basic representations

In this chapter we define the symbiosis of a logic and a predicate of set theory, and prove the main result about symbiosis (Theorem 2.4). The chapter ends with some remarks on the non-absolute nature of the Δ-operation.

By its very definition an abstract logic determines two predicates of set theory: Stc and $\models$. It is convenient for our purposes to establish a converse relation, that is, associate every predicate with a generalized quantifier.

Suppose $P = P(x_1,\ldots,x_n)$ is a predicate of set theory. Let

$$K[P] = \{M \mid M \cong \langle M, \in, a_1,\ldots,a_n\rangle \text{ such that } M \text{ is transitive and } P(a_1,\ldots,a_n)\}.$$

Let Q_P be the generalized quantifier associated with K[P] and

$$L[P] = L_{\omega\omega}(Q_P).$$

Lemma 2.1.

(1) $\models_{L[P]}$ *is* $\Delta_1(P)$.

(2) K[P] *is* $\Delta(L[P]_{\omega\omega})$*-definable.*

Proof. An elaboration of the proof of the well-known fact that $\models_{L_{\infty\omega}}$ is Δ_1 (see e.g. [3] p. 83) gives (1). (2) is trivial as K[P] is even $L[P]_{\omega\omega}$-definable. □

The above lemma shows that L[P] and P are in a sense definable from each other. We take a slight weakening of this property as the definition of symbiosis.

Definition 2.2. *Suppose* L* *is an abstract logic,* P *a predicate of set theory and* A *a transitive class. We say that* L* *and* P *are* symbiotic on A *if the following two conditions hold:*

(S1) If $\varphi \in L^*$, *then* Mod(φ) *is* $\Delta_1(P)$ *w.p.i.* $\{\varphi, L\}$

(S2) K[P] *is* $\Delta(L^*_A)$*-definable.*

The logic L* *is* symbiotic on A *if there is a predicate* $P \neq \emptyset$ *such that* L* *and* P *are symbiotic on* A. *If* A = HF, *we omit the clause "on* A".

Note that symbiosis on A implies symbiosis on any transitive $A' \supseteq A$.

Examples 2.3. *The following pairs are symbiotic on* HF *for any rudimentary* A:

(1) $L[P]_A$ *and* P,
(2) LQ_A *and* Q, *if* Q *is a generalized quantifier and* LQ *is unbounded,*
(3) LW_A *and* On,
(4) LI_A *and* Cd,
(5) LR_A *and* Rg,
(6) L_A^{II} *and* Pw,
(7) $L_{\omega\omega_n}$ *and* Pw_{ω_n}.

The following pairs are symbiotic on $H(\kappa)$ *for any rundimentary* $A \supseteq H(\kappa)$ *and* $\kappa = \lambda^+$, $\lambda \geq \omega$:

(8) $L_{A\kappa}$ *and* Pw_κ,
(9) L_{AG} *and* On.

Proof. The proof of (S1) is similar to the proof of 2.1(1) in any of (1)-(9). As a typical example of the proof of (S2), let us consider (4). Recall that W is $\sum(LI)$. Now

$$\begin{aligned}
&\langle M,E,a\rangle \in K[d] \leftrightarrow \\
&\langle M,E,a\rangle \models Wxy(xEy) \wedge \forall xy(yEx \wedge xEa \to yEa) \wedge \\
&\qquad \wedge \forall xyz(zEy \wedge yEx \wedge xEa \to zEx) \wedge \\
&\qquad \wedge \forall z(zEa \to \neg\, Ixy(xEz)(yEa)) \wedge \\
&\qquad \wedge \forall xy(\forall z(zEx \leftrightarrow zEy) \to x=y). \quad \square
\end{aligned}$$

Note that, if L^* is symbiotic with P $(\neq \emptyset)$ on A then L^*_A is unbounded, in fact, as $K[P]$ is $\Delta(L^*_A)$-definable, it suffices to observe that

$$Wxy(xAy) \leftrightarrow \text{there are } M, E \text{ and } a_1,\dots,a_n \text{ such that}$$
$$\langle M,E,a_1,\dots,a_n\rangle \in K[P] \text{ and}$$
$$\langle M,E,a_1,\dots,a_n\rangle \models \forall xy(xAy \to xEy).$$

Conversely, it is by no means the case that every unbounded logic is symbiotic on some A. We shall indicate later why the logic $L(W,Q_1,\dots,Q_n,\dots)_{n<\omega}$ is not symbiotic.

The next theorem is the basic result about symbiosis and about relation between logic and set theory in general. It was proved in the author's Ph.D. thesis [13] and appeared later, but independently in [10]. We repeat the proof here for completeness. The proof clearly owes a great deal to [2].

Theorem 2.4. *Suppose* A *and* $A_o \subseteq A$ *are transitive classes,* P *a predicate, and* L^* *an abstract logic extending* L_A *and symbiotic with* P *on* A_o. *Then the following are equivalent for any model class* K *of type* $L \in A$:

(1) K *is* $\sum(L^*_A)$*-definable,*
(2) K *is* $\sum_1(P)$ *w.p.i.* A.

Proof. (1) → (2): Suppose $\varphi \in L^*_A$ $\sum$-defines K. Let $L_o \in A$ such that $\varphi \in L^*_o$. Now

$$K = \{M \in \mathrm{Str}(L) \mid \exists N \in \mathrm{Str}(L_o)(N\big|_L = M \;\&\; N \models^* \varphi)\}.$$

Hence K is $\sum_1(P)$ w.p.i. A.

(2) → (1): Suppose

$$\forall x(x \in K \leftrightarrow \varphi(x,a))$$

where $\varphi(x,y)$ is $\sum_1(P)$ and $a \in A$. Let $a' = TC(\{a\})$ $(\in A)$. Let E be a binary predicate symbol not in L and i a sort not in L. For any formula $\psi(x_1,\dots,x_n)$ of set theory let $[\psi(x_1,\dots,x_n)]_E$ be obtained from $\psi(x_1,\dots,x_n)$ by replacing atomic formulae $t\in u$ by tEu (i.e. $E(t,u)$) and changing all bound variables to variables of sort i. Let L_o be L extended with E and i. As L^* is symbiotic, there is an $L_1 \supseteq L_o$ and $\theta \in L^*_{1A}$ such that θ $\sum$-defines the class of well-founded extensional structures $\langle M_i,E\rangle$. By (S2) there is an $L_2 \supseteq L_1$ and $\eta \in L^*_{2A}$ such that η $\sum$-defines the class $K[P]$. Suppose $P = P(x_1,\dots,x_n)$ and $c_1,\dots,c_n$ are the constant symbols in the type of $K[P]$. Let ζ be the L^*_{2A}-sentence obtained from

$$[P(c_1,\dots,c_n)]_E \leftrightarrow \eta(c_1,\dots,c_n)$$

by universally quantifying over $c_1,\ldots,c_n$ using variables of sort i. Let $\varphi_b(x)$ be the formula $\forall y(yEx \leftrightarrow \bigvee_{c\in b} \varphi_c(x))$ for every element b of a'. Let $\psi(x)$ be the L_o^*-sentence which says, using E instead of $\in$, that x is a structure of type L in which any atomic $R(x_1,\ldots,x_n)$ is satisfied by elements $a_1,\ldots,a_m$ if and only if $R(a_1,\ldots,a_m)$ is true. Finally, let ξ be the conjunction of

$$\theta \;\&\; \zeta$$

$$\bigwedge_{b\in a} \exists x\varphi_b(x) \;\&\; \varphi_a(c_1)$$

$$\exists x^i([\varphi(x^i,c_1)]_E \;\&\; \psi(x^i)).$$

Now $\xi \in L^*_{2A}$ and we prove that it $\sum$-defines K. Suppose at first that $M \in K$. Hence $\varphi(M,a)$ is true. Let N be a transitive set which reflects $\varphi(M,a)$ and P. M can be expanded to a model of ξ by letting N serve as the universe of sort i elements. For the converse, suppose $N \models \xi$. Let $N \cong M = \langle\ldots,M_i,\in,\ldots\rangle$ such that M_i is a transitive set. As $M \models \zeta$, M_i reflects P. Clearly c_1 is interpreted as a in M. Let $A \in M_i$ such that

$$M \models [\varphi(A,c_1)]_E \;\&\; \psi(A).$$

Then $A \in \mathrm{Str}(L)$ and $\varphi(A,a)$, whence $A \in K$. As K is closed under isomorphism, $M|_L \in K$. □

Corollary 2.5. *For any rudimentary class* A:

(1) $\Delta(LW_A) = \{K|$*the model class* K *is* Δ_1 *w.p.i.* A$\}$,
$= \Delta(L_{AG})$

(2) $\Delta(LI_A) = \{K|$*the model class* K *is* $\Delta_1(\mathrm{Cd})$ *w.p.i.* A$\}$,

(3) $\Delta(L_A^{II}) = \{K|$*the model class* K *is* Δ_2 *w.p.i.* A$\}$.

