Arch. Math. Logic 43, 19–30 (2004)
Digital Object Identifier (DOI):
10.1007/s00153-003-0171-5

Mathematical Logic

Jouko Väänänen · Boban Veličković

Games played on partial isomorphisms

Received: 5 September 2000 / Revised: 14 August 2002 /
Published online: 6 August 2003 –

1. Introduction

Suppose $\mathcal{A}$ and $\mathcal{B}$ are structures for the same countable relational vocabulary. We denote the universe of $\mathcal{A}$ by A and the universe of $\mathcal{B}$ by B. A partial mapping $p : A \to B$ is a *partial isomorphism* (p.i.) $\mathcal{A} \to \mathcal{B}$ if p is an isomorphism between $\mathcal{A} \restriction \mathrm{dom}(p)$ and $\mathcal{B} \restriction \mathrm{rng}(p)$. Let

$$\mathcal{I}_\kappa = \{p : p \text{ is a p.i. } \mathcal{A} \to \mathcal{B} \text{ and } |p| < \kappa\}.$$

We study the poset $\langle \mathcal{I}_\kappa, \subseteq \rangle$ as a measure of how similar the structures $\mathcal{A}$ and $\mathcal{B}$ are to each other.

A subset X of $\mathcal{I}_\kappa$ has the κ - *back-and-forth* property if for all $\lambda < \kappa$

$$\forall p \in X[\forall a \in {}^\lambda A \exists b \in {}^\lambda B \ (p \cup \{\langle a(i), b(i)\rangle : i < \lambda\} \in X)$$
$$\wedge \forall b \in {}^\lambda B \exists a \in {}^\lambda A (p \cup \{\langle a(i), b(i)\rangle : i < \lambda\} \in X)].$$

It is obvious that there is a largest κ-back-and-forth set which we denote by $\mathcal{I}^*_\kappa$. The structures $\mathcal{A}$ and $\mathcal{B}$ are said to be *partially isomorphic*, $\mathcal{A} \simeq_p \mathcal{B}$, if $\mathcal{I}^*_2 \neq \emptyset$. We get stronger criteria by demanding that $\mathcal{I}^*_\kappa$ is not just non-empty but “large”. This leads naturally to the condition:

$(\sigma)_\kappa$ There is a set $D \subseteq \mathcal{I}^*_\kappa$ which has the κ-back-and-forth property and is σ-closed.

By D being σ - *closed* we mean that if $p_0 \subseteq p_1 \subseteq ... \subseteq p_n \subseteq \cdots (n < \omega)$ are elements of D, then $\bigcup_n \ p_n \in D$. Structures $\mathcal{A}$ and $\mathcal{B}$ satisfying $(\sigma)_{\aleph_1}$ are said in [1] to be strongly partially isomorphic $\mathcal{A} \simeq^s_p \mathcal{B}$. Kueker [5] mentions this concept, too. Not much is known about $(\sigma)_\kappa$. Just as the classical back-and-forth argument gives

$$(\mathcal{A} \simeq_p \mathcal{B} \ \& \ |A|, |B| \leq \aleph_0) \Rightarrow \mathcal{A} \cong \mathcal{B}$$

J. Väänänen*: Department of Mathematics, University of Helsinki, Helsinki, Finland

B. Veličković: UFR de mathèmatiques, Université de Paris 7, Paris, France

*Partially supported by the Academy of Finland grant #1011049.

we have

$$(\mathcal{A} \simeq_p^s \mathcal{B} \,\&\, |A|, |B| \leq \aleph_1) \Rightarrow \mathcal{A} \cong \mathcal{B}.$$

If $\mathcal{A}$ and $\mathcal{B}$ are homogeneous, then $\mathcal{A} \simeq_p \mathcal{B}$ and $\mathcal{A} \simeq_p^s \mathcal{B}$ are both equivalent to $\mathcal{A} \equiv \mathcal{B}$. If $\mathcal{A}$ and $\mathcal{B}$ are η_1-real closed fields, then $(\sigma)_{\aleph_1}$ holds.

The main open question concerning $(\sigma)_{\aleph_1}$ is whether it is transitive, that is, whether

$$(\mathcal{A} \simeq_p^s \mathcal{B} \,\&\, \mathcal{B} \simeq_p^s \mathcal{C}) \Rightarrow \mathcal{A} \simeq_p^s \mathcal{C}?$$

We get some partial answers to this problem. We show that $(\sigma)_{\aleph_1}$ is transitive in the class of structures of size $\leq 2^{\aleph_0}$ and in various classes of structures of size $\leq \aleph_2$, and that $(\sigma)_{\kappa^+}$ is transitive in the class of structures of size κ^+.

The condition $\mathcal{A} \simeq_p \mathcal{B}$ can be characterized in terms of the Ehrenfeucht-Fraïssé game. It is natural to investigate connections between this game and $\mathcal{A} \simeq_p^s \mathcal{B}$. The game EF_δ^κ on $\mathcal{A}$ and $\mathcal{B}$ has two players $\forall$ and $\exists$. The game has δ rounds. During round α player $\forall$ picks $\lambda_\alpha < \kappa$, one of the models $\mathcal{A}$ and $\mathcal{B}$, say $\mathcal{A}$, and a sequence $x_\alpha \in {}^{\lambda_\alpha}A$. Then player $\exists$ picks $y_\alpha \in {}^{\lambda_\alpha}B$. In this case we denote x_α by a_α and y_α by b_α. If $\forall$ picked $\mathcal{B}$ instead of $\mathcal{A}$ and $x_\alpha \in {}^{\lambda_\alpha}B$, then x_α would be denoted by b_α and the choice $y_\alpha \in {}^{\lambda_\alpha}A$ of $\exists$ would be denoted by a_α. After δ rounds have been played we have

$$p = \{\langle a_\alpha(i), b_\alpha(i)\rangle : i < \lambda_\alpha,\ \alpha < \delta\}.$$

If p is a partial isomorphism $\mathcal{A} \to \mathcal{B}$, then $\exists$ won. Otherwise $\forall$ won. We get the following criterion:

$(\exists)_\delta^\kappa$ Player $\exists$ has a winning strategy in the game EF_δ^κ on $\mathcal{A}$ and $\mathcal{B}$.

