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# Unary Quantifiers on Finite Models

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**Abstract.** In this paper (except in Section 5) all quantifiers are assumed to be so called simple unary quantifiers, and all models are assumed to be finite. We give a necessary and sufficient condition for a quantifier to be definable in terms of monotone quantifiers. For a monotone quantifier we give a necessary and sufficient condition for being definable in terms of a given set of bounded monotone quantifiers. Finally, we give a necessary and sufficient condition for a monotone quantifier to be definable in terms of a given monotone quantifier. Our analysis shows that the quantifier "at least one half" and its relatives behave differently than other monotone quantifiers.

**Key words:** Generalized quantifiers, finite model, Ehrenfeucht–Fraïssé games

## 1. Introduction

A *simple unary (or monadic)* quantifier, in this paper just a *quantifier*, is a class $Q$ of structures $(A, R)$, where $R \subseteq A$, which is closed under isomorphisms. This concept was introduced by Mostowski (1996). A more general concept of a quantifier was introduced by Lindström (1966) and a vast literature has emerged on the topic. We consider only quantifiers on finite models. The first to consider quantifiers on finite models seems to have been Hájek (1977), and recently most new work seems to be in the finite context. Generalized quantifiers on finite models have found applications in natural language semantics (see, e.g., Hella et al., 1997; Westerståhl, 1989) and descriptive complexity theory (see, e.g., Hella, 1996).

Here are some examples of quantifiers:

$$\exists = \{(A, R) : R \neq \emptyset\}$$

$$\text{HALF} = \{(A, R) : |R| \geq |A|/2\}$$

$$\text{EVEN} = \{(A, R) : |R| \text{ even}\}$$

$$Q = \{(A, R) \,:\, (|A| \text{ even and } R \neq \emptyset) \text{ or } (|A| \text{ odd and } |R| \geq |A|/2)\} \quad (1)$$

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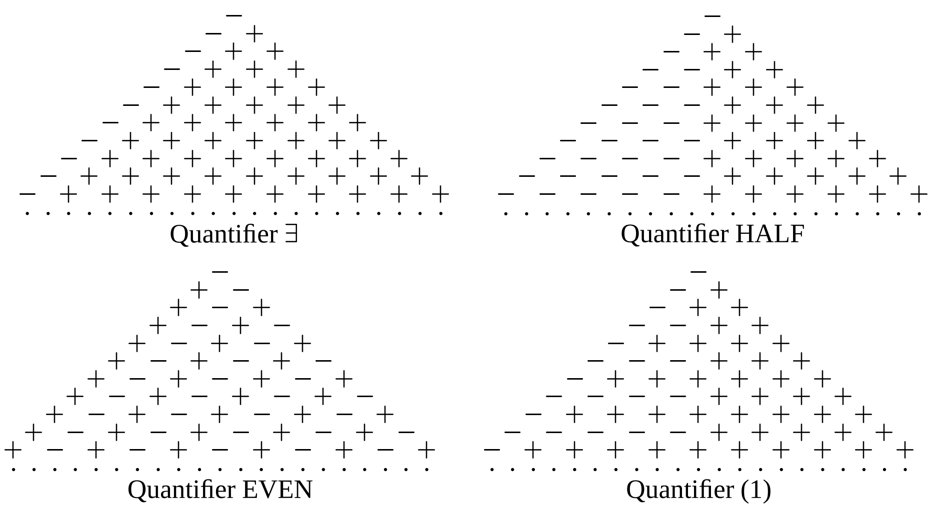


*Figure 1.* Number-triangles of quantifiers.

The *code* of a structure $\mathbf{A} = (A, R)$, where $R \subseteq A$, is the pair

$$\tau(\mathbf{A}) = (|A|, |R|).$$

The point of coding is that $\mathbf{A} \cong \mathbf{B} \iff \tau(\mathbf{A}) = \tau(\mathbf{B})$. If $Q$ is a quantifier, let

$$\tau(Q) = \{\tau(\mathbf{A}) : \mathbf{A} \in Q\}.$$

Thus quantifiers correspond to subsets of $\{(n, m) : m \leq n\}$. Following van Benthem (1984) we may use this correspondence to visualize quantifiers and their properties by means of the *number triangle*:

$$\begin{array}{ccccccc} & & & (0,0) & & & \\ & & (1,0) & & (1,1) & & \\ & (2,0) & & (2,1) & & (2,2) & \\ (3,0) & & (3,1) & & (3,2) & & (3,3) \\ \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

The number-triangle of a quantifier is obtained by replacing $(n, m)$ in the number-triangle by $+$ if $(n, m) \in \tau(Q)$ and by $\Leftrightarrow$otherwise (Figure 1).

If $Q$ is a quantifier, we may define an extension of first order logic FO by adding to the syntactic rules of FO the new rule: if $\phi(x, \mathbf{y})$ is a formula, then so is Q$x\phi(x, \mathbf{y})$. The semantics is defined by

$$\mathbf{A} \models \mathrm{Q}x\phi(x, \mathbf{a}) \iff (A, \{b \in A : \mathbf{A} \models \phi(b, \mathbf{a})\}) \in Q,$$

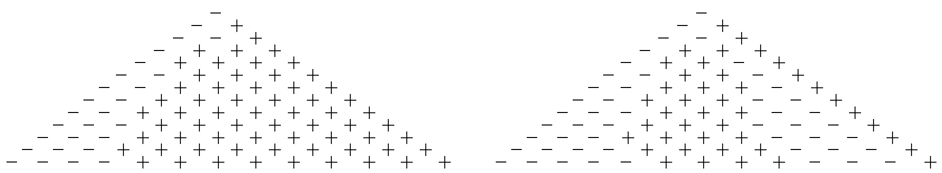

*Figure 2.* A monotone quantifier and a boundedly oscillating quantifier.

where $A$ is the universe of $\mathbf{A}$. The extension of FO by the new quantifier Q is denoted by FO$(Q)$. If $\mathbf{Q}$ is a set of quantifiers, the extension FO$(\mathbf{Q})$ of FO is defined analogously. We say that a quantifier $Q$ is *definable* in terms of the quantifiers in $\mathbf{Q}$ if $Q$ is the class of models of a sentence of FO$(\mathbf{Q})$. In such a case we also say that $Q$ is *definable in* FO$(\mathbf{Q})$. Note, that definability is transitive: if $Q$ is definable in FO$(Q')$ and $Q'$ is definable in FO$(Q'')$, then $Q$ is definable in FO$(Q'')$. If every quantifier in $\mathbf{Q}$ is definable in FO$(\mathbf{Q}')$, we say that FO$(\mathbf{Q})$ is a *sublogic of* FO$(\mathbf{Q}')$ and write FO$(\mathbf{Q}) \leq$ FO$(\mathbf{Q}')$. If FO$(\mathbf{Q}) \leq$ FO$(\mathbf{Q}')$ and FO$(\mathbf{Q}') \leq$ FO$(\mathbf{Q})$, we say that FO$(\mathbf{Q})$ and FO$(\mathbf{Q}')$ are *equivalent*, in symbols FO$(\mathbf{Q}) \equiv$ FO$(\mathbf{Q}')$.

FO$^r(Q)$ denotes the fragment of FO$(Q)$ consisting of formulas with quantifier-rank $\leq r$. The quantifier-rank of Q$x\phi(x, \mathbf{y})$ is the quantifier-rank of $\phi(x, \mathbf{y})$ plus one. It is obvious how FO$^r(\mathbf{Q})$ for a set $\mathbf{Q}$ of quantifiers is defined.

A quantifier is *monotone,* if $(A, R) \in Q$ and $R \subseteq R' \subseteq A$ imply $(A, R') \in Q$. In the number-triangle of a monotone quantifier each row consists of a homogeneous block of minuses and a homogeneous block of pluses with the minuses before the pluses (Figure 2). If $Q$ is monotone, we define

$$f_Q(n) = \begin{cases} \text{least } m \text{ such that } (n, m) \in \tau(Q), & \text{if such } m \text{ exists} \\ n + 1, & \text{otherwise.} \end{cases}$$

On the other hand, if $f : \mathbf{N} \to \mathbf{N}$ such that $f(n) \leq n + 1$ for all $n$, we denote by $Q_f$ the unique monotone quantifier $Q$ with $f_Q = f$. Monotonicity is a very reasonable assumption about a quantifier, especially if the quantifier is a formal counterpart of "largeness." On the other hand, in a database query language we may want to ask if there are an even number of elements with some property, and the quantifier EVEN is of course not monotone.

Logics of the form FO$(\mathbf{Q})$, where $\mathbf{Q}$ is a set of monotone quantifiers, play an important role in this paper. We denote the family of all such logics by **Mon**.

We now introduce a weakening of the concept of monotonicity. A point $(n, m)$ of the number-triangle is called an *oscillation point* of $Q$ if $m < n$ and

$$(n, m) \in \tau(Q) \iff (n, m + 1) \notin \tau(Q).$$

Thus there is on row $n$ at point $(n, m)$ a change from $+$ to $\Leftrightarrow$or from $\Leftrightarrow$to $+$. A quantifier is *boundedly oscillating,* if there is a uniform bound for the number of oscillation points on any row (Figure 2). In Section 2 we prove the following result:

THEOREM 8. *A quantifier is definable in terms of monotone quantifiers if and only if it is boundedly oscillating.*

Thus the family of boundedly oscillating quantifiers is the closure of the family of monotone quantifiers under definability. To prove Theorem 8 we analyze in detail definability among logics in **Mon**.