(4) $\Delta(L_{\omega\omega_n}) = \{K|$*the model class* K *is* $\Delta_1(Pw_{\omega_n})\}$,

If $A \supseteq H(\kappa)$, $\kappa = \lambda^+$, $\lambda \geq \omega$, *then:*

(5) $\Delta(L_{A\kappa}) = \{K|$*the model class* K *is* $\Delta_1(Pw_\kappa)$ *w.p.i.* A$\}$.

In [11] a predicate P(x) is called local if it is of the form $\exists\alpha(R_\alpha \models \varphi(x))$ for some formula $\varphi(x)$ of set theory, and a proof is sketched to the effect that a predicate is local if and only if it is equivalent to a $\sum_2$-predicate. Combining this with Theorem 2.4 yields:

Corollary 2.6. *A model class is* $\sum(L_A^{II})$*-definable if and only if it is defined by a local property w.p.i.* A.

Theorem 2.4 can be immediately iterated to yield a result about $\sum_n$-defina-

bility. A form of the following corollary was first proved by J. Oikkonen in [10] with a different proof.

Corollary 2.7. *Suppose* A *and* $A_o \subseteq A$ *are transitive classes,* P *a predicate, and* L_A^* *a Boolean logic extending* L_A *and symbiotic with* P *on* A_o. *Then the following are equivalent for any model class* K *of type* $L \in A$ *and for any* $n < \omega$:

(1) K *is* $\Sigma_{n+1}(L_A^*)$*-definable,*

(2) K *is* $\Sigma_{n+1}(P)$ *w.p.i.* A.

Proof. We use induction on n. If $n = 0$, the claim follows from 2.4. Suppose then K is $\sum_{n+1}(L_A^*)$-definable and $n > 0$. Let $\varphi \in \Pi_n(L_A^*)$ $\sum$-define K. By induction hypothesis $\mathrm{Mod}(\varphi)$ is $\Pi_n(P)$ w.p.i. A. Now

$$K = \{M \in \mathrm{Str}(L) \mid \exists N \in \mathrm{Mod}(\varphi)(N\restriction_L = M \wedge N \models \varphi)\},$$

and therefore K is $\sum_{n+1}(P)$ w.p.i. A.

For the converse, suppose (2) holds. Let S be a $\Pi_n(P)$-predicate such that K is $\sum_1(S)$ w.p.i. A. By 2.4 K is $\sum$-defined by some $\varphi \in L[S]_A$. By 2.1 $\mathrm{Mod}(\varphi)$ is $\Delta_1(S)$ w.p.i. A, and therefore $\Delta_{n+1}(L_A^*)$-definable. Hence K is $\Sigma_{n+1}(L_A^*)$-definable. □

Corollary 2.8. *For* $n > 0$ *and for any rudimentary class* A:

(1) $\Delta_n(LW_A) = \{K \mid$ *the model class* K *is* Δ_n *w.p.i.* $A\}$,

(2) $\Delta_n(L_A^{II}) = \{K \mid$ *the model class* K *is* Δ_{n+1} *w.p.i.* $A\}$.

Note that $K[Pw]$ is $\Pi(L_{\omega\omega})$-definable, whence $L[Pw] \leq \Delta_2(L_{\omega\omega})$ and therefore $\Delta(L_A^{II}) \leq \Delta_2(L_A)$. On the other hand $\Delta_2(L_A) \leq \Delta_2(LW_A) \leq \Delta(L_A^{II})$. Hence in fact $\Delta(L_A^{II}) \sim \Delta_2(L_A)$, and therefore $\Delta_n(L_A^{II}) \sim \Delta_{n+1}(L_A)$ for all $n > 0$. If this is combined with 2.8, the following obtains:

Corollary 2.9. *For* $n > 1$,

$$\Delta_n(L_A) = \{K \mid \textit{the model class } K \textit{ is } \Delta_n \textit{ w.p.i. } A\}.$$

Therefore in MKM:

$$L_A^s = \{K \mid \textit{the model class } K \textit{ is definable in set theory w.p.i. } A\}.$$

The second part of the above corollary was stated on page 174 of [8].

As $\Delta_1(L_{\omega\omega})$ is just the usual first order logic $L_{\omega\omega}$, and $\Delta_2(L_{\omega\omega})$ is $\Delta(L^{II})$, that is, essentially second order logic, it would be tempting to conjecture that $\Delta_3(L_{\omega\omega})$ is essentially third order logic. This is not the case, however. By familiar methods (see e.g. [9]) one can prove that for any analytical

(this can be improved, see [9]) ordinal α the α'th order logic in Δ-equivalent to second order logic. It seems plausible to put $\Delta_3(L_A)$ above the whole notion of higher order logic and consider it rather as a fragment of a quite new powerful logic, sort logic. Similarly it seems implausible to call $\sum$ a second order quantifier, or even a generalized second order quantifier, as $\sum$ leads to something far beyond second and higher order logic, viz. set theory. We return to the problematics of second order logic in the next chapter.

The Δ-operation can be used to give a very near characterization of symbiosis:

Proposition 2.10. *Suppose* L_A^* *is a Boolean logic extending* L_A *and* P *a predicate. Then* $\Delta(L_A^*) \sim \Delta(L[P]_A)$ *if and only if* (S2) *and* $(S1)_A$: *If* $\phi \in L_A^*$, *then* $\mathrm{Mod}(\phi)$ *is* $\Delta_1(P)$ *w.p.i.* A.

Proof. Suppose at first that $(S1)_A$ and (S2) hold. By Theorem 2.1 every $L[P]_A$-definable model class is $\Delta_1(P)$ w.p.i. A, whence by 2.4, $L[P]_A \leq \Delta(L_A^*)$. Therefore $\Delta(L[P]_A) \leq \Delta(L_A^*)$. On the other hand, if K is L_A^*-definable, then by $(S1)_A$ K is $\Delta_1(P)$ w.p.i. A, whence by 2.4 K is $\Delta(L[P]_A)$-definable. Hence $\Delta(L_A^*) \sim \Delta(L[P]_A)$. The converse is immediate in view of 2.4. □

We can use 2.10 to show that the logic $L^* = L(W,Q_1,\ldots,Q_n,\ldots)_{n<\omega}$ is not symbiotic on any A. Indeed, suppose L_A^* is symbiotic with P on A. As $L_A^* = L_{HF}^*$, we may assume A = HF. By 2.10, $\Delta(L^*) \sim \Delta(L[P]_{\omega\omega})$. Let $n < \omega$ such that K[P] is Δ-definable in $L^+ = L(W,Q_1,\ldots,Q_n)$. Now $\Delta(L^*) \sim \Delta(L[P]_{\omega\omega}) \sim \Delta(L^+)$, a contradiction.

The existence on non-symbiotic unbounded logics may seem to limit the applicability of Theorem 2.4. However, if L^* is the union (in the obvious sense) of the logics L^{+n} $(n < \omega)$, where L^{+n} is symbiotic with P_n on A_n, then for $A = \bigcup_n A_n$,

$$\Delta(L_A^*) = \{K \mid \text{the model class } K \text{ is } \Delta_1(P_n) \text{ w.p.i. } A_n \text{ for some } n < \omega\}.$$

Thus the range of Theorem 2.4 extends to many non-symbiotic logics. For example:

$$\begin{aligned}\Delta(L(W,Q_1,\ldots,Q_n,\ldots)_{n<\omega}) &= \{K \mid \text{the model class } K \text{ is } \Delta_1(\aleph_1,\ldots,\aleph_n) \\ &\qquad \text{for some } n < \omega\} \\ &= \{K \mid \text{the model class } K \text{ is } \Delta_1 \text{ w.p.i.} \\ &\qquad \{\aleph_1,\ldots,\aleph_n,\ldots\}\}.\end{aligned}$$

Note however, that $L_{\omega_1\omega}(W,Q_1,\ldots,Q_n,\ldots)_{n<\omega}$ is symbiotic on HC.

In the next results we investigate the absoluteness of symbiotic logics. Let us consider the following three properties of an abstract logic L^* and a predi-

cate P:

(A1) Stc* is $\Sigma_1(P)$ and $\varphi \vee \psi$, $\varphi \wedge \psi$, $\exists c\varphi$ and $\forall c\varphi$ in (L5) and (L6) can be found with $\Sigma_1(P)$-functions.

(A2) There is a $\Delta_1(P)$-predicate S such that if $\varphi \in L^*$, then

$$\forall M \in Str(L)(M \models^* \varphi \leftrightarrow S(M,\varphi)).$$

The conditions (A1) and (A2) together form a natural notion of P-absoluteness of L^* generalizing the notion of an absolute logic in [2]. Note that (A2) $\rightarrow$ (S1). The following lemma is obvious:

Lemma 2.11.