It is well-known that

$$\mathcal{I}_\omega^* \neq \emptyset \;\Leftrightarrow\; (\exists)_\omega^2,$$

and easy to see that

$$(\sigma)_{\aleph_1} \;\Rightarrow\; (\exists)_{\omega_1}^{\aleph_1}.$$

The question whether

$$(\sigma)_{\aleph_1} \;\Leftrightarrow\; (\exists)_{\omega_1}^{\aleph_1}?$$

is open. Note that a "yes" would imply that $(\sigma)_{\aleph_1}$ is transitive, as $(\exists)_\delta^\kappa$ is clearly transitive. We establish $(\sigma)_{\aleph_1} \Leftrightarrow (\exists)_{\omega_1}^{\aleph_1}$ in the case that $|A|, |B| \leq 2^{\aleph_0}$ as well as for special classes of $\mathcal{A}$ and $\mathcal{B}$ of size $\aleph_2$. Although we cannot prove $(\exists)_{\omega_1}^{\aleph_0} \Rightarrow (\sigma)_{\aleph_1}$ even for all models of size $\aleph_2$ we can prove $(\exists)_{\omega_1 \cdot \omega_1}^{\aleph_1} \Rightarrow (\sigma)_{\aleph_2}$ and $(\exists)_{\omega_1}^{\aleph_2} \Leftrightarrow (\sigma)_{\aleph_2}$ for models of size $\aleph_2$.

We can look at the largeness of $\mathcal{I}_\kappa^*$ also in terms of the Banach-Mazur game $\mathcal{G}(\mathcal{I}_\kappa^*)$ on poset $\langle \mathcal{I}_\kappa^*, \supseteq\rangle$. For this game, see [3], [2] and [6]. The game $\mathcal{G}(P)$ has two players called Empty and Nonempty. They alternately play descending sequence of P:

Empty	p_0		p_2		...	
Nonempty		p_1		p_3		...

Nonempty wins the run of the game if there is $p \in P$ such that $p \leq p_n$ for all $n < \omega$. We show that if Nonempty has a winning strategy in $\mathcal{G}(\mathcal{I}_{\aleph_1}^*)$ and either A

or B has cardinality $< \aleph_\omega$, then the poset $\langle \mathcal{I}^*_{\aleph_1}, \supseteq \rangle$ has a σ-closed dense set. Previously this was known for posets that are trees [6] and for posets whose regular-open algebra contains a dense subset of size $\leq 2^{\aleph_0}$ [2]. As an application of our result we can show that if Nonempty has a winning strategy in $\mathcal{G}(\mathcal{I}^*_{\aleph_1})$, then $(\sigma)_{\aleph_1}$ holds.

2. Models of size 2^ω

One approach to getting $(\sigma)_{\aleph_1}$ from $(\exists)^{\aleph_1}_{\omega_1}$ is to organize the partial isomorphisms arising from positions in the Ehrenfeucht-Fraïssé -game in such a way that they uniquely determine the position they come from. In this section we use this approach to prove that $(\exists)^{\aleph_1}_{\omega_1}$ implies $(\sigma)_{\aleph_1}$ for models $\mathcal{A}$ and $\mathcal{B}$ of size $\leq 2^{\aleph_0}$. First we note a simple lemma:

Lemma 1. *The following conditions are equivalent for any $\mathcal{A}$ and $\mathcal{B}$:*

(1) *$\exists$ has a winning strategy in $EF^{\aleph_1}_{\omega_1}$ on $\mathcal{A}$ and $\mathcal{B}$.*
(2) *There is a countably closed notion of forcing P such that $\Vdash_P \check{\mathcal{A}} \cong \check{\mathcal{B}}$.*

Theorem 2. *Suppose $\mathcal{A}$ and $\mathcal{B}$ have size $\leq 2^{\aleph_0}$. Then $(\exists)^{\aleph_1}_{\omega_1} \Leftrightarrow (\sigma)_{\aleph_1}$.*

Proof. Suppose $(\exists)^{\aleph_1}_{\omega_1}$. Let us assume for simplicity that $A = B = 2^\omega$. By Lemma 1 there is a countably closed notion of forcing P and a name $\tilde{f}$ such that

$$\Vdash_P \tilde{f} : \check{\mathcal{A}} \cong \check{\mathcal{B}}.$$

Case 1. There is a $p \in P$ such that for all α there is a unique β with $r \Vdash \tilde{f}(\check{\alpha}) = \check{\beta}$ for some $r \leq p$, and a unique β with $r \Vdash \tilde{f}(\check{\beta}) = (\check{\alpha})$ for some $r \leq p$. Let

$$g = \{\langle \alpha, \beta \rangle \mid (\exists q \leq p)(q \Vdash \tilde{f}(\check{\alpha}) = \check{\beta})\}.$$

Now it is clear that $g : \mathcal{A} \cong \mathcal{B}$, so $(\sigma)_{\aleph_1}$ holds.

Case 2. For all $p \in P$ there is α such that for two different β we have $r \Vdash \tilde{f}(\check{\alpha}) = \check{\beta}$ for some $r \leq p$ or for two different β we have $r \Vdash \tilde{f}(\check{\beta}) = \check{\alpha}$ for some $r \leq p$.

Thus every $p \in P$ has continuum many incompatible extensions all deciding mutually contradictory things about $\tilde{f}$. These extensions can then be further extended to p_α, $\alpha < 2^\omega$, such that each p_α decides $\tilde{f}(\check{\alpha})$ and $\tilde{f}^{-1}(\check{\alpha})$ but for $\alpha \neq \beta$ p_α and p_β differ on $\tilde{f}$. By iterating this ω_1 times we get sets

$$\{f_s : s \in {}^{<\omega_1}2\}$$
$$\{p_s : s \in {}^{<\omega_1}2\}$$

so that

(C1) $p_s \Vdash \check{f}_s \subseteq \tilde{f}$
(C2) $s \leq s' \Leftrightarrow p_{s'} \leq p_s$
(C3) $s \leq s' \Leftrightarrow f_s \subseteq f_{s'}$

(C4) For every $s \in {}^{<\omega_1}2$ and every $\alpha < 2^\omega$ there are $\beta < 2^\omega$ and $s' \in {}^{<\omega_1}2$ such that $s \le s'$ and $f_s \cup \{\langle \alpha, \beta \rangle\} \subseteq f_{s'}$.
(C5) For every $s \in {}^{<\omega_1}2$ and every $\beta < 2^\omega$ there are $\alpha < 2^\omega$ and $s' \in {}^{<\omega_1}2$ such that $s \le s'$ and $f_s \cup \{\langle \alpha, \beta \rangle\} \subseteq f_{s'}$.