In Section 3 we study the family of bounded monotone quantifiers. A function $f : \mathbf{N} \to \mathbf{N}$ with $f(n) \leq n + 1$ for all $n$ is called (here) *bounded* if there is a $k_f \in \mathbf{N}$ such that for all $n \in \mathbf{N}$

$$f(n) \leq k_f \text{ or } f(n) \geq n \Leftrightarrow k_f.$$

A monotone quantifier $Q$ is bounded if $f_Q$ is bounded. So the only oscillation point of $Q$ on any row of the number-triangle is always within $k_{f_Q}$ of one of the sides of the triangle. An example of a non-trivial bounded quantifier is the quantifier $Q$ such that

$$f_Q(n) = \begin{cases} 1 & \text{if } n \equiv 0 \bmod 3 \\ n \Leftrightarrow 1 & \text{if } n \equiv 1 \bmod 3 \\ n + 1 & \text{if } n \equiv 2 \bmod 3. \end{cases}$$

In this paper we often talk about *colorings*. By a coloring of a set $X$ we simply mean a mapping $\mu$ defined on $X$. A *color class* of $\mu$ is then the set of elements with a fixed color. Suppose $\mu$ and $\mu'$ are colorings of $\mathbf{N}$. We say that $\mu$ *eventually refines* $\mu'$, if there is an $m$ such that elements $\geq m$ with the same $\mu$-color have also the same $\mu'$-color. This terminology comes from the idea that the color-classes form a partition of the set. A coloring which generates a finer partition than $\mu$ is said to refine $\mu$.

Suppose $\mathbf{Q}$ is a finite set of bounded monotone quantifiers. In Section 3 we show how the set $\mathbf{Q}$ gives rise to a canonical coloring $\beta_{\mathbf{Q}}$ of $\mathbf{N}$. For a single quantifier $Q$ this coloring is denoted by $\beta_Q$. We prove:

THEOREM 14. *Suppose* $\mathbf{Q}$ *is a finite set of bounded monotone quantifiers, and* $Q'$ *a monotone quantifier. Then* $Q'$ *is definable in* $\mathrm{FO}(\mathbf{Q})$ *if and only if* $Q'$ *is bounded and* $\beta_{\mathbf{Q}}$ *eventually refines* $\beta_{Q'}$.

By means of this theorem we are able to describe completely the sublogic structure of the family **Mon** below any $\mathrm{FO}(\mathbf{Q})$, $\mathbf{Q}$ a set of bounded monotone quantifiers. A similar result for a different class of quantifiers is in (Corredor, 1986).

The question, when a monotone quantifier $Q_g$ is definable in terms of another monotone quantifier $Q_f$, can be answered completely (Theorem 17). The result looks a little complicated as there are many different cases to consider, but it is simpler if we assume more than mere monotonicity. We prove in Section 4:

THEOREM 21. *Suppose $Q_f$ is a monotone quantifier such that*

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty}(n \Leftrightarrow f(n)) = \infty$$

*and*

$$\lim_{n\to\infty}(f(n) \Leftrightarrow \lfloor n/2 \rfloor) = \infty \quad \textit{or} \quad \lim_{n\to\infty}(\lfloor n/2 \rfloor \Leftrightarrow f(n)) = \infty.$$

*Then a monotone quantifier $Q_g$ is definable in* $\mathrm{FO}(Q_f)$ *if and only if $Q_g$ is first-order definable or there is a constant $a \in$ **Z** and a number $m \in$ **N** s.t.*

$$\forall n \geq m(g(n) = f(n) + a) \quad \textit{or} \quad \forall n \geq m(g(n) = n \Leftrightarrow f(n) + a).$$

In Section 5 we consider the more general case of non-simple non-monotone unary quantifiers. We give a general criterion for the definability of a unary quantifier in terms of a given set of unary quantifiers. The criterion is vastly more complicated than in previous sections, which limits its applicability. However, we use it to prove that the generalized Rescher quantifier

$$\mathrm{MORE}_f x P_1(x) P_2(x) \leftrightarrow |P_1| > f(|P_2|)$$

is definable in terms of simple unary quantifiers if and only if there is an $m \in N$ such that $f(n) \leq m$ for all $n \in N$. This extends a result of (Kolaitis and Väänänen, 1995).

## 2. Monotone Quantifiers

We develop some methods for studying definability by monotone quantifiers, and then use these to prove Theorem 8. Let $\mathbf{F} = (f_0, \ldots, f_{u-1})$ be a sequence of functions such that $f_i(n) \leq n + 1$ for $0 \leq i < u$ and $n \in \mathbf{N}$. When we study definability in FO(**F**) $=_{\mathrm{df}}$ $\mathrm{FO}(\{Q_{f_0}, \ldots, Q_{f_{u-1}}\})$, it is useful to assume that **F** satisfies some closure properties. Let the *dual* $\breve{f}$ of a function $f$ be defined by

$$\breve{f}(n) = n \Leftrightarrow f(n) + 1.$$

Then $Q_{\breve{f}}$ is the *dual* of $Q_f$ in the sense that

$$(A, R) \in Q_{\breve{f}} \iff (A, A \Leftrightarrow R) \notin Q_f.$$

That is,

$$\models \mathrm{Q}_{\breve{f}} x P(x) \leftrightarrow \neg \mathrm{Q}_f x \neg P(x).$$

Thus $Q_{\breve{f}_i}$ is definable in $\mathrm{FO}^1(\mathbf{F})$ for all $i = 0, \ldots, u \Leftrightarrow 1$. Therefore, there is no loss of generality from a definability point of view in assuming that all sequences **F** that we consider satisfy:

**(F1)** $f \in \mathbf{F} \Rightarrow \forall n \in \mathbf{N}(f(n) \leq n+1)$.

**(F2)** $f \in \mathbf{F} \Rightarrow \breve{f} \in \mathbf{F}$.

**(F3)** $f_\exists \in \mathbf{F}$, where $f_\exists(n) \equiv 1$.

We shall now define for each $\mathbf{F}$ and each $r \in \mathbf{N}$ a coloring $\mu_{r,\mathbf{F}}$ of the number-triangle. The color $\mu_{r,\mathbf{F}}(n,m)$ of the pair $(n,m)$ is defined as follows:

$$\begin{aligned}\mu_{r,\mathbf{F}}(n,m) \;=\; &\{(0,i,j) : m \geq f_i(n)+j,\ 0 \leq i < u,\ -r < j < r\} \cup \\ &\{(1,i,j) : f_i(n) \leq j,\ 0 \leq i < u,\ 0 \leq j < r\}.\end{aligned}$$

The point of $\mu_{r,\mathbf{F}}(n,m)$ is that it collects systematically the information about the functions in $\mathbf{F}$ that we really need and nothing more. The following lemma is a coherency feature that we need later.

LEMMA 1. *Suppose* $n, m_1, m_2, m_3 \in \mathbf{N}$ *are such that*

*(i)* $\mu_{r,\mathbf{F}}(n,m_1) = \mu_{r,\mathbf{F}}(n,m_3)$.
*(ii)* $m_1 \leq m_2 \leq m_3$.

*Then* $\mu_{r,\mathbf{F}}(n,m_2) = \mu_{r,\mathbf{F}}(n,m_1)$.

*Proof.* Suppose first $(0,i,j) \in \mu_{r,\mathbf{F}}(n,m_2)$. Then $m_2 \geq f_i(n)+j$. Hence $m_3 \geq f_i(n)+j$, whence $(0,i,j) \in \mu_{r,\mathbf{F}}(n,m_3)$. Now (i) implies $(0,i,j) \in \mu_{r,\mathbf{F}}(n,m_1)$. Conversely, suppose $(0,i,j) \in \mu_{r,\mathbf{F}}(n,m_1)$. Then $m_1 \geq f_i(n)+j$. Hence $(0,i,j) \in \mu_{r,\mathbf{F}}(n,m_2)$. □

Lemma 1 tells us that $\mu_{r,\mathbf{F}}$ divides the number-triangle into monochromatic areas that are intervals on every row. Thus these areas look like strips. We call them $\mu_{r,\mathbf{F}}$*-strips*. The following lemma is equally easy to prove:

LEMMA 2. *If* $r \in \mathbf{N}$ *and* $k \in \mathbf{Z}$, *then* $\mu_{r+k,\mathbf{F}}(n,m) = \mu_{r+k,\mathbf{F}}(n',m')$ *implies* $\mu_{r,\mathbf{F}}(n,m+k) = \mu_{r,\mathbf{F}}(n',m'+k)$.

Next we shall show that every $\mathrm{FO}^r(\mathbf{F})$-definable quantifier is a union of $\mu_{r,\mathbf{F}}$-strips.

LEMMA 3. *If* $\mu_{r,\mathbf{F}}(\tau(\mathbf{A})) = \mu_{r,\mathbf{F}}(\tau(\mathbf{B}))$, *then* $\mathbf{A} \equiv_{\mathrm{FO}^r(\mathbf{F})} \mathbf{B}$.

First we introduce a game and prove two auxiliary lemmas. Given two structures $\mathbf{A}$ and $\mathbf{B}$ of the same vocabulary and a set $\mathbf{F}$ of functions satisfying conditions (F1)–(F3), the $(r,\mathbf{F})$*-Ehrenfeucht–Fraïssé game* on $\mathbf{A}$ and $\mathbf{B}$ is defined as follows: The game has two players: I and II. The game starts with a move of Player I. He chooses one of the models, say $\mathbf{A}$, one of the quantifiers $Q_f$, $f \in \mathbf{F}$, and a subset $X$ of $A$. Then Player II chooses a subset $Y$ of $B$. Next Player I chooses an element

$y$ of $Y$. Finally, Player II chooses an element $x$ of $X$. This sequence of moves is repeated $r$ times. The sets $X \subseteq A$ played by I (or II) have to satisfy $X \geq f(|A|)$ and the sets $Y$ played by II (or I) have to satisfy $Y \geq f(|B|)$. Suppose the players played an element $x_i$ of $A$ and an element $y_i$ of $B$ on round $i$, $i = 1, \ldots, r$. Player II wins if the relation $\{(x_i, y_i) : i = 1, \ldots, r\}$ is a partial isomorphism between $\mathbf{A}$ and $\mathbf{B}$. Otherwise Player I wins. Games like this have been studied in detail, e.g. in (Kolaitis and Väänänen, 1995). There it is also shown that if II has a winning strategy in the modified game, where the sets chosen by Player I are invariant under automorphisms of the model that fix the elements chosen so far, then he has a winning strategy in the game itself. This fact is essential in applications.