(1) If L^* *and* P *are symbiotic on* A *and* $L^+ \leq \Delta(L^*_A)$, *then* L^+ *and* P *satisfy* $(S1)_A$.

(2) If L^*_A *and* P *satisfy (A2) and* $L^+ \leq \Delta(L^*_A)$, *then* L^+ *and* P *satisfy* $(S1)_A$.

The next result generalizes a theorem by Burgess (Theorem 2.2 in [8]) which says that no unbounded absolute logic is Δ-closed. The proof remains almost the same.

Theorem 2.12. *Suppose* L^* *is a Boolean logic symbiotic with* P *on* A *and* $L^+ \sim \Delta(L^*_A)$. *Then* L^+ *and* P *satisfy* $(S1)_A$ *but not (A2).*

Proof. Suppose S is a $\Delta_1(P)$-predicate such that if $\varphi \in L^+$, then $\forall M \in Str(L)(M \models^+ \varphi \leftrightarrow S(M,\varphi))$. Let

$$K = \{M \mid M \cong \langle TC(\{a\}), \in\rangle \text{ for some } a \text{ such that } \neg S(M,a)\}.$$

K is clearly $\Delta_1(P)$. By 2.4 there is a $\varphi \in L^+$ such that $K = Mod(\varphi)$. Let $M = \langle TC(\{\varphi\}), \in\rangle$. Then $M \in K \leftrightarrow M \models^+ \varphi \leftrightarrow \neg S(M,\varphi) \leftrightarrow M \notin K$, a contradiction. □

Corollary 2.13. *Suppose* L^* *is a Boolean logic,* L^* *and* P *are symbiotic on* A *and they satisfy (A2). Then* L^*_A *is not* Δ*-closed.*

Corollary 2.14. *The following logics are not* Δ*-closed:*

(1) $L[P]_A$ *where* P *is a predicate of set theory,*

(2) LQ_A *where* Q *is a generalized quantifier such that* LQ *is unbounded.*

(3) LW_A, LI_A, LR_A, L^{II}_A, $L_{\omega_1\omega_1}$, $L_{\omega_1 G}$.

Hence, if L^*_A *is a symbiotic logic extending* L_A, *there are no* $Q^1,\ldots,Q^n$ *such that*

$$\Delta(L^*_A) \sim LQ^1 \ldots Q^n_A.$$

The following theorem gives another aspect of the failure of syntactical methods in constructing Δ-extensions. Recall the definition of $\mathrm{Stc}_{\Delta(L^*)}$ in §1.

Theorem 2.15. *Suppose* A *and* $A_o \subseteq A$ *are transitive classes,* P *a predicate and* L_A^* *an abstract logic such that there is a* $\sum_1(P)$*-function embedding* L_A *into* L_A^* *and* L_A^* *is symbiotic with* P *on* A_o*. Then the predicate* $\mathrm{Stc}_{\Delta(L_A^*)}$ *is* $\Pi_2(P)$ *but not* $\sum_2(P)$*. Therefore* $\Delta(L_A^*)$ *and* P *do not satisfy (A1).*

Proof. By definition

$$\mathrm{Stc}_{\Delta(L_A^*)}(L,x) \leftrightarrow \exists\varphi\psi L_o L_1 \in \mathrm{TC}(x)$$

$$(\mathrm{Stc}^*(L_o,\varphi) \;\&\; \mathrm{Stc}^*(L_1,\psi) \;\&\; x = \langle\varphi,L_o,\psi,L_1\rangle \;\&$$

$$\forall M \in \mathrm{Str}(L)(\exists N \in \mathrm{Str}(L_o)(N|_L = M \;\&\; N \models^* \varphi) \leftrightarrow$$

$$(\forall N \in \mathrm{Str}(L_1)(N|_L = M \rightarrow N \models^* \psi)))).$$

This proves that $\mathrm{Stc}_{\Delta(L_A^*)}$ is $\Pi_2(P)$. To prove that $\mathrm{Stc}_{\Delta(L_A^*)}$ is not $\sum_2(P)$, let $R(x,y)$ be a $\Pi_2(P)$-predicate which is not $\sum_2(P)$. We construct $\sum_1(P)$-functions f and g such that

$$(*) \qquad \forall x\forall y(R(x,y) \leftrightarrow \mathrm{Stc}_{\Delta(L_A^*)}(f(x,y),g(x,y))).$$

From this it follows that $\mathrm{Stc}_{\Delta(L_A^*)}$ is not $\sum_2(P)$.

Let $T(x,y,z)$ be a $\sum_1(P)$-predicate such that

$$\forall x\forall y(R(x,y) \leftrightarrow \forall z T(x,y,z)).$$

$K[T]$ is $\sum_1(P)$ whence by 2.4 there is an L_A^*-sentence $\varphi(c_1,c_2,c_3)$ which $\sum$-defines $K[T]$. For any x let $\psi_x(y,E)$ be the $L_{\infty\omega}$-formula (see the proof of 2.4)

$$\bigwedge_{a \in \mathrm{TC}(\{x\})} \exists z\varphi_a(z) \;\&\; \varphi_x(y).$$

Let $\theta(E)$ be an L_A^*-sentence which $\sum$-defines the class of models $\langle \mathrm{dom}(E),E,c_3\rangle$ where E is well-founded and extensional. For any x and y let η_{xy} be the sentence

$$(\theta(E) \;\&\; \psi_x(c_1,E) \;\&\; \psi_y(c_2,E)) \rightarrow \varphi(c_1,c_2,c_3).$$

By (A1) we may assume there is a type L'_{xy} such that $\eta_{xy} \in L'^*_{xyA}$ and the predicates $z = \eta_{xy}$ and $z = L'_{xy}$ are $\sum_1(P)$. Let L_{xy} be the subtype of L'_{xy} associated with c_3,c_2,c_1 and E. Let ξ be an arbitrary valid L^*-sentence. We define

$$f(x,y) = L_{xy}$$

$$g(x,y) = \langle \eta_{xy}, L'_{xy}, \xi, L \rangle .$$

Now (*) holds as is not too difficult to see. □

It follows, for example, that there is no Σ_3-formula which decides whether a given $L^{II}_{\infty\omega}$-formula Σ-defines the same model class as another given $L^{II}_{\infty\omega}$-sentence Π-defines. So, although $L^{S}_{\infty\omega}$ has a primitive recursive syntax, it seems unlikely that any similar syntax can be found for its fragment $\Delta(L^{II}_{\infty\omega})$ or for $\Delta_2(L_A)$.

We end this chapter with some remarks on decision problems of symbiotic logics. For simplicity we only consider logics of the form $L^*_{\omega\omega}$. The decision problem of $L^*_{\omega\omega}$ is the set $Val(L^*_{\omega\omega}) = \{\varphi \in HF \mid \varphi \in L^*_{\omega\omega}$ and φ is valid$\}$. It is known (see [13] and [12]) that $Val(L^{II}_{\omega\omega})$ is the complete Π_2-subset of HF. More generally, if L^* is symbiotic with P and L^* is sufficiently syntactic (e.g. $L^* = LQ$ for some Q), then $Val(L^*_{\omega\omega})$ is the complete $\Pi_1(P)$-subset of HF. A proof of this can be found in [13]. For results about $Val(LI_{\omega\omega})$ and $Val(LR_{\omega\omega})$ see [14].

§ 3. Flat definability and second order logic

In this chapter we construct the part of set theory which coincides with second order logic in the same way as the whole set theory coincides with sort logic.

Definition 3.1. *Quantifiers of the form*

(1) $\exists x(HC(x) \leq HC(y_1 \cup \ldots \cup y_n) \ \&\ \varphi(x,y_1,\ldots,y_n))$

(2) $\forall x(HC(x) \leq HC(y_1 \cup \ldots \cup y_n) \rightarrow \varphi(x,y_1,\ldots,y_n))$

are called flat quantifiers. The set of flat formulae of set theory is the smallest set containing Σ_0-formulae and closed under $\&, \vee, \neg$ and flat quantification.

The $\Sigma^b_n(P)$- and $\Pi^b_n(P)$-formulae are defined by induction on n as follows: $\Sigma^b_0(P)$- and $\Pi^b_0(P)$-formulae are just the $\Sigma_0(P)$-formulae. $\Sigma^b_{n+1}(P)$-formulae are formulae of the form (1) where $\varphi(x,y_1,\ldots,y_n)$ is $\Pi^b_n(P)$. $\Pi^b_{n+1}(P)$-formulae are formulae of the form (2) where $\varphi(x,y_1,\ldots,y_n)$ is $\Sigma^b_n(P)$. $\Sigma^b_n(P)^{ZFC}$- and $\Pi^b_n(P)^{ZFC}$-formulae are defined as usual.

It is easy to see that the set of $\Sigma^b_n(P)^{ZFC}$-formulae is closed under $\&, \vee, \neg, \exists x \in y, \forall x \in y$ and (1) above. Note that by Levy's theorem ([4] p. 104) every Σ_1-formula is Σ^{bZFC}_1.