Now it is clear that $D = \{f_s : s \in {}^{<\omega_1}2\}$ is a σ-closed $\aleph_1$-back-and-forth set. □

3. Models of different cardinality

In this section we deduce $(\sigma)_{\aleph_1}$ from $(\exists)^{\aleph_1}_{\omega_1}$ in all cases where the models are of different cardinality.

Theorem 3. *Suppose $\mathcal{A}$ and $\mathcal{B}$ have different cardinality. Then* $(\exists)^{\aleph_1}_{\omega_1} \Leftrightarrow (\sigma)_{\aleph_1}$.

Proof. Suppose $(\exists)^{\aleph_1}_{\omega_1}$. Pick a countably closed notion of forcing P and a name $\tilde{f}$ such that

$$\Vdash_P \tilde{f} : \check{\mathcal{A}} \cong \check{\mathcal{B}}.$$

Suppose $|A| = \kappa$ and $|B| = \lambda > \kappa$. We may assume $\kappa > \omega$. We assume λ is singular. The regular case is similar but easier. Let λ_ξ, $\xi < cf(\lambda)$, be an increasing cofinal sequence in λ such that $cf(\lambda) < \lambda_\xi$ for all ξ. Let us call a set of partial functions $A \to B$ an *antichain* if the union of any two of them fails to be a partial function. Let T be the tree of sequences $s : \omega_1 \to \lambda$ such that for all limit ν and all $n < \omega$ we have

$$s(\nu + 3n) < cf(\lambda), s(\nu + 3n + 1) < \lambda_{s(\nu+3n)}, s(\nu + 3n + 2) < \kappa.$$

We will use repeatedly the following fact:

(*) If $\kappa < \mu^+ < \lambda$ and $p \in P$ such that $p \Vdash \check{g} \subseteq \tilde{f}$, then there are extensions q_ξ, $\xi < \mu^+$, of p and extensions g_ξ of g such that $q_\xi \Vdash \check{g}_\xi \subseteq \tilde{f}$ and $\{g_\xi : \xi < \mu^+\}$ is an antichain.

By iterating this way of extending a condition, we get conditions p_s, $s \in T$, and functions f_s, $s \in T$ such that:

(D1) $p_s \Vdash \check{f}_s \subseteq \tilde{f}$
(D2) $s \le s' \Leftrightarrow p_{s'} \le p_s$
(D3) $s \le s' \Leftrightarrow f_s \subseteq f_{s'}$
(D4) If $s \in T$ and $\alpha \in A$, then there is $\beta \in B$ and $s' \in T$ such that $s \le s'$ and $f_s \cup \{\langle \alpha, \beta \rangle\} \subseteq f_{s'}$
(D5) If $s \in T$ and $\beta \in B$ then there is $\alpha \in A$ and $s' \in T$ such that $s \le s'$ and $f_s \cup \{\langle \alpha, \beta \rangle\} \subseteq f_{s'}$.

Let

$$D = \{f_s : s \in T\}.$$

By construction, D is an $\aleph_1$-back-and-forth set which is σ-closed. So we have $(\sigma)_{\aleph_1}$. □

4. Models of the same cardinality: Trees

For models of the same size we do not have a general proof for the equivalence of $(\sigma)_{\aleph_1}$ and $(\exists)^{\aleph_1}_{\omega_1}$. However, we show now that for trees of height ω_1 the condition $(\exists)^{\aleph_1}_{\omega_1}$ implies $(\sigma)_{\aleph_1}$.

Theorem 4. *If the trees T_0 and T_1 are of height ω_1 and satisfy $(\exists)^2_{\omega_1}$, then they satisfy $(\sigma)_{\aleph_1}$.*

Before we start the proof we present a simple lemma about bipartite graphs which may have some interest in its own. Recall that a bipartite graph is a triple $G = (A, B, E)$ where A and B are nonempty sets and $E \subseteq A \times B$ is the set of edges between A and B. A *matching* is a (possibly partial) injective function f from A to B such that for every $x \in \mathrm{dom}(f)$ $(x, f(x)) \in E$. A *perfect matching* is a bijection between A and B which is also a matching. Let $\mathcal{M}$ be a family of partial matchings. We say that $\mathcal{M}$ has *the extension property* if for every $f \in \mathcal{M}$, $x \in A$ and $y \in B$ there is $g \in \mathcal{M}$ with $f \subseteq g$ such that $x \in \mathrm{dom}(g)$ and $y \in \mathrm{ran}(g)$. Thus if $\mathcal{M}$ has the extension property and we consider it as a forcing notion ordered by reverse inclusion then forcing with $\mathcal{M}$ introduces a perfect matching in G.

Lemma 5. *Let $G = (A, B, E)$ be a bipartite graph and assume there is a σ-closed forcing notion $\mathcal{P}$ which adds a perfect matching in G. Then there is a σ-closed family $\mathcal{M}$ of countable partial matchings which has the extension property.*

Proof. We shall prove this by induction on the size$(G) = |A| + |B|$. First, note that if size$(G) \leq \aleph_1$ then G has a perfect matching f and thus we can take for $\mathcal{M}$ the family of all countable partial submatchings of f.