LEMMA 4. *If* $\mu_{r,\mathbf{F}}(\tau(\mathbf{A})) = \mu_{r,\mathbf{F}}(\tau(\mathbf{B}))$, *then Player II has a winning strategy in the* $(r, \mathbf{F})$*-Ehrenfeucht–Fraïssé game on* $\mathbf{A}$ *and* $\mathbf{B}$.

*Proof.* Let $\mathbf{A} = (A, R)$ and $\mathbf{B} = (B, S)$. Suppose distinct elements $a_1, \ldots, a_t$ of $A$ and elements $b_1, \ldots, b_t$ of $B$ have been played already, and $a_i \mapsto b_i$ is a partial isomorphism between $\mathbf{A}$ and $\mathbf{B}$. Suppose player I chooses $f_k \in \mathbf{F}$ and plays a subset $X$ of, say, $A$ so that $|X| \geq f_k(|A|)$. We may assume that $X$ is invariant under automorphisms that fix $a_1, \ldots, a_t$.

*Case 1.* $X \subseteq \{a_1, \ldots, a_t\}$. We let Player II choose $Y = \{b_i : a_i \in X, 1 \leq i \leq t\}$. Let $s = f_k(|A|)$. Then $s \leq |X|$, so $(1, k, |X|) \in \mu_{r,\mathbf{F}}(\tau(\mathbf{A}))$. Since $\mu_{r,\mathbf{F}}(\tau(\mathbf{A})) = \mu_{r,\mathbf{F}}(\tau(\mathbf{B})), (1, k, |Y|) \in \mu_{r,\mathbf{F}}(\tau(\mathbf{B}))$, whence $|Y| \geq f_k(|B|)$. Next Player I chooses some $y \in Y$, say $y = b_j$. The strategy of Player II is obviously to play $x = a_j$. Then $x \in X$ by construction and trivially $a_i \mapsto b_i$ remains a partial isomorphism.

*Case 2.* $A \Leftrightarrow X \subseteq \{a_1, \ldots, a_t\}$. We let Player II choose $Y = B \Leftrightarrow \{b_i : a_i \notin X\}$. Let $s = f_k(|A|)$; so $|X| \geq s$. Let $f_l = \check{f}_k$. Then $|A \Leftrightarrow X| \not\geq f_l(|A|)$, whence $(1, l, |A \Leftrightarrow X|) \notin \mu_{r,\mathbf{F}}(\tau(\mathbf{A}))$. By assumption, $(1, l, |A \Leftrightarrow X|) \notin \mu_{r,\mathbf{F}}(\tau(\mathbf{B}))$, whence $|B \Leftrightarrow Y| = |A \Leftrightarrow X| \not\geq f_l(|B|)$ and $|Y| \geq f_k(|B|)$ follows.

*Case 3.* $X$ meets $R \Leftrightarrow \{a_1, \ldots, a_t\}$ and $A \Leftrightarrow X$ meets $A \Leftrightarrow (R \cup \{a_1, \ldots, a_t\})$. By an automorphism argument, $X \Leftrightarrow \{a_1, \ldots, a_t\} = R \Leftrightarrow \{a_1, \ldots, a_t\}$. Player II chooses

$$Y = (S \Leftrightarrow \{b_1, \ldots, b_t\}) \cup \{b_i : a_i \in X, 1 \leq i \leq t\}.$$

Let $s \in [\Leftrightarrow t, t]$ so that $|X| = |R| + s$. Thus $|R| + s \geq f_k(|A|)$, whence $(0, k, \Leftrightarrow s) \in \mu_{r,\mathbf{F}}(\tau(\mathbf{A}))$. By assumption, $(0, k, \Leftrightarrow s) \in \mu_{r,\mathbf{F}}(\tau(\mathbf{B}))$, whence $|S| + s \geq f_k(|B|)$ and, therefore, $|Y| \geq f_k(|B|)$. Suppose then Player I picks $y \in Y$. If $y \in S \Leftrightarrow \{b_1, \ldots, b_t\}$, we let Player II choose some $x$ from $R \Leftrightarrow \{a_1, \ldots, a_t\}$. (This set is non-empty, by assumption.) Now $x \in X$ and clearly

$$\{(a_1, b_1), \ldots, (a_t, b_t), (x, y)\}$$

is a partial isomorphism $\mathbf{A} \to \mathbf{B}$. If, on the other hand, $y = b_i$ with $a_i \in X$, then Player II can simply play $x = a_i$.

*Case 4.* $X$ meets $A \Leftrightarrow (R \cup \{a_1, \ldots, a_t\})$ and $A \Leftrightarrow X$ meets $R \Leftrightarrow \{a_1, \ldots, a_t\}$. This case is symmetrical to Case 3. □

LEMMA 5. *Player II has a winning strategy in the* $(r, \mathbf{F})$*-Ehrenfeucht–Fraïssé game on* $\mathbf{A}$ *and* $\mathbf{B}$ *if and only if* $\mathbf{A} \equiv_{\mathrm{FO}^r(\mathbf{F})} \mathbf{B}$.

*Proof.* This is essentially proved in (Kolaitis and Väänänen, 1995). The proof of Lemma 31 below gives an idea of how to prove the direction of this lemma that we actually use. □

Lemma 3 now follows from Lemmas 4 and 5.

PROPOSITION 6. *The following conditions are equivalent for any models* $\mathbf{A} = (A, R), R \subseteq A$, *and* $\mathbf{B} = (B, S), S \subseteq B$, *for any* $r \in \mathbf{N}$ *and for any* $\mathbf{F}$ *satisfying (F1)–(F3):*

*(i)* $\mathbf{A} \equiv_{\mathrm{FO}^r(\mathbf{F})} \mathbf{B}$.
*(ii)* $\mu_{r,\mathbf{F}}(\tau(\mathbf{A})) = \mu_{r,\mathbf{F}}(\tau(\mathbf{B}))$.

*Proof.* (ii) $\rightarrow$ (i) by Lemma 3. For the converse, let $(0, i, j) \in \mu_{r,\mathbf{F}}(\tau(\mathbf{A}))$. Thus $|R| \geq f_i(|A|) + j$.

*Case 1.* $j < 0$: The model $\mathbf{A}$ satisfies the following sentence of $\mathrm{FO}^r(\mathbf{F})$ (we use $P$ as a name for the relations $R$ and $S$).

$$\Phi^-_{i,j} : \forall x_1 \ldots \forall x_{|j|} \left( \left( \bigwedge_{1 \leq s < s' \leq |j|} \neg x_s = x_{s'} \wedge \bigwedge_{1 \leq s \leq |j|} \neg P(x_s) \right) \rightarrow \mathrm{Q}_{f_i} x \left( P(x) \vee \bigvee_{1 \leq s \leq |j|} x = x_s \right) \right).$$

By (i), $\mathbf{B}$ also satisfies $\Phi^-_{i,j}$, and, therefore, $|S| \geq f_i(|B|) + j$, i.e. $(0, i, j) \in \mu_{r,\mathbf{F}}(\tau(\mathbf{B}))$.

*Case 2.* $j \geq 0$ : The model $\mathbf{A}$ satisfies the following sentence of $\mathrm{FO}^r(\mathbf{F})$:

$$\Phi^+_{i,j} : \forall x_1 \ldots \forall x_j \left( \left( \bigwedge_{1 \leq s < s' \leq j} \neg x_s = x_{s'} \wedge \bigwedge_{1 \leq s \leq j} P(x_s) \right) \rightarrow \mathrm{Q}_{f_i} x \left( P(x) \wedge \bigwedge_{1 \leq s \leq j} \neg x = x_s \right) \right).$$

By (i), $\mathbf{B}$ satisfies $\Phi^+_{i,j}$, and $(0, i, j) \in \mu_{r,\mathbf{F}}(\tau(\mathbf{B}))$ follows.

Suppose then $(1, i, j) \in \mu_{r,\mathbf{F}}(\tau(\mathbf{A}))$. Thus $f_i(|A|) \leq j$. The model $\mathbf{A}$ satisfies the following sentence of $\mathrm{FO}^r(\mathbf{F})$:

$$\Psi_{i,j} : \forall x_1 \ldots \forall x_j \left( \left( \bigwedge_{1 \leq s < s' \leq j} \neg x_s = x_{s'} \right) \to \mathrm{Q}_{f_i} x \bigvee_{1 \leq s \leq j} x = x_s \right).$$

By (i), also $\mathbf{B}$ satisfies this sentence and therefore $f_i(|B|) \leq j$, whence $(1, i, j) \in \mu_{r,\mathbf{F}}(\tau(\mathbf{B}))$. We have proved $\mu_{r,\mathbf{F}}(\tau(\mathbf{A})) \subseteq \mu_{r,\mathbf{F}}(\tau(\mathbf{B}))$. By symmetry, $\mu_{r,\mathbf{F}}(\tau(\mathbf{A})) = \mu_{r,\mathbf{F}}(\tau(\mathbf{B}))$. □

COROLLARY 7. *Suppose $Q$ is a quantifier and $\mathbf{F}$ satisfies (F1)–(F3). Then the following conditions are equivalent:*

*(i) $Q$ is $\mathrm{FO}^r(\mathbf{F})$-definable.*
*(ii) $Q$ is closed under the equivalence relation*

$$\mathbf{A} \sim \mathbf{B} \iff \mu_{r,\mathbf{F}}(\tau(\mathbf{A})) = \mu_{r,\mathbf{F}}(\tau(\mathbf{B})). \tag{2}$$

*(iii) $Q$ is definable by a Boolean combination of sentences of $\mathrm{FO}^r(\mathbf{F})$ of the form*

$$\forall \mathbf{x}(\psi(\mathbf{x}) \to \mathrm{Q}_i y \theta(y, \mathbf{x})),$$

*where $\psi(\mathbf{x})$ and $\theta(y, \mathbf{x})$ are quantifier-free.*

*Proof.* (i) → (ii) follows from Proposition 6. (ii) → (iii): Let $C_{u,r}$ consist of $(0, i, j)$ for $0 \leq i < u$, ⇔$r < j < r$, and of $(1, i, j)$, for $0 \leq i < u, 0 \leq j < r$. Let $\phi_D, D \subseteq C_{u,r}$, be the conjunction of

$\Phi^+_{i,j}$, for $(0, i, j) \in D, j \geq 0$
$\Phi^-_{i,j}$, for $(0, i, j) \in D, j < 0$
$\Psi_{i,j}$, for $(1, i, j) \in D$
$\neg\Phi^+_{i,j}$, for $(0, i, j) \in C_{u,r}$ ⇔$D, j \geq 0$
$\neg\Phi^-_{i,j}$, for $(0, i, j) \in C_{u,r}$ ⇔$D, j < 0$
$\neg\Psi_{i,j}$, for $(1, i, j) \in C_{u,r}$ ⇔$D$.