The whole point of flat formulae is the following reflection principle:

Lemma 3.2. *Suppose* $\varphi(x_1,\ldots,x_n)$ *is a flat formula and* a *an arbitrary set. Then there is a transitive set* M *such that* $a \in M$, $HC(M) = HC(a)$ *and* M *reflects* $\varphi(x_1,\ldots,x_n)$.

Proof. By the usual reflection principle ([4] p. 99) there is a transitive set N containing a such that N reflects $\varphi(x_1,\ldots,x_n)$. For any subformula $\psi(y_1,\ldots,y_m)$ of $\varphi(x_1,\ldots,x_n)$ and for any $b_2,\ldots,b_m \in N$ let $f = f_{\psi(y_1,\ldots,y_m)}(b_2,\ldots,b_m) \in N$ such that $HC(f) \leq HC(b_2 \cup \ldots \cup b_m)$ and

$$N \models \exists y_1(HC(y_1) \leq HC(b_2,\ldots,b_m) \ \&\ \psi(y_1,b_2,\ldots,b_m)) \rightarrow \psi(f,b_2,\ldots,b_m).$$

Choose M to be the smallest transitive set containing a and closed under the functions $f_{\psi(y_1,\ldots,y_m)}$, where $\psi(y_1,\ldots,y_m)$ runs through the subformulae of $\varphi(x_1,\ldots,x_n)$. □

The above lemma shows, among other things, that every flat formula is Δ_2^{ZFC} (using Theorem 3.7.2 of [4]).

Definition 3.3. *Suppose* L^* *is an abstract logic,* A *a transitive class and* P *a predicate of set theory.* L^* *and* P *are* strongly symbiotic on A *if the following two conditions are satisfied*

(SS1) If $\varphi \in L^*$, *then* $Mod(\varphi)$ *is* $\Delta_1^b(P)$ *w.p.i.* $\{\varphi,L\}$.

(SS2) $K[P]$ *is* $\Delta_1^1(L_A^*)$.

Strong symbiosis is harder to come by than symbiosis. For example, W is not $\Delta_1^1(LI)$-definable (essentially because in countable domains I is redudent and Theorem 7.3 of [3] can be used), whence LI is not strongly symbiotic. This failure can be regarded as an indication of the incompleteness of the definition of LI, rather than as a characteristic property of LI. The situation is different with second order logic which seems to resist strong symbiosis in an essential way, as we shall prove in a moment.

Examples 3.4. *The following pairs are strongly symbiotic on* HF *for any rudimentary* A:

(1) $L[P]_A$ *and* P,

(2) $L_A(W,Q)$ *and* Q, *if* Q *is any generalized quantifier,*

(3) $L_A(W)$ *and* On,

(4) $L_A(W,I)$ *and* Cd,

(5) $L_A(W,R)$ *and* Rg.

Theorem 3.5. *Suppose* $A \subseteq HC$ *and* $A_o \subseteq A$ *are transitive sets,* P *a predicate, and* L_A^* *an abstract logic extending* L_A *and strongly symbiotic with*

P *on* A_0. *Then the following are equivalent for any model class* K *of type* $L \in A$:

(1) K *is* $\sum^1_1(L^*_A)$*-definable,*

(2) K *is* $\sum^b_1(P)$ *w.p.i.* A.

Proof. We follow the proof of 2.4. The implication (1) → (2) is obvious. For (2) → (1), suppose $\varphi(x,y)$ is $\sum^b_1(P)$, $a \in A$ and

$$\forall x(x \in K \leftrightarrow \varphi(x,a)).$$

Let $a' = TC(\{a\})$. Let μ be the conjunction of ξ (as it is defined in the proof of 2.4) and the first order sentence which says that there is a bijection which maps all elements of the sorts in L_3 one-one to elements of the sorts in L. Using Lemma 3.2 one can prove that μ still $\sum$-defines K. But every model of μ has the same power as its L-reduct. Hence the new universes introduced by L_3 can be dispensed with in favour of new predicates, and therefore μ can be converted into a $\Delta^1_1(L^*_A)$-definition of K. □

The proof of Corollary 2.7 carries over immediately and we have:

Corollary 3.6. *Suppose* $A \subseteq HC$ *and* $A_0 \subseteq A$ *are transitive sets,* P *a predicate, and* L^*_A *a Boolean logic extending* L_A *and strongly symbiotic with* P *on* A_0. *Then the following are equivalent for any model class* K *of type* $L \in A$ *and for any* $n < \omega$:

(1) K *is* $\sum^1_{n+1}(L^*_A)$*-definable,*

(2) K *is* $\sum^b_{n+1}(P)$ *w.p.i.* A.

Corollary 3.7. *For* $A \subseteq H(\omega_1)$:

(1) $\Delta^1_n(L_A(W)) = \{K \mid$ *the model class* K *is* Δ^b_n *w.p.i.* A$\}$,

(2) $\Delta^1_n(L_A(W,I)) = \{K \mid$ *the model class* K *is* $\Delta^b_n(Cd)$ *w.p.i.* A$\}$.

The following corollary is proved mutatis mutandis as Proposition 2.10:

Corollary 3.8. *Suppose* L^*_A *is a Boolean logic extending* L_A *and* P *a predicate. Then* $\Delta^1_1(L^*_A) \sim \Delta^1_1(L[P]_A)$ *if and only if (SS2) and* $(SS1)_A$: *If* $\phi \in L^*_A$, *then* $Mod(\phi)$ *is* $\Delta^b_1(P)$ *w.p.i.* A.

It follows that second order logic is not strongly symbiotic, because there is no Q such that L^{II} $(\sim \Delta^1_1(L^{II})) \sim \Delta^1_1(LQ)$. Second order logic is rather the closure of first order logic under the $\sum^1_1$-operation. More exactly, let us define for any abstract logic L^*:

$$\Delta^1_{(\omega)}(L^*) = \{K \mid \text{the model class } K \text{ is } \Delta^1_n(L^*)\text{-definable for some } n < \omega\}.$$

Clearly, $L^{II} = \Delta^1_{(\omega)}(L_{\omega\omega}) = \Delta^1_{(\omega)}(LW) = \Delta^1_{(\omega)}(LI)$. Note that $\Delta^1_{(\omega)}(L^*)$ satisfies the single-sorted interpolation theorem. The following characterization of $\Delta^1_{(\omega)}(L^*)$ and second order logic follows from 3.6:

Corollary 3.9. *Suppose* L^*_A *is a Boolean logic extending* L_A *and* P *a predicate such that* L^*_A *and* P *are strongly symbiotic on* $A \subseteq H(\omega_1)$. *Then the following hold:*

(1) $\Delta^1_{(\omega)}(L^*_A) = \{K \mid$ *the model class* K *is definable with a flat formula of the language* $\{\in,P\}$ *w.p.i.* A$\}$.

(2) $L^{II}_A = \{K \mid$ *the model class* K *is definable with a flat formula w.p.i.* A$\}$.

To sum up, second order definability corresponds to flat definability in set theory, implicit second order (that is $\Delta(L^{II})$-) definability corresponds to Δ_2-definability in set theory, and finally, definability in sort logic corresponds to definability in set theory. Recall that by Theorem 3.7 of [2], first order definability corresponds to Δ^{KP}_1-definability.

It is well-known (see e.g. [9]) that the Π^1_1-part of second order logic has already the whole implicit strength of second order logic. Another way of saying the same is $L^{II} \leq \Delta(LQ_H)$, because Q_H is Π^1_1-definable (see [7]). This fact has the following more general analogue:

Proposition 3.10. *Suppose* $A \subseteq HC$ *and* $A_o \subseteq A$ *are transitive classes and* L^* *is a Boolean logic symbiotic on* A_o. *If* $\Pi^1_1(L_{\omega\omega}) \leq \Delta(L^*_A)$, *then* $\Delta^1_{(\omega)}(L^*_A) \subseteq \Delta(L^*_A)$.

Proof. Suppose L^* is symbiotic with P on A_o. In view of 3.9 it suffices to prove that if K is definable by a flat formula of set theory in the language $\{\in,P\}$ w.p.i. A, then K is $\Delta_1(P)$ w.p.i. A. Suppose $\varphi(x,y)$ is a flat formula in the language $\{\in,P\}$ and $a \in A$. Then $\varphi(a,b)$ holds if and only if there is a strong limit α such that $R(\alpha)$ reflects P, a and b are in $R(\alpha)$, and $R(\alpha) \models \varphi(a,b)$. As the assumption $\Pi^1_1(L_{\omega\omega}) \leq \Delta(L^*_A)$ implies that Pw is $\Delta_1(P)$ w.p.i. A, the above equivalence shows that $\varphi(a,y)$ is $\sum_1(P)$ w.p.i. A. Similarly $\neg\varphi(a.y)$ is $\sum_1(P)$ w.p.i. A. □

Proposition 3.10 can be improved by considering suitably defined $\Delta^\alpha_{(\omega)}$-operations, where α is an ordinal definable in finite order logic or $\alpha \in A$ (see [9]).