Assume now size$(G) = \kappa > \aleph_1$ and fix a $\mathcal{P}$-name $\dot{f}$ for a perfect matching. For a given condition $p \in \mathcal{P}$ let $\mathcal{M}_p$ be the collection of all countable partial matchings s such that there is a condition $q \leq p$ with $q \Vdash s \subseteq \dot{f}$. Let us say that a set of partial matchings is an *antichain* if the union of any two of them fails to be a partial matching. Let λ be the least cardinal such that for some $p\mathcal{M}_p$ has no antichains of size λ. If $\lambda = \kappa^+$ we can build a required family of partial matchings as in the proof of Theorem 3. Now, assume $\lambda \leq \kappa$ and fix a condition p such that $\mathcal{M}_p$ has no antichains of size λ. Since $\mathcal{P}$ is σ-closed we can easily show that $\mathrm{cof}(\lambda) > \aleph_0$. Let $E_p = \bigcup \mathcal{M}_p$ and let $G_p = (A, B, E_p)$ be the resulting subgraph. Now note that the connected components of G_p are of size $< \lambda$. Therefore we can find decompositions

$$A = \bigcup_{i \in I} A_i \text{ and } B = \bigcup_{i \in I} B_i$$

and such that $|A_i| + |B_i| < \lambda$, for each i, and $E_p \subseteq \bigcup_{i \in I} A_i \times B_i$. Now let E_i be the restriction of E_p to $A_i \times B_i$ and consider bipartite graphs $G_i = (A_i, B_i, E_i)$, for $i \in I$. Each of them has size $< \lambda$ and p forces that $\dot{f} \restriction A_i$ is a perfect matching of G_i. We can therefore apply our induction hypothesis to find a family of countable partial matchings $\mathcal{M}_i$ in G_i which have the extension property. Finally, let $\mathcal{M}$ consist of all partial matchings g of the form $g = \bigcup_{i \in I_0} g_i$, where $I_0 \subseteq I$ is a countable set, and $g_i \in \mathcal{M}_i$, for each $i \in I_0$. Then $\mathcal{M}$ is as required. $\square$

Proof of Theorem 4. Let us fix a σ-closed forcing notion $\mathcal{P}$ which forces T_0 and T_1 to be isomorphic and a $\mathcal{P}$-name for an isomorphism $\dot{f}$. We shall define a relation $R \subseteq T_0 \times T_1$ and for each $(x, y) \in R$ a condition $p_{x,y} \in \mathcal{P}$ such that $p_{x,y} \Vdash \dot{f}(x) = y$. In particular, we will have that if $(x, y) \in R$ then x and y are of the same height. Moreover, if $x < x'$, $y < y'$ and both (x, y) and (x', y') are in R then we will have $p_{x',y'} \leq p_{x,y}$.

To begin, assume for simplicity that T_0 and T_1 both have roots, say r_0 and r_1. Put (r_0, r_1) in R and let p_{r_0,r_1} be any condition in $\mathcal{P}$ (which necessarily forces that $\dot{f}(r_0) = r_1$.) Assume now, we have put (x, y) in R and let $p_{x,y}$ be the associated condition. Let S_x be the set of successors of x in T_0 and S_y the set of successors of y in T_1. Let

$$E_{x,y} = \{(x', y') \in S_x \times S_y : \text{ there is } q \leq p_{x,y} \text{ such that } q \Vdash \dot{f}(x') = y'\}.$$

We put all elements of $E_{x,y}$ into R and for each $x, y \in E_{x',y'}$ we pick $p_{x',y'} \leq p_{x,y}$ such that $p_{x',y'} \Vdash \dot{f}(x') = y'$. Finally, if $x \in T_0$ and $y \in T_1$ are elements of the same height which is a limit ordinal and for every $x' < x$ and $y' < y$ of the same height $(x', y') \in R$ then we put (x, y) into R and we define $p_{x,y}$ to be any condition extending the corresponding conditions $p_{x',y'}$. This completes the construction of the relation R and the assignment of a condition to each pair $(x, y) \in R$.

Notice now that since $p_{x,y} \Vdash \dot{f}(x) = y$ and $\dot{f}$ is forced to be an isomorphism, forcing with $\mathcal{P}$ below $p_{x,y}$ introduces a perfect matching in the graph $G_{x,y} = (S_x, S_y, E_{x,y})$. By Lemma 5 we can find, for each $(x, y) \in R$, a family of partial matchings $\mathcal{M}_{x,y}$ in the graph $G_{x,y}$ with the extension property. We may, of course, assume that $\mathcal{M}_{x,y}$ contains the empty set.

We now describe a σ-closed family $\mathcal{F}$ of partial isomorphisms between T_0 and T_1 which has the back and forth property. We put a countable partial function g into $\mathcal{F}$ if $\mathrm{dom}(g)$ is an initial segment of T_0, $\mathrm{ran}(g)$ is an initial segment of T_1, g is a partial isomorphism between the two, and whenever $g(x) = y$ then $g \restriction S_x \in \mathcal{M}_{x,y}$. It is straightforward to check that $\mathcal{F}$ is as required. □

5. Models of the same cardinality: The decomposition property

In this section we consider models of size at most $(2^{\aleph_0})^+$. If two models satisfy $(\exists)^{\aleph_1}_{\omega_1}$, it is always possible to express the models as unions of smaller submodels which again satisfy $(\exists)^{\aleph_1}_{\omega_1}$. If these smaller submodels are of size $\aleph_1$, they are actually pairwise isomorphic. We consider now a condition which states that such smaller submodels can be chosen to be mutually disjoint. The models $\mathcal{A}$ and $\mathcal{B}$ are said to satisfy the *decomposition property*, or $(D)_\kappa$ for short, if

(1) $|A| = |B| = \kappa$.
(2) $A = \bigcup_{\alpha<\kappa} A_\alpha,\ B = \bigcup_{\alpha<\kappa} B_\alpha$.
(3) $|A_\alpha| < \kappa,\ |B_\alpha| < \kappa$ for $\alpha < \kappa$.
(4) $A_\alpha \cap A_\beta = B_\alpha \cap B_\beta = \emptyset$ for $\alpha < \beta < \kappa$.
(5) $(\sigma)_{\aleph_1}$ holds for $\mathcal{A}_\alpha$ and $\mathcal{B}_\alpha$ for $\alpha < \kappa$,

A simple example of the failure of $(D)_\kappa$ is a pair $(\mathcal{A}, \mathcal{B})$ where some definable subset has cardinality $< \kappa$ in $\mathcal{A}$ but cardinality κ in $\mathcal{B}$.