Notice that for all $\mathbf{A}$ and $D \subseteq C_{u,r}$:

$$\mathbf{A} \models \phi_D \iff \mu_{r,\mathbf{F}}(\tau(\mathbf{A})) = D. \tag{3}$$

Let $\phi$ be the disjunction of all sentences $\phi_D, D \subseteq C_{u,r}$, for which there is some $\mathbf{A} \in Q$ with $\mu_{r,\mathbf{F}}(\tau(\mathbf{A})) = D$. Now $\phi$ is of the form required by (iii), so it suffices to show that $\phi$ defines $Q$. Suppose $\mathbf{A} \in Q$. By (3), $\mathbf{A} \models \phi_{\mu_{r,\mathbf{F}}(\tau(\mathbf{A}))}$, whence $\mathbf{A} \models \phi$. On the other hand, if $\mathbf{A} \models \phi$, then $\mathbf{A} \models \phi_{\mu_{r,\mathbf{F}}(\tau(\mathbf{B}))}$ for some $\mathbf{B} \in Q$, whence $\mu_{r,\mathbf{F}}(\tau(\mathbf{A})) = \mu_{r,\mathbf{F}}(\tau(\mathbf{B}))$ by (3), and finally $\mathbf{A} \in Q$ by (ii).

(iii) → (i): This implication is trivial. □

THEOREM 8. *A quantifier is definable in terms of monotone quantifiers if and only if it is boundedly oscillating.*

*Proof.* Suppose $Q$ is FO$^r$($\mathbf{F}$)-definable, where $\mathbf{F}$ satisfies (w.l.o.g.) (F1)–(F3). By Lemma 1, $\mu_{r,\mathbf{F}}$ divides rows of the number-triangle of $Q$ into monochromatic intervals. On each row the set of pluses is, by Corollary 7, a union of these intervals. Therefore, $Q$ can oscillate only at the end-points of these intervals. There are at most $2^{|C_{u,r}|}$ different sets $\mu_{r,\mathbf{F}}(\tau(\mathbf{A}))$ and therefore at most $2^{|C_{u,r}|}+1$ oscillation points on any row. Hence $Q$ is boundedly oscillating.

For the other direction, suppose $k$ is the maximum number of oscillation points on any row of the number-triangle of $Q$. Let $f_1, \dots, f_{k+2}$ be functions so that

$$
\begin{aligned}
& m < f_1(n) \Rightarrow (n,m) \in \tau(Q) \\
& f_1(n) \le m < f_2(n) \Rightarrow (n,m) \notin \tau(Q) \\
& f_2(n) \le m < f_3(n) \Rightarrow (n,m) \in \tau(Q) \\
& \dots \\
& f_{k+1}(n) \le m < f_{k+2}(n) \Rightarrow \begin{cases} (n,m) \notin \tau(Q) & \text{if } k \text{ even} \\ (n,m) \in \tau(Q) & \text{if } k \text{ odd} \end{cases} \\
& f_{k+2}(n) = n+1.
\end{aligned}
$$

The idea is that the functions $f_1, \dots, f_{k+1}$ pick the oscillation points on each row so that $f_1(n)$ picks the first minus. If there are less than $k$ oscillation points on a row, then $f_i(n)$ reaches its maximum $n+1$ already for some $i < k+2$. It follows that

$$(n,m) \in \tau(Q) \iff m < f_1(n) \text{ or}$$

$$f_i(n) \le m < f_{i+1}(n) \text{ for some even } i \le k+1.$$

From this it follows immediately, that $Q$ is closed under the equivalence relation

$$\mathbf{A} \sim \mathbf{B} \iff \mu_{1,\mathbf{F}}(\tau(\mathbf{A})) = \mu_{1,\mathbf{F}}(\tau(\mathbf{B})),$$

where $\mathbf{F}$ is the closure of $(f_1, \dots, f_{k+1}, f_\exists)$ under duals. Now $Q$ is FO($\mathbf{F}$)-definable by Corollary 7. □

*Example 9.* For a rational $a$ let $\lfloor a \rfloor$ be the largest integer $\le a$, and $\lceil a \rceil$ the least integer $\ge a$. Let $f(n) = \lfloor n/2 \rfloor$ and $g(n) = \lceil n/2 \rceil$. Then $\breve{f}(n) = g(n) + 1$ and $\breve{g}(n) = f(n) + 1$, so the quantifiers $Q_f$ and $Q_g$ are definable from each other. Let

$$h(n) = \begin{cases} n/2 + 1 & \text{if } n \text{ is even} \\ \lfloor n/2 \rfloor + 5 & \text{if } n \text{ is odd.} \end{cases}$$

It can be seen fairly easily that $Q_h$ is closed under the equivalence relation

$$\mathbf{A} \sim \mathbf{B} \iff \mu_{5,\mathbf{F}}(\tau(\mathbf{A})) = \mu_{5,\mathbf{F}}(\tau(\mathbf{B})),$$

where $\mathbf{F} = (f_\exists, f, \breve{f}, \breve{f}_\exists)$. Hence $Q_h$ is FO($Q_f$)-definable by Corollary 7.

*Example 10.* Let $f(n) = \lfloor n/3 \rfloor$ and $g(n) = \lceil n/3 \rceil$. Now $Q_g$ is not definable in terms of $Q_f$, which can be seen as follows. Let $\mathbf{F} = (f_\exists, f, \breve{f}, \breve{f}_\exists)$. If $Q_g$ were FO$(Q_f)$-definable, there would be, by Corollary 7, an $r \in \mathbf{N}$ such that $Q_g$ is closed under the equivalence relation (2). But a direct calculation shows that $\mu_{r,\mathbf{F}}(3r, r) = \mu_{r,\mathbf{F}}(3r+1, r)$, while $(3r, r) \in \tau(Q_g)$ and $(3r+1, r) \notin \tau(Q_g)$. Similarly, if

$$h(n) = \begin{cases} n/3 + 1 & \text{if 3 divides } n \\ \lfloor n/3 \rfloor + 5 & \text{otherwise} \end{cases}$$

then $Q_h$ is not closed under (2) and hence not definable in FO$(Q_f)$.

Theorem 8 can be used to show that various quantifiers are not definable in terms of monotone quantifiers. All we have to do is to show that no uniform bound can be put on the number of oscillation points on an arbitrary but fixed row. The most obvious example is EVEN.

## 3. Bounded Monotone Quantifiers

In this section we present a proof of Theorem 14. We start with a useful necessary condition for definability:

LEMMA 11. *Suppose* **F** *satisfies (F1)–(F3) and* $Q_g$ *is* FO$^r(\mathbf{F})$*-definable. Then for all* $n \in \mathbf{N}$*:*

$$g(n) \in \{f_i(n) + j : 0 \leq i < u, \Leftrightarrow r < j < r\}.$$

*Proof.* Note that if $0 < g(n) \leq n$, then $(n, g(n) \Leftrightarrow 1)$ is an oscillation point of $Q_g$, hence $(n, g(n))$ is the left endpoint of a $\mu_{r,\mathbf{F}}$-monochromatic interval. Such left endpoints must have the form $(n, f_i(n) + j)$, $\Leftrightarrow r < j < r$. (This also holds when $g(n) = 0$ or $g(n) = n + 1$.) □

*Example 12.* With the criterion of Lemma 11 it is easy to exhibit non-definability results. Let $f(n) = \lfloor n/2 \rfloor$ and $g(n) = \lfloor n/3 \rfloor$. $Q_f$ is not definable in terms of $Q_g$ because for every $r \in \mathbf{N}$, letting $n = 12r$, we have

$$\lfloor n/3 \rfloor + r < \lfloor n/2 \rfloor < n \Leftrightarrow \lfloor n/3 \rfloor \Leftrightarrow r.$$

Similarly, for any other familiar functions $f$ and $g$ we can (try to) prove non-definability of $Q_f$ in terms of $Q_g$ by simply writing down some inequalities and then appealing to Lemma 11.