The results about absoluteness of symbiotic logics in the previous chapter carry over to strongly symbiotic logics as follows: Let us consider the following properties:

(SA1) Stc* is $\sum^b_1(P)$ and $\varphi\vee\psi$, $\varphi\wedge\psi$, $\exists c\varphi$ and $\forall c\varphi$ in (L5) and (L6) can be found with $\sum^b_1(P)$-functions.

(SA2) There is a $\Delta_1^b(P)$-predicate S such that if $\varphi \in L^*$, then

$$\forall M \in \mathrm{Str}(L)(M \models^* \varphi \leftrightarrow S(M,\varphi)).$$

The condition (SA1) and (SA2) together from a notion of strong P-absoluteness of L^*. Note that if P is omitted, strong absoluteness coincides with the notion of absoluteness, because every Σ_1-predicate is Σ_1^b. So the difference comes only when some non-trivial predicates P are considered. For example, second order logic is Pw-absolute but not strongly Pw-absolute (see the remarks after 3.8).

The following theorem is proved as 2.12:

Theorem 3.11. *Suppose Boolean* L^* *and* P *are strongly symbiotic on* A *and* $L^+ \leq \Delta_1^1(L_A^*)$. *Then* L^+ *and* P *satisfy* $(SS1)_A$ *but not (SA2).*

Corollary 3.12. *Suppose Boolean* L^* *and* P *are strongly symbiotic on* A *and satisfy (SA2). Then* L_A^* *is not* Δ_1^1*-closed.*

Corollary 3.13. *The following logics are not* Δ_1^1*-closed:*

(1) $L[P]_A$, *where* P *is a predicate of set theory,*

(2) $L_A(W,Q)$, *where* Q *is a generalized quantifier.*

Hence, if L_A^* *is a strongly symbiotic logic on* A *extending* L_A, *there are no generalized quantifiers* $Q^1 \ldots Q^n$ *such that*

$$\Delta_1^1(L_A^*) \sim L_A(Q^1,\ldots,Q^n).$$

Also the proof of Theorem 2.15 carries over:

Theorem 3.14. *Suppose* L^* *and* P *are strongly symbiotic on* V, *satisfy (SA1), and there is a* $\Sigma_1^b(P)$*-function which embedds* $L_{\infty\omega}$ *into* L^*. *Then the predicate* $\mathrm{Stc}_{\Delta_1^1(L^*)}$ *is* $\Pi_2^b(P)$ *but not* $\Sigma_2^b(P)$.

This theorem shows how difficult it is to find a syntax for Δ_1^1-extensions, whereas the full $\Delta_{(\omega)}^1$-extension has a simple primitive recursive syntax. The situation is hence similar as in the case of Δ-extension.

§ 4. Löwenheim numbers

The purpose of this chapter is to transfer the definability results of § 2 from the level of model classes to the level of spectra and in particular minima of spectra, that is Löwenheim numbers.

Definition 4.1. *Suppose* L^* *is an abstract logic and* $\sigma \in L^*$. *The spectrum*

of φ, $Sp(\varphi)$, *is the class*

$$\{\mathrm{card}(M) \mid M \models^* \varphi\}.$$

The indexed family

$$Sp(L^*) = \{Sp(\varphi) \mid \varphi \in L^*\}$$

is called the family of L*-*spectra.*

Examples 4.2.

(1) The class of successor cardinals and the class of limit cardinals are LI*-spectra.*

(2) $\{\lambda \mid \exists\kappa(\kappa^+ < \lambda \leq 2^\kappa)\}$ *is an* LI*-spectrum.*

(3) The class of regular cardinals and the class of weakly inaccessible cardinals are LR*-spectra.*

(4) $\{2^\kappa \mid \kappa$ *a cardinal*$\}$ *is an* L^{II}*-spectrum.*

(5) $\{2^\kappa \mid \kappa$ *is measurable*$\}$ *is an* L^{II}*-spectrum.*

For other examples of spectra see [13] *and* [14].

The following problem is called the spectrum problem for L^*: Is the complement of an arbitrary L*-spectrum again an L*-spectrum? The spectrum problem for LW, for example, has a negative solution because $\{\kappa \mid \kappa \leq \aleph_0\}$ is an LW-spectrum but $\{\kappa \mid \kappa > \aleph_0\}$ is not. The spectrum problem for LI can have a negative answer - this will be discussed later. The spectrum problem for L^{II} has a positive solution for a rather trivial reason: if C is an L^{II}-spectrum, then $C = Sp(\varphi)$ for some identity-sentence φ and the complement of C is just $Sp(\neg\varphi)$. This fact has a more general analogue. At first we note the following trivial lemma:

Lemma 4.3. *Suppose* C *is a class of cardinals and* C' *is the class of structures* $\langle A\rangle$*, where* $\mathrm{card}(A) \in C$. *Then* C *is an* L*-*spectrum if and only if* C' *is* $\sum^1_1(L^*)$*-definable.*

If this is combined with Theorem 3.5 and Corollary 3.9, the following characterization of spectra yields:

Theorem 4.4. *Suppose* $A \subseteq HC$ *and* $A_0 \subseteq A$ *are transitive classes,* P *a predicate, and* L^*_A *a Boolean logic extending* L_A *and strongly symbiotic with* P *on* A_0. *Then*

$$Sp(L^*_A) = \{C \subseteq Cd \mid C \text{ is } \sum^b_1(P) \text{ w.p.i. } A\},$$

$$Sp(\Delta^1_{(\omega)}(L_A)) = \{C \subseteq Cd \mid C \text{ is definable by a flat formula in the language } \{\varepsilon, P\} \text{ w.p.i. } A\},$$

$$Sp(L_A^{II}) = \{C \subseteq Cd \mid C \text{ is definable with a flat formula of set theory w.p.i. } A\}.$$

Definition 4.5. *Suppose* L^* *is an abstract logic. The Löwenheim-number* $l(L^*)$ *of* L^* *is the least cardinal* κ *such that* $\min(C) \leq \kappa$ *for every* $C \in Sp(L^*)$, *if any such* κ *exist. Equivalently,* $l(L^*)$ *is the least cardinal* κ *such that if* $\varphi \in L^*$ *has a model, then* φ *has a model power* $\leq \kappa$.

It is well-known that $l(L_A^*)$ exists if A is a set.

Theorem 4.6. *Suppose* A *and* $A_0 \subseteq A$ *are transitive classes,* P *a predicate, and* L_A^* *a Boolean logic extending* L_A *and symbiotic with* P *on* A_0. *Then for any* $n < \omega$:

$$l(\Delta_n(L_A^*)) = \sup \{\kappa \mid \kappa \text{ is } \Pi_n(P)\text{-definable w.p.i. } A\}.$$

If $l(\Delta_n(L_A^*))$ *is a limit cardinal (e.g.* $n > 1$ *or* $LI \leq \Delta(L_A^*)$*), then moreover*

$$l(\Delta_n(L_A^*)) = \sup \{\alpha \mid \alpha \text{ is } \Pi_n(P)\text{-definable w.p.i. } A\}.$$

Proof. Suppose at first that α is $\Pi_n(P)$-definable w.p.i. A. Suppose $\varphi(x,y)$ is a $\sum_n(P)$-formula and $a \in A$ such that

$$\forall\beta(\beta \geq \alpha \leftrightarrow \varphi(\beta,a)).$$

Let K be the class of linearly ordered structures the ordertype of which is an ordinal $\geq \alpha$. K is clearly $\sum_n(P)$ w.p.i. A, whence K is $\sum_n(L_A^*)$-definable. But every model of K has power $\geq \mathrm{card}(\alpha)$. Hence $\alpha \leq \min\{\mathrm{card}(M) \mid M \in K\} < l(\Delta_n(L_A^*))$. For the converse, suppose $\kappa < l(\Delta_n(L_A^*))$. Let $\varphi \in L_A^*$ such that $\kappa \leq \lambda = \min(Sp(\varphi)) < l(\Delta_n(L_A^*))$. Now

$$\forall\beta(\beta > \lambda \leftrightarrow \exists\gamma \leq \beta\exists M(|M| = \gamma \;\&\; M \models^* \varphi))$$

whence λ is $\Pi_n(P)$-definable w.p.i. A. □

Corollary 4.7. *For any rudimentary set* A *and* $n > 1$:

(1) $l(LI_A) = \sup \{\alpha \mid \alpha \text{ is } \Pi_1(Cd)\text{-definable w.p.i. } A\}$,

(2) $l(L_A^{II}) = \sup \{\alpha \mid \alpha \text{ is } \Pi_2\text{-definable w.p.i. } A\}$,

(3) $l(\Delta_n(L_A)) = \sup \{\alpha \mid \alpha \text{ is } \Pi_n\text{-definable w.p.i. } A\}$,

(4) $l(L_A^s) = \sup \{\alpha \mid \alpha \text{ is definable in set theory w.p.i. } A\}$ *(in MKM).*

Part (2) of the above corollary was proved earlier but independently in [6].