Theorem 6. *If $\mathcal{A}$ and $\mathcal{B}$ have cardinality $\le (2^{\aleph_0})^+$ and satisfy $(\exists)^{\aleph_1}_{\omega_1}$, then they satisfy $(\sigma)_{\aleph_1}$ or $(D)_{(2^{\aleph_0})^+}$.*

Proof. Let P be a σ-closed poset forcing that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic and let $\Vdash_P \tilde{f} : \tilde{\mathcal{A}} \cong \tilde{\mathcal{B}}$. For simplicity, assume $A, B \subseteq \mu$, where $\mu = (2^{\aleph_0})^+$.

Case 1. For every p one of the following holds:

(1.1) There is an $\alpha \in A$ such that for arbitrarily large $\beta < \mu$ we have $\beta \in B$ and for some $q \le p,\ q \Vdash \tilde{f}(\tilde{\alpha}) = \tilde{\beta}$.
(1.2) There is an $\alpha \in B$ such that for arbitrarily large $\beta < \mu$ we have $\beta \in A$ and for some $q \le p,\ q \Vdash \tilde{f}(\check{\beta}) = \check{\alpha}$.

In particular, every $p \in P$ has a continuum of incompatible extensions all deciding mutually contradictory things about $\tilde{f}$. These extensions can then be further extended to $p_\alpha,\ \alpha < 2^{\aleph_0}$, such that each p_α decides $\tilde{f}(\gamma_\alpha)$ and $\tilde{f}^{-1}(\gamma_\alpha)$ for some preassigned ordinals γ_α. On the other hand, every $p \in P$ has μ incomparable extensions p_ξ each deciding a value $\tilde{f}(\alpha_\xi) = \beta_\xi$ for some fixed α_ξ with varying $\beta_\xi \ge \xi$. By iterating these two ways of extending a condition, we get conditions $p_s, s \in {}^{<\omega_1}\mu$, and functions $f_s,\ s \in {}^{<\omega_1}\mu$ such that:

(G1) $p_s \Vdash \check{f}_s \subseteq \tilde{f}$
(G2) $s \le s' \Leftrightarrow p_{s'} \le p_s$
(G3) $s \le s' \Leftrightarrow f_s \subseteq f_{s'}$
(G4) If $s \in {}^{<\omega_1}\mu$ and $\alpha \in A$, then there is $\beta \in B$ and $s' \in {}^{<\omega_1}\mu$ such that $s \le s'$ and $f_s \cup \{\langle \alpha, \beta \rangle\} \subseteq f_{s'}$
(G5) If $s \in {}^{<\omega_1}\mu$ and $\beta \in B$ then there is $\alpha \in A$ and $s' \in {}^{<\omega_1}\mu$ such that $s \le s'$ and $f_s \cup \{\langle \alpha, \beta \rangle\} \subseteq f_{s'}$.

Let
$$D = \{f_s : s \in {}^{<\omega_1}\mu\}.$$
By construction, D is an $\aleph_1$-back-and-forth set which is σ-closed. So we have $(\sigma)_{\aleph_1}$.

Case 2. There is a $p \in P$ such that the following conditions both hold:

(2.1) For every $\alpha \in A$ there is $\beta_\alpha < \mu$ such that if $q \le p$ and $q \Vdash \tilde{f}(\check{\alpha}) = \check{\gamma}$, then $\gamma < \beta_\alpha$.
(2.2) For every $\alpha \in B$ there is $\delta_\alpha < \mu$ such that if $q \le p$ and $q \Vdash \tilde{f}(\check{\gamma}) = \check{\alpha}$, then $\gamma < \delta_\alpha$.

Let C be the cub of $\alpha < \mu$ closed under the functions $\xi \mapsto \beta_\xi$ and $\xi \mapsto \delta_\xi$. Let $C = \{c_\alpha : \alpha < \mu\}$ in ascending order and

$$A_\alpha = A \cap [c_\alpha, c_{\alpha+1})$$
$$B_\alpha = B \cap [c_\alpha, c_{\alpha+1}).$$

It is clear that $(\exists)^{\aleph_1}_{\omega_1}$ holds for $\mathcal{A}_\alpha$ and $\mathcal{B}_\alpha$. Since $|A_\alpha|, |B_\alpha| \le 2^{\aleph_0}$ we have by Theorem 2.2 that $(\sigma)^{\aleph_1}_{\aleph_1}$ holds for $\mathcal{A}_\alpha$ and $\mathcal{B}_\alpha$. So we have $(D)_{(2^{\aleph_0})^+}$. □

Theorem 7. *If $\mathcal{A}$ is an $\aleph_2$-like dense linear order, and $\mathcal{B}$ has cardinality $\leq \aleph_2$, then $(\exists)^2_{\omega_1}$ implies $(\sigma)_{\aleph_1}$.*

Proof. Suppose $(\exists)^2_{\omega_1}$ but not $(\sigma)_{\aleph_1}$. Then also $\mathcal{B}$ is an $\aleph_2$-like dense linear order. In the proof of Theorem 3.1 we way now choose the decompositions $\langle \mathcal{A}_\alpha : \alpha < \omega_2 \rangle$ and $\langle \mathcal{B}_\alpha : \alpha < \omega_2 \rangle$ so that each $\mathcal{A}_\alpha$ is an interval of $\mathcal{A}$ and each $\mathcal{B}_\alpha$ an interval of $\mathcal{B}$. Since the intervals are of cardinality $\leq \aleph_1$ and satisfy $(\exists)^2_{\omega_1}$, they are isomorphic. Hence $\mathcal{A} \cong \mathcal{B}$ and $(\sigma)_{\aleph_1}$ after all. □

6. The Banach-Mazur game

The game $\mathcal{G}_\delta(P)$ is defined like $\mathcal{G}(P)$ except that there are δ moves. Nonempty moves first at limits and he has to play an extension of all previous moves. Nonempty wins $\mathcal{G}_\delta(P)$ if he can play all δ moves without breaking the rules. So $\mathcal{G}(P)$ is $\mathcal{G}_{\omega+1}(P)$. The following lemma is well-known.