Suppose **F** satisfies (F1)–(F3) and every function in **F** is bounded. Note that the dual of a bounded function is bounded. Thus there is a $k_\mathbf{F} \in \mathbf{N}$ such that for $0 \leq i < u$:

$$\forall n \in \mathbf{N}(f_i(n) \leq k_\mathbf{F} \quad \text{or} \quad f_i(n) \geq n \Leftrightarrow k_\mathbf{F}).$$

We define a finite coloring of $\mathbf{N}$ as follows:

$$\beta_{\mathbf{F}}(n) = \{(i,j) : f_i(n) = j \le k_{\mathbf{F}}\}.$$

If $\beta_{\mathbf{F}}$ and $\beta_{\mathbf{F}'}$ are two such colorings, we say that $\beta_{\mathbf{F}}$ *eventually refines* $\beta_{\mathbf{F}'}$ if there is an $m \in \mathbf{N}$ such that

$$\forall n, n' \ge m(\beta_{\mathbf{F}}(n) = \beta_{\mathbf{F}}(n') \to \beta_{\mathbf{F}'}(n) = \beta_{\mathbf{F}'}(n')).$$

If $\mathbf{Q}$ is a set of monotone quantifiers, we let $\beta_{\mathbf{Q}}$ be the coloring $\beta_{\mathbf{F}}$, where $\mathbf{F}$ consists of $f_{\exists}$, each $f$ with $Q_f \in \mathbf{Q}$, and the duals of these. The functions $\beta_{\mathbf{F}}$ and $\mu_{r,\mathbf{F}}$ both extract information out of $\mathbf{F}$, and they are obviously related to each other. We cannot quite calculate $\mu_{r,\mathbf{F}}$ from $\beta_{\mathbf{F}}$ alone but we can do the following:

LEMMA 13. *Suppose* $\mathbf{F}$ *is a finite set of bounded functions which satisfies (F1)–(F3),* $r > k_{\mathbf{F}}$ *and* $n, n' \ge k_{\mathbf{F}} + r$*. Then the following conditions are equivalent:*

*(i)* $\beta_{\mathbf{F}}(n) = \beta_{\mathbf{F}}(n')$.
*(ii)* $\mu_{r,\mathbf{F}}(n,0) = \mu_{r,\mathbf{F}}(n',0)$.

*Proof.* (i) $\to$ (ii) Suppose $(0,i,j) \in \mu_{r,\mathbf{F}}(n,0)$, where $0 \le i < u$ and $\Leftrightarrow r < j < r$. Then $0 \ge m' = f_i(n) + j$. Since $f_i$ is bounded, either $f_i(n) \le k_{\mathbf{F}}$ or $n \Leftrightarrow f_i(n) \le k_{\mathbf{F}}$.

*Case 1.* $f_i(n) \le k_{\mathbf{F}}$. Thus $(i, m' \Leftrightarrow j) \in \beta_{\mathbf{F}}(n)$. By (i), $(i, m' \Leftrightarrow j) \in \beta_{\mathbf{F}}(n')$, whence $f_i(n') + j = m'$. It follows that $(0,i,j) \in \mu_{r,\mathbf{F}}(n',0)$.

*Case 2.* $n \Leftrightarrow f_i(n) \le k_{\mathbf{F}}$. Now $n \le k_{\mathbf{F}} \Leftrightarrow j < k_{\mathbf{F}} + r$ contrary to the assumption that $n \ge k_{\mathbf{F}} + r$.

Suppose then $(1,i,j) \in \mu_{r,\mathbf{F}}(n,0)$, where $0 \le i < u$ and $\Leftrightarrow r < j < r$. Now $f_i(n) = j' \le j$. Again boundedness of $f_i$ implies that one of the following cases holds:

*Case 1.* $f_i(n) \le k_{\mathbf{F}}$. Thus $(i,j') \in \beta_{\mathbf{F}}(n)$, whence by (i), $(i,j') \in \beta_{\mathbf{F}}(n')$ and therefore $f_i(n') = j' \le j$. It follows that $(1,i,j) \in \mu_{r,\mathbf{F}}(n',0)$.

*Case 2.* $n \Leftrightarrow f_i(n) \le k_{\mathbf{F}}$. Now $n < k_{\mathbf{F}} + r$ contrary to the assumption $n \ge k_{\mathbf{F}} + r$.

We have proved $\mu_{r,\mathbf{F}}(n,0) \subseteq \mu_{r,\mathbf{F}}(n',0)$. By symmetry, $\mu_{r,\mathbf{F}}(n,0) = \mu_{r,\mathbf{F}}(n',0)$.

(ii) $\to$ (i) Suppose $(i,j) \in \beta_{\mathbf{F}}(n)$, that is, $f_i(n) = j \le k_{\mathbf{F}}$. Then $j < r$ and by (ii),

$$\begin{aligned} j = f_i(n) &\Rightarrow (1,i,j) \in \mu_{r,\mathbf{F}}(n,0) \\ &\Rightarrow (1,i,j) \in \mu_{r,\mathbf{F}}(n',0) \\ &\Rightarrow j \ge f_i(n'). \end{aligned}$$

If $j = 0$, this implies $f_i(n') = 0 \le k_{\mathbf{F}}$ and, therefore, $(i,j) \in \beta_{\mathbf{F}}(n')$. Otherwise,

$$j \Leftrightarrow 1 \not\ge f_i(n) \Rightarrow (1, i, j \Leftrightarrow 1) \notin \mu_{r,\mathbf{F}}(n,0)$$

$$\Rightarrow\ (1, i, j \Leftrightarrow 1) \notin \mu_{r,\mathbf{F}}(n', 0)$$
$$\Rightarrow\ j \Leftrightarrow 1 \ngeq f_i(n'),$$

whence $f_i(n') = j \leq k_{\mathbf{F}}$, and $(i, j) \in \beta_{\mathbf{F}}(n')$ follows. We have proved $\beta_{\mathbf{F}}(n) \subseteq \beta_{\mathbf{F}}(n')$. By symmetry, $\beta_{\mathbf{F}}(n) = \beta_{\mathbf{F}}(n')$. □

THEOREM 14. *Suppose* **Q** *is a finite set of bounded monotone quantifiers, and $Q'$ a monotone quantifier. Then $Q'$ is definable in* FO(**Q**) *if and only if $Q'$ is bounded and $\beta_{\mathbf{Q}}$ eventually refines $\beta_{Q'}$.*

*Proof.* Suppose $Q' = Q_g$. We start by assuming that $Q_g$ is definable in $\mathrm{FO}^r(\mathbf{F})$. By Lemma 11 we have for every $n \in \mathbf{N}$

$$g(n) \in \{f_i(n) + j : 0 \leq i < u, \Leftrightarrow r < j < r\}.$$

Suppose now $g(n) = f_i(n) + j$. If $f_i(n) \leq k_{\mathbf{F}}$, then $g(n) < k_{\mathbf{F}} + r \Leftrightarrow 1$. If $n \Leftrightarrow f_i(n) \leq k_{\mathbf{F}}$, then $g(n) \geq n \Leftrightarrow (k_{\mathbf{F}} + r)$. We have proved that $Q'$ is bounded. To prove that $\beta_{\mathbf{F}}$ eventually refines $\beta_{Q'}$, suppose $\beta_{\mathbf{F}}(n) = \beta_{\mathbf{F}}(n')$ where $n, n' \geq k_{\mathbf{F}} + 2r$. By Lemma 13, $\mu_{2r,\mathbf{F}}(n, 0) = \mu_{2r,\mathbf{F}}(n', 0)$ ($2r > k_{\mathbf{F}}$ can be assumed). By Lemma 2, $\mu_{r,\mathbf{F}}(n, r) = \mu_{r,\mathbf{F}}(n', r)$. Similarly, $\mu_{r,\mathbf{F}}(n, m) = \mu_{r,\mathbf{F}}(n', m)$ for all $m \leq r$. Let $\mathbf{F}' = (g_0, g_1, g_2, g_3) = (f_\exists, g, \breve{g}, \breve{f}_\exists)$.

*Claim.* $\mu_{r,\mathbf{F}'}(n, 0) = \mu_{r,\mathbf{F}'}(n', 0)$. Suppose $(0, i, j) \in \mu_{r,\mathbf{F}'}(n, 0)$, that is $0 \geq g_i(n) + j$, where $\Leftrightarrow r < j < r$.

*Case 1.* $g_i = f_\exists$ or $g_i = \breve{f}_\exists$. In the first case clearly $0 \geq g_i(n') + j$ whence $(0, i, j) \in \mu_{r,\mathbf{F}'}(n', 0)$. The second case is impossible.

*Case 2.* $g_i = g$. Then $(n, \Leftrightarrow j) \in \tau(Q')$ and $0 \leq \Leftrightarrow j < r$. Since $\mu_{r,\mathbf{F}}(n, \Leftrightarrow j) = \mu_{r,\mathbf{F}}(n', \Leftrightarrow j)$ and $Q'$ is closed under the equivalence relation (2) of Corollary 7, we know that $(n', \Leftrightarrow j) \in \tau(Q')$, whence $0 \geq g_i(n') + j$. It follows that $(0, i, j) \in \mu_{r,\mathbf{F}'}(n', 0)$.

*Case 3.* $g_i = \breve{g}$. If $Q_g$ is definable in $\mathrm{FO}^r(\mathbf{F})$, then so is $Q_{\breve{g}}$. Hence this case follows from Case 2.

Similarly, one proves that $(1, i, j) \in \mu_{r,\mathbf{F}'}(n, 0)$ implies $(1, i, j) \in \mu_{r,\mathbf{F}'}(n', 0)$. By symmetry, the claim follows. From the claim and Lemma 13 we get $\beta_{\mathbf{F}'}(n) = \beta_{\mathbf{F}'}(n')$, as desired.

To prove the other half of Theorem 14, suppose $Q' = Q_g$ is bounded and $\beta_{\mathbf{F}}$ eventually refines $\beta_{\mathbf{F}'}$, where $\mathbf{F}' = (f_\exists, g, \breve{g}, \breve{f}_\exists)$. Choose $m$ so that $m \geq \max(k_{\mathbf{F}}, k_{\mathbf{F}'})$ and

$$\forall n, n' \geq m(\beta_{\mathbf{F}}(n) = \beta_{\mathbf{F}}(n') \Rightarrow \beta_{\mathbf{F}'}(n) = \beta_{\mathbf{F}'}(n')).$$

This means that on $[m, \infty)$ every $\beta_{\mathbf{F}'}$-class is a union of $\beta_{\mathbf{F}}$-classes. Intuitively, we can use FO(**F**) to tell in a universe of size $n$, what $\beta_{\mathbf{F}'}(n)$ is, and from this we can read what $g(n)$ is. We shall now see how this works in detail.