Löwenheim-numbers can also be characterized in terms of a notion of describability. This notion is related to the notion of indiscribability (see [4] p. 268) but differs mainly in that less parameters are allowed.

Definition 4.8. *Let* D *be a set of formulae of set theory. An ordinal* α *is* D-*describable w.p.i.* A *if there are a* $\varphi(x) \in D$ *and an* $a \in R_\alpha \cap A$ *such that*

$$R_\beta \models \varphi(a) \quad \textit{for} \quad \beta \geq \alpha$$

and

$$R_\beta \not\models \varphi(a) \quad \textit{for} \quad rk(a) \leq \beta < \alpha.$$

The predicate P *is* R-*absolute if every* R_α *reflects* P.

Lemma 4.9. *Suppose* P *is* R-*absolute and* α *is* $\Pi_1(P)$-*definable w.p.i.* A. *Then there is a* $\beta > \alpha$ *such that* β *is* $\sum_1(P)$-*describable w.p.i.* A.

Proof. Suppose $\varphi(x,y)$ is $\sum_1(P)$-formula and $a \in A$ such that

$$\forall\beta(\beta \geq \alpha \leftrightarrow \varphi(\beta,a)).$$

Let $\psi(y)$ be the $\sum_1(P)$-formula $\exists x\varphi(x,y)$ and β the least β such that $R_\beta \models \psi(a)$. Then $\gamma \geq \beta \to R_\gamma \models \psi(a)$. Hence $\psi(a)$ describes β. $R_\beta \models \psi(a)$ clearly implies $\beta > \alpha$. □

Lemma 4.10. *Suppose* P *is* R-*absolute and* α *is* $\sum_1(P)$-*describable w.p.i.* A. *Then* $\alpha + 1$ *is* $\Pi_1(P)$-*definable w.p.i.* A.

Proof. Suppose $\varphi(x)$ is a $\sum_1(P)$-formula and $a \in A$ such that $R_\beta \models \varphi(a)$ if and only if $\beta \geq \alpha$. Let $\psi(y,x)$ be the $\sum_1(P)$-formula which says that $\varphi(x)$ is true in a transitive set which reflects P and the ordinal of which is $< y$. If $\psi(\beta,a)$, then (because P reflects) for some $\gamma < \beta$ $R_\gamma \models \varphi(a)$, whence $\beta > \alpha$. On the other hand, if $\beta > \alpha$, then $\psi(\beta,a)$ as one can choose R_α as the required transitive set. □

Corollary 4.11. *Suppose* A *and* $A_0 \subseteq A$ *are transitive sets,* P *an* R-*absolute predicate, and* L_A^* *an abstract logic extending* L_A *and symbiotic with* P *on* A_0, *and* $l(L_A^*)$ *is a limit cardinal. Then*

$$l(L_A^*) = \sup \{\alpha \mid \alpha \textit{ is } \sum_1\textit{-describable w.p.i. A}\}.$$

Proof. The claim follows immediately from 4.9, 4.10 and 4.6. □

Lemma 4.12. *Suppose* α *is first order describable (that is described by*

some formula of set theory) w.p.i. A. *Then* α *is* Π_2*-definable w.p.i.* A.

Proof. Suppose $\varphi(x)$ is a formula and $a \in A$ such that for $\beta \geq \mathrm{rk}(a)$

$$R_\beta \models \varphi(a) \text{ if and only if } \beta \geq \alpha.$$

Let $\psi(y,x)$ be the Σ_2-formula "$R_y \models \varphi(a)$". Then $\psi(\beta,a)$ if and only if $\beta \geq \alpha$.

Corollary 4.13. *For any set* A:

$$l(L_A^{II}) = \sup \{\alpha \mid \alpha \text{ is first order describable w.p.i. } A\}.$$

Another way of formulating Corollary 4.11 is the following:

Proposition 4.14. *Suppose* A *and* $A_0 \subseteq A$ *are rudimentary sets,* P *an* R*-absolute predicate, and* L_A^* *an abstract logic extending* L_A *and symbiotic with* P *on* A_0*, and* $l(L_A^*)$ *is a limit cardinal. Then*

$$l(L_A^*) = \text{the least } \alpha \text{ such that } \langle R_\alpha, \in, a\rangle_{a\in A\cap R_\alpha} \equiv_{\Sigma_1(P)} \langle V, \in, a\rangle_{a\in A}.$$

In particular,

$$l(L_A^{II}) = \text{the least } \alpha \text{ such that } \langle R_\alpha, \in, a\rangle_{a\in A\cap R_\alpha} \equiv_{\Sigma_2} \langle V, \in, a\rangle_{a\in A}.$$

The above result suggest the study of ordinals α such that

$$(*) \qquad \langle R_\alpha, \in\rangle \prec_{\Sigma_n} \langle V, \in\rangle.$$

Let us denote the predicate (*) of α by $D_n(\alpha)$. The following lemma will be most useful:

Lemma 4.15. *The predicate* $D_n(\alpha)$ *is* Π_n*, for* $n > 1$.

Proof. Let $S(x,y)$ be the Σ_n-predicate which is universal for Σ_n-formulae with one free variable y (see e.g. [4] p. 272). Let $F(z)$ by the Δ_1-predicate "z is a Σ_n-formula with one free variable y". If $F(z)$, let $f(z,\alpha)$ be the relativization of z to R_α. f is clearly Δ_2. Let $S_0(x,y)$ be the Δ_1-predicate which is universal for Σ_0-formulae with one free variable y. Now we have:

$$D_n(\alpha) \leftrightarrow \forall y \in R_\alpha \forall z \in \omega(F(z) \rightarrow (S_0(f(z,\alpha),y) \vee \neg S(z,y))),$$

and therefore $D_n(\alpha)$ is Π_n. □

Proposition 4.16. *If* α *is* Π_n*-definable w.p.i.* A*, then there is a* $\beta \geq \alpha$ *such that* β *is* Δ_n*-definable w.p.i.* A $(n > 1)$.

Proof. Let $\varphi(x,y)$ be a Π_n-formula and $a \in A$ such that

$$\forall\beta(\beta \in \alpha \leftrightarrow \varphi(\beta,a)).$$

Let $\psi(u,v)$ be the Δ_n-predicate "$D_{n-1}(u)$ & $R_u \models \exists x\, \varphi(x,v)$". Note that $D_1(\alpha)$ may not be Π_1 but by the proof of 4.15 it is Δ_2. Let $\theta(w,v)$ be the Δ_n-predicate $\exists u \leq w\psi(u,v)$. We claim that $\neg\,\theta(w,a)$ defines an ordinal $\geq \alpha$. By reflection there is an ordinal β such that $\psi(\beta,a)$. Let β be the least such β. If $\gamma \geq \beta$ then $\theta(\gamma,a)$. On the other hand, if $\theta(\gamma,a)$, then $\psi(\delta,a)$ for some $\delta \leq \gamma$, whence $\beta \leq \delta \leq \gamma$. □

Corollary 4.17. *For any rudimentary set* A *and* n > 1:

$l(L_A^{II}) = sup\ \{\alpha|\alpha$ *is* Δ_2*-definable w.p.i.* A$\}$.

$l(\Delta_n(L_A)) = sup\ \{\alpha|\alpha$ *is* Δ_n*-definable w.p.i.* A$\}$.

The predicate $D_n(\alpha)$ is actually equivalent to a Löwenheim-Skolem-theorem, as the following theorem shows:

Theorem 4.18. *The following are equivalent for any* n > 1:

(1) $l(\Delta_n(L_{\kappa\omega})) = \kappa$,

(2) $\langle R,\in\rangle \prec_{\sum_n} \langle V,\in\rangle$.

Proof. Note that both (1) and (2) imply $\kappa = \beth_\kappa$. If (2) holds and $\varphi \in \Delta_n(L_{\kappa\omega})$ has a model, then $R_\kappa \models$ "φ has a model", whence φ has a model of power $< \beth_\kappa = \kappa$. So (2) implies (1). Suppose then (1) holds. We may assume thet $D_{n-1}(\kappa)$ holds because if n = 2, it follows from $\kappa = \beth_\kappa$, and if n > 2, it follows from a suitable induction hypothesis. Suppose $\varphi(x)$ is a $\sum_n$-formula and $a \in R_\kappa$ such that $\varphi(a)$ holds. Let K be the class of ordinals α such that $D_{n-1}(\alpha)$ and $R_\alpha \models \varphi(a)$. By Theorem 2.4 and (1), there is a $\beta \in K$ such that $\beta \in \kappa$. As $D_{n-1}(\kappa)$, we have $R_\kappa \models \varphi(a)$, as required. □

Corollary 4.19. *If* κ *is supercompact, then* $l(\Delta_2(L_{\kappa\omega})) = \kappa$. *If* κ *is extendible, then* $l(\Delta_3(L_{\kappa\omega})) = \kappa$.