Lemma 8. *If Nonempty has a winning strategy in $\mathcal{G}(P)$, he has one in $\mathcal{G}_{\omega_1}(P)$.*

Lemma 9. *If Nonempty has a winning strategy in $\mathcal{G}_{\omega_1}(\mathcal{I}')$ for some $\aleph_1$-back-and-forth set $\mathcal{I}' \subseteq \mathcal{I}^*_{\aleph_1}$, then $(\exists)^{\aleph_1}_{\omega_1}$ holds.*

Proof. We describe a winning strategy of $\exists$ in $EF^{\aleph_1}_{\omega_1}$. During the game $EF^{\aleph_1}_{\omega_1}$ player $\exists$ maintains a sequence $\langle p_\alpha : \alpha \leq 2\nu \rangle$ in $\mathcal{I}'$ such that conditions (1) – (3) below are satisfied. Let $\langle \tau_\alpha : \alpha < \omega_1 \rangle$ be a winning strategy of Nonempty in $\mathcal{G}_{\omega_1}(\mathcal{I}')$.

(1) $\alpha < \beta < 2\nu$ implies $p_\alpha \leq p_\beta$
(2) $p_{2\mu} = \tau_{2\mu}(\langle p_\xi : \xi < 2\mu \rangle)$, for all $\mu \leq \nu$
(3) If $\langle a_\xi, b_\xi \rangle$, $\xi \leq 2\nu$, is the game so far, $p_\xi(a_\xi(i)) = b_\xi(i)$ for $i < \omega$.

Suppose we are at some stage ν and $\forall$ plays, say $x \in A^\omega$. Since $\mathcal{I}'$ is an $\aleph_1$-back-and-forth set, there is $p_{2\nu+1} \supseteq p_{2\nu}$ such that $\mathrm{ran}(x) \subseteq \mathrm{dom}(p_{2\nu+1})$. Let $p_{2\nu+2} = \tau_{2\nu+2}(\langle p_\xi : \xi \leq 2\nu + 1 \rangle)$. Now $\exists$ plays $y = \langle p_{2\nu+2}(x(i)) : i < \omega \rangle$ in $EF^{\aleph_1}_{\omega_1}$. □

It is known that the following conditions are equivalent: [3]

(1) Empty does not have a winning strategy in $\mathcal{G}(P)$.
(2) P is *Baire* i.e. if $D_n \subseteq P$ are dense and open for $n < \omega$, then so is $\bigcap_{n<\omega} D_n$.

It is also known that $\mathcal{G}(P)$ can be nondetermined [3]. What is not known is whether the following conditions are equivalent:

(3) Nonempty has a winning strategy in $\mathcal{G}(P)$
(4) P contains a σ-closed dense subset.

Jech and Shelah [4] prove that it is consistent that (3) does not imply (4). We can settle the equivalence of (3) and (4) for the poset $\mathcal{I}^*_{\aleph_1}$:

Theorem 10. *Suppose A and B are sets with $card(A) < \aleph_\omega$ and $\mathbb{P}$ is a poset of countable partial functions $A \to B$ ordered by $\subseteq$. Assume that $\mathbb{P}$ is closed under restrictions and has the following extension property: for every $p \in \mathbb{P}$ and every countable $X \subset A$ there is $q \in \mathbb{P}$ extending p such that $X \subseteq dom(q)$. Then the following conditions are equivalent:*

(i) Nonempty has a winning strategy in $\mathcal{G}(\mathbb{P})$
(ii) $\mathbb{P}$ contains a σ-closed dense subset.

Proof. Suppose $|A| = \aleph_n$. Let S be cofinal in $P^{<\omega_1}(\omega_n)$ with $card(S) = \aleph_n$. Let $S = \{E_\alpha : \alpha < \omega_n\}$. Let $\pi(E_0) = \sup(E_0) + 1$ and

$$\pi(E_\alpha) = \sup\{\{\pi(E_\beta) : \beta < \alpha\} \cup E_\alpha\} + 1$$
$$E'_\alpha = E_\alpha \cup \{\pi(E_\alpha)\}$$
$$S' = \{E'_\alpha : \alpha < \omega_n\}.$$

Now S' is a "bursting family" $\subseteq P^{<\omega_1}(\omega_n)$ i.e. cofinal, of cardinality $\aleph_n$ and $\{E \in S' : E \subseteq x\}$ is countable for all $x \in P^{<\omega_1}(\omega_n)$. Moreover we have a function π such that π is one-one and $\pi(x) \in x$ for all $x \in S'$. We may assume S' is closed under finite unions. We use this family to prove $(i) \to (ii)$.

A mapping $p \in \mathbb{P}$ is called *good* if $\mathrm{dom}(p) \in S'$. Suppose $\tau = \langle \tau_n : n < \omega\rangle$ is a winning strategy of Nonempty in $\mathcal{G}(\mathbb{P})$. A sequence $s = \langle p_0, \ldots, p_{2n+1}\rangle$ is a *partial τ-play* if for all $i \in \{0, \ldots, n\}$, $p_{2i} \in \mathbb{P}$, $p_{2i+1} = \tau(\langle p_0, \ldots, p_{2i}\rangle)$, and $p_{2i+2} \supseteq p_{2i+1}$ if $i < n$. A partial τ-play $\langle p_0, \ldots, p_{2n+1}\rangle$ is *good* if the mappings p_{2i}, $0 \le i \le n$, are good. Let the set D consist of $f \in \mathbb{P}$ such that if $s = \langle p_0, \ldots, p_{2n+1}\rangle$ is a good partial τ-play and $q \in \mathbb{P}$ is good with

$$p_{2n+1} \subseteq q \subsetneq f,$$

then s can be extended to a good partial τ-play $\langle p_0, \ldots, p_{2n+1}, p_{2n+2}, p_{2n+3}\rangle$ so that $q \subseteq p_{2n+3} \subseteq f$.

Claim 1. D is dense in $\mathbb{P}$.