Let $r = m + 1$ and $E = [0, u \Leftrightarrow 1] \times [0, k_{\mathbf{F}}]$. For $(i, j) \in E$ let (cf. the proof of Proposition 6)

$$\psi_{i,j} = \begin{cases} \Psi_{i,j} \wedge \neg\Psi_{i,j-1} & \text{if } j \geq 1 \\ \Psi_{i,0} & \text{if } j = 0. \end{cases}$$

Thus $\mathbf{A} \models \psi_{i,j}$ if and only if $f_i(|A|) = j$, and $\psi_{i,j} \in \mathrm{FO}^r(\mathbf{F})$. If $C \subseteq E$, let

$$\Theta_C = \bigwedge_{(i,j)\in C} \psi_{i,j} \wedge \bigwedge_{(i,j)\in E-C} \neg\psi_{i,j}.$$

Then $\mathbf{A} \models \Theta_C$ if and only if $\beta_{\mathbf{F}}(|A|) = C$. Since every $\beta_{\mathbf{F}'}$-class is a union of $\beta_{\mathbf{F}}$-classes on $[m, \infty)$, we can use disjunctions of the sentences $\Theta_C$ to write for each $\beta_{\mathbf{F}'}$-color $D$ a sentence $\Xi_D$ of $\mathrm{FO}^r(\mathbf{F})$ such that for $|A| \geq m$ : $\mathbf{A} \models \Xi_D$ if and only if $\beta_{\mathbf{F}'}(|A|) = D$. Furthermore, we can use disjunctions of the sentences $\Xi_D$ to write for each $j \leq k_{\mathbf{F}'}$ sentences $\Gamma_j$ and $\Delta_j$ of $\mathrm{FO}^r(\mathbf{F})$ such that for $|A| \geq m$:

$$\mathbf{A} \models \Gamma_j \iff g(|A|) = j,$$

$$\mathbf{A} \models \Delta_j \iff g(|A|) = |A| \Leftrightarrow j.$$

Let $\Lambda_j$ and $\Upsilon_j$, $j \leq k_{\mathbf{F}'}$, be sentences of $\mathrm{FO}^r$ with one unary predicate $P$ such that

$$(A, R) \models \Lambda_j \iff |R| \geq j,$$

$$(A, R) \models \Upsilon_j \iff |R| \geq |A| \Leftrightarrow j.$$

Finally, we can define $Q_g$ in $\mathrm{FO}^r(\mathbf{F})$ in models of size $\geq m$:

$$\models \mathrm{Q}_g x P(x) \Leftrightarrow \bigwedge_{j \leq k_{\mathbf{F}'}} ((\Gamma_j \wedge \Lambda_j) \vee (\Delta_j \wedge \Upsilon_j)).$$

The models of size $< m$ are all definable in $\mathrm{FO}^r$, so the ones that are in $Q_g$ can be listed separately. □

We can use Theorem 14 to describe completely the sublogics of $\mathrm{FO}(\mathbf{F})$ in **Mon**:

$$\mathrm{FO}(\mathbf{F}') \text{ is a sublogic of } \mathrm{FO}(\mathbf{F}) \iff \beta_{\mathbf{F}} \text{ eventually refines } \beta_{\mathbf{F}'}.$$

Let $p(n)$ denote the number of partitions of a set of $n$ elements. Thus e.g. $p(0) = 0$, $p(1) = 1$, $p(2) = 2$, and $p(3) = 5$. The so called Hardy–Ramanujan asymptotic formula says

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{2/3}\sqrt{n}}.$$

Suppose now that $\mathbf{F}$ satisfies (F1)–(F3) and the functions in $\mathbf{F}$ are bounded, but $\mathrm{FO}(\mathbf{F})$ is not first order logic. Then there are $n, m \in \mathbf{N}$ such that $m > 2n > 2$

and $\beta_{\mathbf{F}}$ divides $[m,\infty)$ into $t$ infinite classes. Let $\beta_1,\ldots,\beta_{p(t)}$ be the complete list of colorings of $[m,\infty)$ which are refined by $\beta_{\mathbf{F}}$ on $[m,\infty)$. For each $\beta_i$ it is easy to define a bounded function $g_i$ such that, if $\mathbf{F}'_i = (f_\exists, g_i, \breve{g}_i, \breve{f}_\exists)$, then $\beta_{\mathbf{F}'_i}$ partitions $[m,\infty)$ exactly as $\beta_i$ (here we use $m > 2n$). Thus, FO$(\mathbf{F}'_i)$ is a sublogic of FO$(\mathbf{F})$ for each $i \in [1, p(t)]$. Moreover, FO$(\mathbf{F}'_i)$ is not equivalent to FO$(\mathbf{F}'_j)$, for $i \neq j$, because $\beta_i$ and $\beta_j$ are not eventually refinements of each other. On the other hand, if FO$(\mathbf{F}')$ is a sublogic of FO$(\mathbf{F})$, then eventually $\beta_{\mathbf{F}}$ refines $\beta_{\mathbf{F}'}$. Thus, eventually $\beta_{\mathbf{F}'}$ and some $\beta_i$ refine each other. Then FO$(\mathbf{F}') \equiv$ FO$(\mathbf{F}'_i)$. Thus, we have a complete description of the sublogics of FO$(\mathbf{F})$, $\mathbf{F}$ a finite set of bounded functions, in the family **Mon**.

Suppose $\mathbf{F}$ is a finite set of bounded functions satisfying (F1)–(F3). We let #$(\mathbf{F})$ denote the number of infinite $\beta_{\mathbf{F}}$-color classes.

COROLLARY 15. *Let* $\mathbf{F}$ *be a finite set of bounded functions satisfying (F1)–(F3). There are exactly* $p(\#(\mathbf{F}))$ *different sublogics of* FO$(\mathbf{F})$ *in* **Mon**.

Note that there are continuum many pairwise noncomparable logics FO$(\mathbf{F})$ with #$(\mathbf{F}) = n$, for every $n > 1$, because there are continuum many partitions of $\mathbf{N}$ into $n$ sets, none of which is a refinement of another.

*Example 16.* Let

$$f(n) = \begin{cases} 0 & \text{if } n \text{ even} \\ n+1 & \text{if } n \text{ odd} \end{cases}$$

and $\mathbf{F} = (f_\exists, f, \breve{f}, \breve{f}_\exists)$. Then #$(\mathbf{F}) = 2$ and $FO(Q_f)$ has no proper sublogic in **Mon** except FO. Suppose $X$ is infinite and co-infinite,

$$f_X(n) = \begin{cases} 0 & \text{if } n \in X \\ n+1 & \text{if } n \notin X \end{cases}$$

and $\mathbf{F}_X = (f_\exists, f_X, \breve{f}_X, \breve{f}_\exists)$. Then #$(\mathbf{F}_X) = 2$. By choosing sets $X$ whose symmetric difference is infinite, we get a continuum of logics FO$(\mathbf{F}_X)$ no two of which are comparable to each other. Suppose

$$g(n) = \begin{cases} 5 & \text{if } n \geq 100 \text{ even} \\ 7 & \text{if } n \geq 100 \text{ odd} \\ n \Leftrightarrow 1 & \text{if } 0 < n < 100 \\ 0 & \text{if n=0} \end{cases}$$

and $\mathbf{F}' = (f_\exists, g, \breve{g}, \breve{f}_\exists)$. Now $\beta_{\mathbf{F}}$ and $\beta_{\mathbf{F}'}$ eventually refine each other, so FO$(Q_f) \equiv$ FO$(Q_g)$.

## 4. Definability by One Monotone Quantifier

The main result of this section is Theorem 17 in which we describe completely which monotone quantifiers are definable from a given monotone quantifier. The result emphasizes the role of $n/2$ in definability by a monotone quantifier. As an application we present the proof of Theorem 22.

Intuitively, it is clear that we can list the monotone quantifiers definable by a given $Q_f$ by simply going through all possible defining formulas. In view of Corollary 7 it suffices to examine formulas of a particularly simple form. Still a lot of different possibilities remain. The point of Theorem 17 below is that we classify these numerous possibilities to just five different categories. After the theorem we consider specific examples in which many of these five categories are irrelevant. Then it is possible to get a clearer picture of the situation.

THEOREM 17. *Suppose $f$ and $g$ are functions on* **N** *such that $f(n) \leq n+1$ and $g(n) \leq n+1$ for all $n$. Let* $\mathbf{F} = (f_\exists, f, \breve{f}, \breve{f}_\exists) = (f_0, f_1, f_2, f_3)$. *Then the following conditions are equivalent:*

*(i)* $Q_g$ *is* FO$(Q_f)$*-definable.*
*(ii) There is an $r \in$* **N** *such that for all $n > 4r$:*

$$g(n) = \begin{cases} F_i^1(n) + a_i^1 & \text{if } f(n) = i \text{ or } f(n) = n \Leftrightarrow i, \Leftrightarrow 1 \leq i < r \\ F^2(n) + a^2 & \text{if } r \leq f(n) \leq \lfloor n/2 \rfloor \Leftrightarrow r \\ F_i^3(n) + a_i^3 & \text{if } n \text{ is even, } f(n) = n/2 + i, \Leftrightarrow r < i < r \\ F_i^4(n) + a_i^4 & \text{if } n \text{ is odd, } f(n) = \lfloor n/2 \rfloor + i, \Leftrightarrow r < i < r \\ F^5(n) + a^5 & \text{if } \lfloor n/2 \rfloor + r \leq f(n) \leq n \Leftrightarrow r, \end{cases}$$

*where* $F_i^1, F^2, F_i^3, F_i^4, F^5 \in$ **F**, $a_i^1, a^2, a_i^3, a_i^4, a^5 \in (\Leftrightarrow r, r)$, *and if one of* $F^2, F_i^3, F_i^4, F^5$ *is in* $\{f_\exists, \breve{f}_\exists\}$, *then they are all equal and the constants* $a^2, a_i^3, a_i^4, a^5$ *are also equal.*