Proof. If κ is supercompact, then $D_2(\kappa)$; if κ is extendible, then $D_3(\kappa)$. These facts are proved in [11]. □

§ 5. Hanf-numbers

Hanf-numbers can be characterized in the same way as Löwenheim-numbers. One has to bear in mind, however, that Δ does not preserve Hanf-numbers (see [15]). Therefore we introduce a new notion of definability, bounded definability, which is neat enough to preserve Hanf-numbers but still almost as powerful as Δ- or Δ_1-

definability. This notion was first studied in [15]. The main result of this chapter is Theorem 5.6. The chapter ends with a discussion on definable ordinals and sort logic.

Definition 5.1. *Let* P *be a predicate of set theory. A predicate* $S(x_1,\dots,x_n)$ *of set theory is* $\sum_1^B(P)$ *w.p.i. A if there are a* $\sum_0(P)$*-formula* $\varphi(x_1,\dots,x_n,x,z)$ *and* $a \in A$ *such that*

$$\forall x_1\dots\forall x_n(S(x_1,\dots,x_n) \leftrightarrow \exists x\varphi(x_1,\dots,x_n,x,a))$$

and

$$\forall x_1\dots\forall x_n(\{x|\varphi(x_1,\dots,x_n,x,a)\} \text{ is a set}).$$

$S(x_1,\dots,x_n)$ *is* $\prod_1^B(P)$ *w.p.i. A if* $-S(x_1,\dots,x_n)$ *is* $\sum_1^B(P)$ *w.p.i. A. S is* Δ_1^B *w.p.i. A if S is both* $\sum_1^B(P)$ *and* $\prod_1^B(P)$ *w.p.i. A.*

An example of a $\Delta_1(Cd)$-predicate which is not (provably) $\Delta_1^B(Cd)$ is given in [15]. Note that every $\sum_1$-predicate is $\sum_1^B$ by Levy's theorem. From the fact

$$\exists x\varphi(x) \leftrightarrow \exists x(\varphi(x) \;\&\; \forall y\&rk(y) < rk(x) \rightarrow -\varphi(y))$$

it follows that every $\sum_1(P)$-predicate is $\sum_1^B(P,Pw)$. Therefore there is no need to define $\sum_n^B(P)$-predicates for $n > 1$ - they would coincide with the $\sum_n(P)$-predicates.

Now we define the model theoretic analogues of the above notions.

Definition 5.2. *Let* L* *be an abstract logic. A model class* K *is* $\sum^B(L^*)$*-definable if it is* $\sum$*-defined by an* L**-sentence* φ *such that*

$$\forall A\ \exists\kappa\forall B \in E(A,\varphi)(\mathrm{card}(B) \leq \kappa).$$

K *is* $\prod^B(L^*)$*-definable if* $\overline{K}$ *is* $\sum^B(L^*)$*-definable.* K *is* $\Delta^B(L^*)$*-definable if it is both* $\sum^B(L^*)$*- and* $\prod^B(L^*)$*-definable.*

Δ^B is a natural operation on logics and resembles Δ-operation so much that it is in fact not at all obvious that there is any difference between them. For a treatment of Δ^B see [15]. We pick up some of the results of [15] to the following lemma (note that (1) below fails for Δ):

Lemma 5.3.

(1) Δ^B *preserves Löwenheim- and Hanf-numbers.*

(2) $\sum^B(L^*) \sim \sum(L^*)$ *if* $L^{II} \leq \Delta^B(L^*)$ *or* L* *is one of the following logics (or a fragment of one)* $L_{\infty\omega}$, $L_{\infty\omega}(W)$, $L_{\infty\omega}(Q_\alpha)$, $L_{\infty\omega}(Q^{MM(n)})$, $L_{\infty G}$. *Hence* $\Delta^B(L^*) \sim \Delta(L^*)$ *for such* L*.

(3) $V = L$ *implies* $\Delta^B(LI) \sim \Delta(LI)$.

(4) If Con(ZF), *then* Con(ZFC + $\Delta^B(LI) \not\prec \Delta(LI)$).

Related to the bounded notions of definability is a new notion of symbiosis as well:

Definition 5.4. *Suppose* L^* *is an abstract logic,* P *a predicate of set theory and* A *a transitive class.* L^* *and* P *are boundedly symbiotic on* A *if the following two conditions are satisfied:*
(BS1) If $\varphi \in L^*$, *then* Mod(φ) *is* $\Delta_1^B(P)$ *w.p.i.* $\{\varphi, L\}$
(BS2) K[P] *is* $\Delta^B(L_A^*)$*-definable.*
The pairs of example 2.3 are all boundedly symbiotic.

We omit the proof of the following theorem because the proof would be mutatis mutandis as that of 2.4.

Theorem 5.5. *Suppose* A *and* $A_o \subseteq A$ *are transitive classes,* P *a predicate, and* L_A^* *an abstract logic extending* L_A *and boundedly symbiotic with* P *on* A_o. *Then the following are equivalent:*
(1) K *is* $\sum^B(L_A^*)$*-definable,*
(2) K *is* $\sum_1^B(P)$ *w.p.i.* A.

Theorem 5.6. *Suppose* A *and* $A_o \subseteq A$ *are transitive classes,* P *a predicate, and* L_A^* *a Boolean logic extending* L_A *and boundedly symbiotic with* P *on* A_o. *Then*

$$h(L_A^*) = sup\ \{\alpha | \alpha \text{ is } \textstyle\sum_1^B(P)\text{-definable w.p.i. A}\}$$

and for n > 1:

$$h(\Delta_n(L_A^*)) = sup\ \{\alpha | \alpha \text{ is } \textstyle\sum_n(P)\text{- definable w.p.i. A}\}.$$

Proof. In order to prove the two claims simultaneously, let us agree that $\sum_n^B(P)$ for n > 1 means $\sum_n(P)$. Now, let n > 0. Suppose that α is $\sum_n^B(P)$-definable w.p.i. A. Let K be the class of linearly ordered structures the order type of which is $< \alpha$. K is $\sum_n^B(P)$ w.p.i. A and therefore $\sum_n^B(L_A^*)$-definable (using 2.4 and 5.4, $\sum_n^B(L_A^*)$ for n > 1 means $\sum_n(L_A^*)$). Hence K is $\sum(\Delta_n^B(L_A))$-definable. If n > 1, then $L^{II} \leq \Delta_n^B(L_A^*)$ whence by 5.3 (2) K is $\sum^B(\Delta_n^B(L_A^*))$-definable. If n = 1, the same conclusion follows trivially. Hence there is a $\psi \in \Delta_n^B(L_A^*)$ which $\sum^B$-defines K. As K has models of power $\leq$ card(α) only, ψ does not have arbitrary large models. But ψ has a model of power $\geq \kappa$ for every $\kappa < \alpha$. Hence card(α) $< h(\Delta_n^B(L_A^*))$. It follows easily that $\alpha < h(\Delta_n^B(L_A^*))$. For the converse, suppose $\kappa < h(\Delta_n^B(L_A^*))$. Let φ be in $\Delta_n^B(L_A)$ such that $\kappa \leq \lambda = \sup Sp(\varphi)$. Now

$$\alpha < \lambda \leftrightarrow \exists\beta\exists M(\alpha \leq \beta \ \&\ |M| = \beta \ \&\ M \models^* \varphi).$$

Hence λ is $\sum_n^B(P)$-definable w.p.i. A. □

Corollary 5.7. *For any rudimentary set* A:

(1) $h(L_A^{II}) = sup\ \{\alpha | \alpha$ *is* $\sum_2$*-definable w.p.i.* A$\}$.

(2) $h(\Delta_n(L_A)) = sup\ \{\alpha | \alpha$ *is* $\sum_n$*-definable w.p.i.* A$\}$ $(n > 1)$.

(3) In MKM: $h(L_A^S) = sup\ \{\alpha | \alpha$ *is definable in set theory w.p.i.* A$\}$
$= l(L_A^S)$.

Part (1) of the above corollary was proved earlier, but independently, in [6] (see also [1]).

Let us write

l_n for sup $\{\alpha | \alpha$ is Π_n-definable$\}$,

h_n for sup $\{\alpha | \alpha$ is $\sum_n$-definable$\}$.

By what we have already proved: (for $n > 1$)

$$l_n = l(\Delta_n^*(L_{HF})) = \sup \{\alpha | \alpha \text{ is } \Delta_n\text{-definable}\},$$

$$h_n = h(\Delta_n(L_{HF})).$$

In the next few lemmas we shall establish the mutual relations of the ordinals $l_n, h_n, n < \omega$. It turns out that the following notation is helpful:

t_n = the least α such that $D_n(\alpha)$

= the least α such that $\langle R_\alpha, \in\rangle \prec_{\sum_n} \langle V, \in\rangle$

= the least α such that $l(\Delta_n(L_{\alpha\omega})) = \alpha$.