Suppose $p_0 \in \mathbb{P}$. Let $\mathbb{P}'$ be the poset of good $p \in \mathbb{P}$ such that $p_0 \subseteq p$. Fix a sufficiently large regular cardinal λ and let $N \prec H(\lambda)$ be countable so that $p_0, S, \mathbb{P}' \in N$. Let G be $\mathbb{P}'$-generic over N and $f = \cup G$. If $g \in N$ is good and $g \subseteq f$, then there is $f' \in G$ so that $g \subseteq f'$, for $\{f' \in \mathbb{P}' : \mathrm{dom}(g) \subseteq \mathrm{dom}(f')\}$ is dense in $\mathbb{P}'$. To prove that $f \in D$, suppose $g_0 \subseteq \ldots \subseteq g_{2n+1}$ is a good partial τ-play such that $g_{2n+1} \subseteq f$ and $g_{2n+1} \subseteq g$ is good with $g \subseteq f$. Let $f' \in G$ with $g \subseteq f'$. Any $h \in \mathbb{P}'$ with $f' \subseteq h$ can be extended to h' such that $g_0 \subseteq \ldots \subseteq g_{2n+1} \subseteq h \subseteq h'$ is a good partial τ-play. Hence there are $f'' \in G$ and h such that we have a good partial τ-play $g_0 \subseteq \ldots \subseteq g_{2n+1} \subseteq h \subseteq f''$. Thus $f \in D$.

Claim 2. D is σ-closed.

Suppose $f_0 \subseteq \ldots \subseteq f_n \subseteq \ldots$ are in D. Let $\mathrm{dom}(f_n) = \bigcup_{i<\omega} E^n_i$ where $E^n_i \in S$. let

$$E^*_n = \bigcup_{i=0}^{n} \bigcup_{j=0}^{n} E^i_j.$$

Since $f_1 \in D$, there is a good partial τ-play $\langle g_0, g_1\rangle$ such that $f_0 \restriction E^*_0 \subseteq g_1 \subseteq f_1$. Since $f_2 \in D$, there is a good partial τ-play $\langle g_0, g_1, g_2, g_3\rangle$ with $f_1 \restriction E^*_1 \subseteq g_3 \subseteq f_2$, and so on. We get a τ-play $\langle g_0, g_1, \ldots\rangle$ such that

$$f_n \restriction E^*_n \subseteq g_{2n+1}$$

Since τ is a winning strategy, $\bigcup_n g_n \in \mathbb{P}'$. By Claim 1 there is $g \supseteq \bigcup_n g_n$ with $g \in D$. $\square$

Corollary 11. *If one of the models $\mathcal{A}$, $\mathcal{B}$ is of size $< \aleph_\omega$, then Nonempty has winning strategy in $G(\mathcal{I}^*_{\aleph_1})$ if and only if $\mathcal{I}^*_{\aleph_1}$ contains a σ-closed dense subset.*

Corollary 12. *If Nonempty has a winning strategy in $\mathcal{G}(\mathcal{I}^*_{\aleph_1})$, then $(\sigma)_{\aleph_2}$ holds.* $\square$

However, it is by no means the case that if $(\sigma)_{\aleph_2}$ holds (or even $\mathcal{A} \cong \mathcal{B}$), then Nonempty has a winning strategy in $\mathcal{G}(\mathcal{I}^*_{\aleph_1})$.

7. Uncountable partial isomorphisms

In this section we study models of cardinality κ^+. It turns out that for $\kappa > \omega$ some new ideas can be used if we consider $\mathcal{I}_{\kappa^+}$ rather than $\mathcal{I}_{\aleph_1}$. Likewise, the game $\mathrm{EF}^{\kappa}_{\omega_1}$ is in some sense easier to deal with than $\mathrm{EF}^{\aleph_1}_{\omega_1}$, when models of size κ^+ are considered.

Indeed, we can now prove $(\exists)^{\aleph_2}_{\omega_1} \leftrightarrow (\sigma)_{\aleph_2}$:

Theorem 13. *The following conditions are equivalent for models $\mathcal{A}$ and $\mathcal{B}$ of cardinality κ^+:*

(i) Player $\exists$ has a winning strategy in the game $EF^{\kappa^+}_{\omega_1}$ on $\mathcal{A}$ and $\mathcal{B}$.
(ii) There is a set $D \subseteq \mathcal{I}^*_{\kappa^+}$ which has the κ^+-back-and-forth property and is σ-closed.

Proof. Suppose $\exists$ has a winning strategy τ in $\mathrm{EF}^{\kappa^+}_{\omega_1}$ on $\mathcal{A}$ and $\mathcal{B}$. We assume that $A = B = \kappa^+$. A *partial τ-play* is a sequence

$$s = \langle \langle x_\alpha, y_\alpha \rangle : \alpha < \delta \rangle$$

of moves x_α of $\forall$ and responses y_α of $\exists$ in $\mathrm{EF}^{\kappa^+}_{\omega_1}$, $\exists$ playing τ. Each partial τ-play s determines a partial isomorphism $p(s)$ of cardinality $\leq \kappa$. Let T be the tree of all partial τ-plays s such that the domain and range of $p(s)$ are equal and are an initial segment of κ^+. Let $s \leq_T s'$ if the partial τ-play s' is obtained by continuing the partial τ-play s. Let P be the set of all $p(s)$, where $s \in T$ is a partial τ-play. Then P is a tree under inclusion. If $s \in T$, let

$$P_s = \{p(s') : s' \in T,\ s \leq_T s'\}$$

Note that P_s may be a proper subset of $\{p \in P : p(s) \subseteq p\}$.

Case 1. There is an $s \in T$ such that if $A \subseteq P_s$ is an antichain in P_s, then $\sup\{\mathrm{dom}(p) : p \in A\} < \kappa^+$. This means that P_s is a κ^+-Souslin tree. We construct a σ-closed dense back-and-forth set $D \subseteq \mathcal{I}^*_{\kappa^+}$. Let us consider the following *persistency game* on P_s:

$$\begin{array}{c|cccccc} \mathrm{I} & \alpha_0 & & \alpha_1 & & \ldots & \\ \hline \mathrm{II} & & p_0 & & p_1 & & \ldots \end{array}$$

There are two players, I and II, and ω moves. Player I starts by choosing $\alpha_0 < \kappa^+$. Then player II chooses $p_0 \in P_s$ with $\mathrm{dom}(p_0) \geq \alpha_0$. Then I chooses $\alpha_1 < \kappa^+$ and

II chooses $p_1 \in P_s$ with $\mathrm{dom}(P_1) \geq \alpha_1$ and $p_0 \leq p_1$, and so on. Player II wins if he can play ω moves $p_n, n < \omega$ so that eventually $\cup_n p_n \subseteq p_\omega$ for some $p_\omega \in P_s$. Now it is clear that II has a winning strategy in this persistency game. All he has to do is to continue the partial τ-play s in an obvious way. Let P'_s be the downward closure in P_s of the set of first moves of II against all possible first moves of I.