*Proof.* (i)$\Rightarrow$ (ii) Suppose $Q_g$ is FO$^s(Q_f)$-definable. Let $r = 2s \Leftrightarrow 1$. By Lemma 11 there are $h_n \in$ **F** and $a_n \in (\Leftrightarrow r, r)$ such that for all $n$: $g(n) = h_n(n) + a_n$. Suppose $n, m > 4r$. If $f(n) = f(m) = i < r$, then

$$\forall l \in \{0,1,2,3\} \forall j \in (\Leftrightarrow r, r)(\mu_{r,\mathbf{F}}(n, f_l(n) + j) = \mu_{r,\mathbf{F}}(m, f_l(m) + j)). \quad (4)$$

Hence by (2) of Corollary 7,

$$\forall l \in \{0,1,2,3\} \forall j \in (\Leftrightarrow r, r)(g(n) = f_l(n) + j \iff g(m) = f_l(m) + j). (5)$$

Thus, we can assume $h_n = h_m = F_i^1$ and $a_n = a_m = a_i^1$ for such $n$ and $m$. The inference that leads to a definition of $h_n = F_i^1$ and an $a_i^1$ for $f(n) = n \Leftrightarrow i, i < r$, is similar. If $r \leq f(n) < \lfloor n/2 \rfloor \Leftrightarrow r$ and $r \leq f(m) < \lfloor m/2 \rfloor \Leftrightarrow r$, then (4) holds. Thus, (5) holds and we may define $F^2$ and $a^2$ as the common $h_n$ and $a_n$ for which both $h_n(n) + a_n = g(n)$ and $h_n(m) + a_n = g(m)$ hold. $F_i^3, F_i^4, F^5, a_i^3, a_i^4$ and $a^5$ are defined similarly. In each case we have a set of numbers $n$ for which $\mu_{r,\mathbf{F}}(n, f_l(n) + j)$ is constant for all $l \in \{0,1,2,3\}$ and $j \in (\Leftrightarrow r, r)$. Since $g(n)$

is one of the numbers $f_l(n) + j$, we know how to define the desired function $(F^3_k, F^4_k, F^5)$ and the desired constant $(a^3_k, a^4_k, a^5)$. One has to distinguish the case that $n$ is even from the case that $n$ is odd, because for example, if $f(n) = \lfloor n/2 \rfloor + 1$, then $f(n) = \breve{f}(n)$ if and only if $n$ is odd. So we could not have (4) for functions near $n/2$ without separating even $n$ from odd $n$. Finally, if one of $F^2, F^3_i, F^4_i, F^5$ is $f_\exists$, then it is easy to use (2) of Corollary 7 to show that they all are $f_\exists$ and that $a^2 = a^3_i = a^4_i = a_5$, independently of $i$. The same happens if one of $F^2, F^3_i, F^4_i, F^5$ is $\breve{f}_\exists$.

(ii) $\Rightarrow$ (i) Let $|A| = n > 4r$ and $R \subseteq A$. We consider first several cases to see how to make use of (ii).

*Case 1*. $|R| = n \Leftrightarrow i$ for some $i \in [0, r)$. In this case $|R| \geq g(n)$ if and only if ($f(n) = j$ or $f(n) = n \Leftrightarrow j$ for some $j \in [\Leftrightarrow 1, r)$ and $|R| \geq F^1_j(n) + a^1_j$), or ($r \leq f(n) \leq n \Leftrightarrow r$, and $F^2 \in \{f_\exists, \breve{f}_\exists\}$ implies $|R| \geq F^2(n) + a^2$).

*Case 2*. $\max\{f(n) + r, \breve{f}(n) + r\} \leq |R| \leq n \Leftrightarrow r$. In this case $|R| \geq g(n)$ if and only if $F^2 \neq \breve{f}_\exists$, $f(n) = i$ implies $F^1_i \in \{f_\exists, f\}$, and $f(n) = n \Leftrightarrow i$ implies $F^1_i \in \{f_\exists, \breve{f}\}$ $(i \in [\Leftrightarrow 1, r))$.

*Case 3*. $f(n) = n/2 + i$, $n$ even and $|R| = n/2 + j$, $(i, j \in (\Leftrightarrow r, r))$. Now $|R| \geq g(n)$ if and only if $F^3_i \neq \breve{f}_\exists$, $F^3_i = f$ implies $a^3_i \leq j \Leftrightarrow i$, and $F^3_i = \breve{f}$ implies $a^3_i \leq j + i \Leftrightarrow 1$.

*Case 4*. $f(n) = \lfloor n/2 \rfloor + i$, $n$ odd and $|R| = \lfloor n/2 \rfloor + j$ $(i, j \in (\Leftrightarrow r, r))$. This case is analogous to Case 3.

*Case 5*. $r \leq |R| \leq \min\{f(n) \Leftrightarrow r, \breve{f}(n) \Leftrightarrow r\}$. In this case $|R| \geq g(n)$ if and only if $F^2 = f_\exists$.

*Case 6*. $|R| = \breve{f}(n) + i$, $i \in (\Leftrightarrow r, r)$, $f(n) \Leftrightarrow \lfloor n/2 \rfloor \notin (\Leftrightarrow r, r)$ and $r \leq f(n) \leq |R|$. So $r \leq f(n) \leq \lfloor n/2 \rfloor \Leftrightarrow r$. Now $|R| \geq g(n)$ if and only if $F^2 = \breve{f}$ and $a^2 \leq i$, or $F^2 \in \{f_\exists, f\}$.

*Case 7*. $|R| = f(n) + i$, $i \in (\Leftrightarrow r, r)$, $\breve{f}(n) \Leftrightarrow \lfloor n/2 \rfloor \notin (\Leftrightarrow r, r)$ and $r \leq \breve{f}(n) \leq |R|$. Analogous to Case 6.

*Case 8*. $f(n) + r \leq |R| \leq \breve{f}(n) \Leftrightarrow r$. Now $|R| \geq g(n)$ if and only if $F^2 \in \{f_\exists, f\}$.

*Case 9*. $\breve{f}(n) + r \leq |R| \leq f(n) \Leftrightarrow r$. Now $|R| \geq g(n)$ if and only if $F^5 \in \{f_\exists, \breve{f}\}$.

*Case 10*. $|R| < r$. Now $|R| \geq g(n)$ if and only if ($f(n) = i$ or $f(n) = n \Leftrightarrow i$ for some $i \in [\Leftrightarrow 1, r)$ and $|R| \geq F^1_i(n) + a^1_i$), or ($r \leq f(n) \leq n \Leftrightarrow r$, $F^2 = f_\exists$, and $|R| \geq 1 + a^2$).

Note that these cases are exhaustive and mutually exclusive. The point of the above analysis is that for each $q = 1, \ldots, 10$ we can find a sentence $\zeta_q$ of FO$(Q_f)$ so that

$$(A, R) \models \zeta_q \iff R \text{ falls into Case } q \text{ and } |R| \geq g(n),$$

where $R \subseteq A$, $n = |A|$. Let us do Case 3 as an example. First, under the assumptions of this case we have $g(n) = F^3_i(n) + a^3_i$, and so $|R| \geq g(n)$ if and only if $n/2 + j \geq F^3_i(n) + a^3_i$, from which we easily get the conditions stated in Case 3 above. Next, we can observe that

$$n \text{ is even and } |R| = n/2 + j \text{ and } f(n) = n/2 + i$$

is equivalent to

$$|R| \Leftrightarrow j + i = f(n) \text{ and } n \Leftrightarrow |R| + j + i = f(n).$$

Also, using ideas of the proof of Proposition 6 it is easy to write a sentence $\theta_{ij}$ in FO($Q_f$) such that

$$(A, R) \models \theta_{ij} \iff |R| \Leftrightarrow j + i = f(n) \text{ and}$$
$$|A \Leftrightarrow R| + j + i = f(n).$$

It follows that the sentence

$$\zeta_3 = \bigvee_{(i,j)\in X} \theta_{ij},$$

where $X = \{(i, j) : i, j \in (\Leftrightarrow r, r)$, and $F_i^3 \neq \breve{f}_\exists$, and $F_i^3 = f$ implies $a_i^3 \leq j \Leftrightarrow i$, and $F_i^3 = \breve{f}$ implies $a_i^3 \leq j + i \Leftrightarrow 1\}$, characterizes Case 3 as we wanted.

After the sentences $\zeta_q$, $q = 1, \ldots, 10$, are found, we can define $Q_g$:

$$(A, R) \in Q_g \iff (A, R) \models \bigvee_{q=1}^{10} \zeta_q.$$

□

Theorem 17 provides an explicit method for exhibiting quantifiers definable by means of a given $Q_f$. One just has to make decisions how to choose the functions $F_i^1, F^2, F_i^3, F_i^4, F^5$ from **F**, the numbers $a_i^1, a^2, a_i^3, a_i^4, a^5$ from **Z**, and what the parameter $r$ is. On the other hand, Theorem 17 provides an effective method for showing that certain monotone quantifiers are not definable in terms of another monotone quantifier.

*Example 18.* Suppose $f : \mathbf{N} \to \mathbf{N}$ such that $f(n) \leq n + 1$ for all $n \in \mathbf{N}$, and

$$g(n) = \begin{cases} n \Leftrightarrow f(n) + 1 & \text{if } f(n) < \lfloor n/2 \rfloor \\ f(n) & \text{if } f(n) \geq \lfloor n/2 \rfloor. \end{cases}$$

Then Theorem 17 implies that $Q_g$ is FO($Q_f$)-definable.

*Example 19.* Let $f(n) = \lfloor n/2 \rfloor$ and

$$g(n) = \begin{cases} \lfloor n/2 \rfloor & \text{if } n \text{ even} \\ \lfloor n/2 \rfloor + 5 & \text{if } n \text{ odd}. \end{cases}$$

Then Theorem 17 implies that $Q_g$ and $Q_f$ are definable from each other. Thus FO($Q_g$) and FO($Q_f$) are equivalent. The only sublogics of FO($Q_f$) in **Mon** are FO and FO($Q_f$) itself. To see why this is the case, suppose FO($Q_{f_1}, \ldots, Q_{f_k}$) is a sublogic of FO($Q_f$). Then each $Q_{f_i}$ is definable in FO($Q_f$). Suppose some $Q_{f_i}$ is not first order definable. Then, as above, FO($Q_{f_i}$) $\equiv$ FO($Q_f$), and therefore $FO(Q_{f_1}, \ldots, Q_{f_k}) \equiv$ FO($Q_f$).