Trivially $l_n \leq t_n \leq h_n$ for $n > 1$.

Lemma 5.8. *For* $n > 0$, $l_n < t_n$.

Proof. Let $S(x,y)$ be the $\sum_n$-formula which is universal for $\sum_n$-formulae with the free variable y. Let

$$a = \{\varphi(y) | \varphi(y) \text{ is a } \textstyle\sum_n\text{-formula such that } \neg\, \varphi(y) \text{ defines an ordinal}\}.$$

$a \in R_{\omega+1}$ and therefore $a \in R_{t_n}$. Let $\psi(x,y)$ be a $\sum_n$-formula equivalent to $\forall u \in x S(u,y)$. Now $\psi(a,y)$ is true for some y, whence $\psi(a,y)$ is true for some $y \in R_{t_n}$. This y is an ordinal which is greater than any Π_n-definable ordinal. Therefore $l_n \leq y < t_n$. □

Lemma 5.9. *If* $n > 1$, *then* t_n *is* $\sum_n$*-definable, and hence* $t_n < h_n$.

Proof. Recall from 4.15 that D_n is Π_n. Hence the claim follows from

$$\forall\alpha(\alpha < t_n \leftrightarrow \neg D_n(\alpha) \ \& \ \forall\beta < \alpha \neg D_n(\beta)). \quad \square$$

Lemma 5.10. *If* $n > 1$, *then* $h_n = l_{n+1}$.

Proof. Suppose α is Π_{n+1}-definable and $\varphi(x,y)$ is a $\sum_n$-formula such that

$$(*) \qquad \forall\beta\ \beta < \alpha \leftrightarrow \forall x\varphi(x,\beta)).$$

Let $\psi(x)$ be a $\sum_n$-formula saying that x is an ordinal and $\varphi(y,\beta)$ holds for all $y \in R_x$ and $\beta < x$. If $\forall x\psi(x)$, then $\forall x\varphi(x,\alpha)$, a contradiction. Therefore there are δ such that $\neg\psi(\delta)$. Let δ be the least of them. Hence if $\beta < \delta$ then $\psi(\beta)$. On the other hand, if $\psi(\beta)$ and $\gamma \leq \beta$ then $\psi(\gamma)$, whence $\gamma \neq \delta$. Therefore $\psi(x)$ $\sum_n$-defines δ. Hence it suffices to prove that $\alpha \leq \delta$. Suppose the contrary, that is $\delta < \alpha$. If $y \in R_\delta$ and $\beta < \delta$, then by (*) $\varphi(y,\beta)$. Hence $\psi(\delta)$ holds, a contradiction. Therefore $\alpha \leq \delta$. $\square$

Corollary 5.11. $l_2 < h_2 = l_3 < h_3 = l_4 < h_4 = l_5 < \ldots$

If the proofs of 5.8-5.10 are carried out with parameters, the following theorem yields:

Theorem 5.12. *Suppose* A *is a rudimentary set and* $n > 1$. *Then*

$$l(\Delta_n(L_A)) < h(\Delta_n(L_A)) = l(\Delta_{n+1}(L_A)).$$

Corollary 5.13. *(MKM)* $l(L^s_{HF}) = h(L^s_{HF})$ = *the least* α *such that* $R_\alpha \prec V$.

Proof. Suppose $\varphi(x_1,\ldots,x_n)$ is a formula of set theory and $a_1,\ldots,a_n$ sets in R_α, $\alpha = l(L^s_{HF})$, such that $\varphi(a_1,\ldots,a_n)$. Let $m < \omega$ such that $\varphi(x_1,\ldots,x_n)$ is equivalent to a $\sum_m$-formula $\varphi(x_1,\ldots,x_n)$. Now $R_{t_k} \models \psi(a_1,\ldots,a_n)$ for a sufficiently large $k < \omega$. We may assume $D_{m-1}(\alpha)$. Therefore $R_\alpha \models \psi(a_1,\ldots,a_n)$. Hence $R_\alpha \models \varphi(a_1,\ldots,a_n)$. For the converse, suppose $R_\alpha \prec V$. Then every definable ordinal must be $< \alpha$. Therefore $\alpha \geq l(L^s_{HF})$. $\square$

Corollary 5.14. *If the required cardinals exist, then*

1st measurable $< l_2 <$ *1st supercompact* $< h_2 = l_3 <$ *1st extendible* $< h_3$.

Proof. The predicate "α is measurable" is $\sum_2$. Hence the 1st measurable is Π_2-definable and therefore $< l_2$. If κ is supercompact, then $D_2(\kappa)$ (see [11] p. 86), and hence $l_2 < t_2 \leq \kappa$. The predicate "α is supercompact" is Π_2. Hence the 1st supercompact is $\sum_2$-definable and therefore $< h_2$. If κ is extendible, then $D_3(\kappa)$ (see [11] p. 103), and hence $l_3 < t_3 \leq \kappa$. The predicate

"α is extendible" is Π_3 and therefore the 1st extendible is Σ_3-definable. Hence the 1st extendible $< h_3$. □

So we see that the Löwenheim- and Hanf-numbers of even the lowest levels of sort logic exhaust a wide range of large cardinals. This would seem to suggest that the logics $\Delta_n(L_A)$ are rather strong indeed. In connection with 5.14, note that the ordinals $l_n, h_n, n < \omega$ exist even if there are no large cardinals; they exist in L, for example.

It seems to be a rather common phenomenon that the Löwenheim-number of a logic is smaller (often substantially) than the Hanf-number (see e.g. 5.12). However, in the second part of this paper we shall construct a model of set theory where the Hanf-number of LI is smaller than the Löwenheim-number of LI. In that model the spectrum problem for LI has a negative solution, because there is a cardinal κ between l(LI) and h(LI) such that $\{\lambda | \lambda \geq \kappa\}$ is a spectrum, but $\{\lambda | \lambda < \kappa\}$ is (obviously) not.

We end this chapter with a remark on another way of characterizing $h(L_A^{II})$.

Definition 5.15. *An ordinal α is weakly first order describable w.p.i.* A *if there are a formula $\varphi(x)$ of set theory and an $a \in R_\alpha \cap A$ such that*

$$R_\beta \models \varphi(a) \quad \textit{for} \quad \beta \geq \alpha$$

and

$$R_\beta \not\models \varphi(a) \quad \textit{for arbitrary large} \quad \beta < \alpha,\ \beta \geq \mathrm{rk}(a).$$

Theorem 5.16. *Suppose* A *is a rudimentary set.*

$h(L_A^{II}) = \sup \{\alpha | \alpha$ *is weakly first order describable w.p.i.* A$\}$.

Bibliography

[1] J. Barwise, The Hanf number of second order logic, J.S.L., 37 (1972), pp. 588-594.

[2] J. Barwise, Absolute logics and $L_{\infty\omega}$, Annals of Math. Logic, 4 (1972), pp. 309-340.

[3] J. Barwise, Admissible sets and structures, Springer-Verlag, Berlin and New York, 1975.

[4] F. Drake, Set theory, North-Holland, Amsterdam-Oxford-New York, 1974.

[5] S. Feferman, Many-sorted interpolation theorems and applications, Proceedings of the Tarski Symposium, A.M.S. Proc. Symp. in Pure Math., 25, pp. 205-223, A.M.S. 1974.

[6] A. Krawczyk and W. Marek, On the rules of proof generated by hierarchies, Lecture Notes in Mathematics, 619, Springer-Verlag, Berlin and New York, 1977, pp. 227-240.

[7] M. Krynicki and A. Lachlan, On the semantics of the Henkin quantifier, J. Symbolic Logic, to appear.

[8] J. Makowsky, S. Shelah and J. Stavi, Δ-logics and generalized quantifiers, Annals of Math. Logic 10 (1976), pp. 155-192.

[9] R. Montague, Reduction of higher-order logic, The theory of models (Addison, Henkin, Tarski, editors), North-Holland, Amsterdam, 1965, pp. 251-264.

[10] J. Oikkonen, Second order definability, game quantifiers and related expressions, Commentationes Physico-Mathematicae, 48 (1978).

[11] R. Solovay, W. Reinhardt and A. Kanamori, Strong axioms of infinity and elementary embeddings, Annals of Math. Logic 13 (1978), pp. 73-116.

[12] L. Tharp, The characterization of monadic logic, J.S.L. 38 (1973), pp. 481-488.

[13] J. Väänänen, Applications of set theory to generalized quantifiers, Ph.D. thesis, University of Manchester, October 1977.

[14] J. Väänänen, Boolean valued models and generalized quantifiers. Reports of the Department of Mathematics, University of Helsinki, 1978.

[15] J. Väänänen, Δ-operation and Hanf-numbers. Preprint, Oslo, 1978.