Let us then consider the game $\mathcal{G}_\infty(P'_s, a)$ from [6]: This game is like $\mathcal{G}(P'_s)$ except that Empty plays antichains $A_0, A_1, \ldots$ and Nonempty plays $a_n \in A_n$. The game starts with Empty choosing $a \in P'_s$ and maximal antichain A_0 of extensions of a. Later, Empty has to choose A_{n+1} so that it is a maximal antichain of extensions of a_n. Again, Nonempty wins if he can play all of his ω moves and then have $a_\omega \in P_s$ with $\bigcup_n a_n \subseteq a_\omega$. Nonempty has the following winning strategy in $\mathcal{G}_\infty(P'_s, a)$. Suppose Empty plays a maximal antichain A_0 in $\{p \in P'_s : a \subseteq p\}$. By assumption, there is $\alpha_0 < \kappa^+$ so that $\sup(\mathrm{dom}(p)) < \alpha_0$ for $p \in A_0$. Now II uses his winning strategy in the persistency game to get $p_0 \in P_s$ of height α_0. Since $\Sigma A_0 = a$, there is $a_0 \in A_0$ such that $a_0 \subseteq p_0$. This is the first move of Nonempty. It is clear that he can go on like this for ω moves and win.

It follows from a result in [6] that Nonempty wins $\mathcal{G}(P'_s)$. Since P'_s is a tree, by another result from [6] it has a σ-closed dense subset D. Now in fact D is a σ-closed back-and-forth subset of $\mathcal{I}^*_{\kappa^+}$.

Case 2. For every $s \in T$, P_s has an antichain A_s such that $\sup\{\mathrm{dom}(p) : p \in A_s\} = \kappa^+$. We define a mapping $\pi : {}^{<\omega_1}\kappa^+ \to T$ such that

(1) $\pi(f \restriction \alpha) <_T \pi(f)$ if $f \in {}^{<\omega_1}\kappa^+$ and $\alpha \in \mathrm{dom}(f)$
(2) $\{p(\pi(f \cup \{\langle \beta, \xi\rangle\})) : \xi < \kappa^+\}$ is an antichain in P if $f \in {}^{<\omega_1}\kappa^+$ and $\beta = \mathrm{dom}(f)$.

Suppose $\pi(g)$ is defined for $\mathrm{dom}(g) < \alpha$, where $\alpha < \omega_1$, and $\mathrm{dom}(f) = \alpha$. If $\alpha = \cup\alpha$, the sequence

$$\langle \pi(f \restriction \beta) : \beta < \alpha\rangle$$

is an ascending chain in T and has therefore a limit which we denote by $\pi(f)$. If $\alpha = \beta + 1$, we let $A_{\pi(f\restriction\beta)} = \{a_\xi : \xi < \kappa^+\}$ and let

$$\pi(f \cup \{\langle \beta, \xi\rangle\}) = a_\xi$$

for $\xi < \kappa^+$. This ends the construction. Now let

$$D = \{p(\pi(f)) : f \in {}^{<\omega_1}\kappa^+\}.$$

We prove that D is an κ^+-back-and-forth set. Let $p(\pi(f)) \in D$ and, say $x \in {}^\kappa A$. Let $\mathrm{dom}(f) = \beta$. We can choose $\xi < \kappa^+$ so that

$$\sup\{x(i) : i < \kappa\} < \mathrm{dom}(\pi(f \cup \langle \beta, \xi\rangle)).$$

Thus $\mathrm{rng}(x) \subseteq \mathrm{dom}(\pi(f \cup \langle \beta, \xi\rangle))$ and $\pi(f \cup \{\langle \beta, \xi\rangle\}) \in D$.

Finally, we show that D is σ-closed. Suppose $p_0 \subsetneq p_1 \subsetneq \ldots$ in D. Let $p_n = p(\pi(f_n))$. By construction, there is $f_\omega \in {}^{<\omega_1}\kappa^+$ such that $f_n = f_\omega \restriction \mathrm{dom}(f_n)$. Now $p_n \subseteq p(\pi(f_\omega))$ for all $n < \omega$. □

Corollary 14. $(\exists)^{\aleph_1}_{\kappa\cdot\kappa} \to (\sigma)_{\kappa^+}$ *on models of cardinality* κ^+.

Corollary 15. $(\sigma)_{\kappa^+}$ *is transitive on models of cardinality* κ^+.

Open problems:

1. Is $(\sigma)_{\aleph_1}$ transitive?
2. Is $(\sigma)_{\kappa^+}$ transitive on all models?
3. $(\exists)^{\aleph_1}_{\omega_1} \leftrightarrow (\sigma)_{\aleph_1}$?

References

[1] Dickmann, M.A.: Large infinitary languages. Model theory. Studies in Logic and the Foundations of Mathematics, Vol. 83. North-Holland Publishing Co., Amsterdam-Oxford; American Elsevier Publishing Co., Inc., New York, xv+464 pp 1975

[2] Foreman, M.: Games played on Boolean algebras. J. Symbolic Logic **48**, 714–723 (1983)

[3] Jech, T.: A game theoretic property of Boolean algebras. Logic Colloquium 77 (Proc. Conf., Wroclaw, 1977), pp. 135–144, Stud. Logic Foundations Math., 96, North-Holland, Amsterdam-New York, 1978

[4] Jech, T., Shelah, S.: On countably closed complete Boolean algebras. J. Symbolic Logic, 1996

[5] Kueker, David W.: Back-and-forth arguments and infinitary logics. Infinitary logic: in memoriam Carol Karp, pp. 17–71. Lecture Notes in Math., Vol. 492, Springer, Berlin, 1975

[6] Veličković, B.: Playful Boolean algebras. Trans. Amer. Math. Soc. **296**, 727–740 (1986)