*Example 20.* Let

$$f(n) = \begin{cases} \lfloor n/2 \rfloor & \text{if 3 divides } n \\ \lfloor n/2 \rfloor + 5 & \text{if 3 does not divide } n \end{cases}$$

and

$$g(n) = \begin{cases} \lfloor n/2 \rfloor & \text{if 3 divides } n \text{ and } n \text{ is even} \\ \lfloor n/2 \rfloor + 6 & \text{if 3 divides } n \text{ and } n \text{ is odd} \\ \lfloor n/2 \rfloor + 17 & \text{if 3 does not divide } n \text{ and } n \text{ is even} \\ \lfloor n/2 \rfloor + 2 & \text{if 3 does not divide } n \text{ and } n \text{ is odd.} \end{cases}$$

Then FO$(Q_g) \equiv$ FO$(Q_f)$, since $Q_g$ and $Q_f$ are definable from each other. The only non-trivial sublogic of FO$(Q_f)$ in **Mon** is FO$(Q_h)$, where $h(n) = \lfloor n/2 \rfloor$.

*Example 21.* Let

$$f(n) = \begin{cases} \lfloor n/2 \rfloor & \text{if } n \text{ is of the form } 3m \\ \lfloor n/2 \rfloor + 4 & \text{if } n \text{ is of the form } 3m+1 \\ \lfloor n/2 \rfloor \Leftrightarrow 1 & \text{if } n \text{ is of the form } 3m+2. \end{cases}$$

The FO$(Q_f)$ has six sublogics in **Mon**. They arise from the trivial sublogic FO and the $p(3) = 5$ partitions that the partition of the definition of $f$ refines.

The previous examples demonstrate how one can analyze the sublogic structure of logics FO$(Q_f)$, where $f(n)$ oscillates between some values close to $n/2$, in the same way as we analyzed the sublogic structure of logics FO$(Q_f)$ with bounded $f$.

We are ready to prove Theorem 22. It represents the special case of Theorem 17 where the function $f(n)$ stays away from both 0, $n/2$ and $n$, and either stays below $n/2$ or above it. Examples of functions like this are $\lfloor n/3 \rfloor$, $\lfloor 7n/8 \rfloor$, $\lfloor \sqrt{n} \rfloor$ and $\lfloor \log n \rfloor$.

THEOREM 22. *Suppose $Q_f$ is a monotone quantifier such that*

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty} (n \Leftrightarrow f(n)) = \infty$$

*and*

$$\lim_{n\to\infty} (f(n) \Leftrightarrow \lfloor n/2 \rfloor) = \infty \textit{ or } \lim_{n\to\infty} (\lfloor n/2 \rfloor \Leftrightarrow f(n)) = \infty.$$

*Then a monotone quantifier $Q_g$ is definable in* FO$(Q_f)$ *if and only if $Q_g$ is first-order definable or there is a constant $a \in$ **Z** and a number $m \in$ **N** s.t.*

$$\forall n \geq m(g(n) = f(n) + a) \textit{ or } \forall n \geq m(g(n) = n \Leftrightarrow f(n) + a). \tag{6}$$

*Proof.* Suppose $Q_g$ is definable in FO$(Q_f)$ and $\lim_{n\to\infty}(f(n) \Leftrightarrow \lfloor n/2 \rfloor) = \infty$. Then $g$ can be recovered from $f$ as in (ii) of Theorem 17. Since $\lim_{n\to\infty} f(n) = \infty$, the case $f(n) = i$ disappears eventually. The same happens to $f(n) = n \Leftrightarrow i$, as $\lim_{n\to\infty}(n \Leftrightarrow f(n)) = \infty$. The case $f(n) = \lfloor n/2 \rfloor + i$, $n$ even, (or odd) disappears since $\lim_{n\to\infty}(f(n) \Leftrightarrow \lfloor n/2 \rfloor) = \infty$. For the same reason the case $r \leq f(n) \leq \lfloor n/2 \rfloor \Leftrightarrow r$ disappears. All we have left for $n \geq m$, $m$ a constant, is $g(n) = F^5(n) + a^5$. Here $F^5 \in \{f, \breve{f}\}$, so the claim follows. The case $\lim_{n\to\infty}(\lfloor n/2 \rfloor \Leftrightarrow f(n)) = \infty$ is similar. □

Theorem 22 shows that if $f$ satisfies the conditions of the Theorem, then FO$(Q_f)$ has no non-trivial sublogics in **Mon**, for (6) implies that $Q_f$ is definable from $Q_g$. We may conclude that whenever a monotone quantifier $Q_f$ is given, the more numbers $k$ there are such that $f(n) \leq k$, $f(n) \leq n \Leftrightarrow k$, or $|f(n) \Leftrightarrow \lfloor n/2 \rfloor| \leq k$ for infinitely many $n$, the more sublogics FO$(Q_f)$ has in **Mon**, while in the extreme opposite case represented by Theorem 22, the logic FO$(Q_f)$ has no non-trivial sublogics in **Mon** what so ever.

The following example demonstrates the difficulties in generalizing Theorem 17 for definability in FO$(\mathbf{F})$ with a more general $\mathbf{F}$.

*Example 23.* Let $f_1(n) = \lfloor \sqrt{n} \rfloor$ and $g(n) = \lfloor \sqrt{n} \rfloor + \lceil n/2 \rceil \Leftrightarrow \lfloor n/2 \rfloor$. Now $g(n) = f_1(n) + a_n$, where $a_n$ is 0 or 1 depending on whether $n$ is even or odd. Let $\mathbf{F} = (f_1, f_2, f_3)$, where

$$f_2(n) = \lfloor \sqrt[3]{n} \rfloor + \lceil n/2 \rceil \Leftrightarrow \lfloor n/2 \rfloor$$
$$f_3(n) = \lfloor \sqrt[3]{n} \rfloor.$$

We can detect whether $n$ is even or odd by comparing $f_2(n)$ and $f_3(n)$. Hence $a_n$ depends only on the mutual order of the functions in $\mathbf{F}$, and yet $Q_g$ is not FO$(\mathbf{F})$-definable. To see this, suppose $Q_g$ were definable in FO$^r(Q_f)$. Let $n$ be even and so large that $\mu_{r,\mathbf{F}}(n, \lfloor \sqrt{n} \rfloor) = \mu_{r,\mathbf{F}}(n+1, \lfloor \sqrt{n+1} \rfloor)$ and $\lfloor \sqrt{n} \rfloor = \lfloor \sqrt{n+1} \rfloor$. Then by Corollary 7, $\lfloor \sqrt{n} \rfloor \geq g(n)$ if and only if $\lfloor \sqrt{n+1} \rfloor \geq g(n+1)$, a contradiction.

## 5. A More General Setting

In this section we develop the ideas of the preceding sections in the framework of non-monotone and non-simple unary quantifiers. The definability criteria that we get are rather complicated and their applicability may be questioned. However, it turns out that even with quite elementary counting methods some non-definability results can be proved (Theorems 34 and 35). More sophisticated combinatorial methods have been used in this setting in (Kolaitis and Väänänen, 1995; Luosto, 1997).

Suppose $k$ is a natural number. Let $L_k$ be the vocabulary $\{R_1, \ldots, R_k\}$ of $k$ unary predicates. A *unary quantifier of type* $(1;k)$ is any class $Q$ of $L_k$-structures $\mathbf{A} = (A, P_1, \ldots, P_k)$ satisfying the *isomorphism condition*:

$$\mathbf{A} \cong \mathbf{B} \Rightarrow (\mathbf{A} \in Q \iff \mathbf{B} \in Q). \tag{7}$$

These quantifiers are called *unary* or *monadic* because their vocabulary is unary. For $k = 1$ these quantifiers are called *simple*. The type $(1;k)$ is often written as $(1, \ldots, 1)$ but this notation hides the number $k$. Therefore we use $(1;k)$.

Examples of non-simple unary quantifiers are the *generalized Härtig quantifier*

$$\mathrm{I}_f = \{(A, P_1, P_2) : |P_1| = f(|P_2|)\} \tag{8}$$

and the *generalized Rescher quantifier*

$$\mathrm{MORE}_f = \{(A, P_1, P_2) : |P_1| > f(|P_2|)\}, \tag{9}$$

where $f : \mathbf{N} \to \mathbf{N}$ is arbitrary.

The isomorphism condition (7) permits an algebraic formulation, because isomorphism of two unary structures can be checked by simply counting cardinalities of sets. To this end, we fix some notation. Let $[k]$ denote the set $\{1, \ldots, k\}$. Every $i \in [2^k]$ has a unique representation

$$i = 1 + \sum_{j=1}^{k} bin(i,j) 2^{j-1},$$

where $bin(i,j) \in \{0,1\}$. This is the standard way of using numbers in $[2^k]$ to code subsets of $[k]$. The *code* of an $L_k$-structure $\mathbf{A} = (A, P_1, \ldots, P_k)$ is the sequence

$$\sigma(\mathbf{A}) = (|\psi^k_{1,\mathbf{A}}|, \ldots, |\psi^k_{2^k,\mathbf{A}}|),$$

where

$$\psi^k_{i,\mathbf{A}} = \bigcap \{P_j : bin(i,j) = 0\} \cap \bigcap \{A \Leftrightarrow P_j : bin(i,j) = 1\}.$$

Now $\mathbf{A} \cong \mathbf{B} \iff \sigma(\mathbf{A}) = \sigma(\mathbf{B})$ and we may identify a quantifier $Q$ with the set $\sigma(Q) = \{\sigma(\mathbf{A}) : \mathbf{A} \in Q\}$ of codes of its elements. In this way the study of quantifiers of type $(1;k)$ becomes a study of subsets of $\mathbf{N}^{2^k}$.

*Example 24.* Suppose $Q$ is of type $(1;1)$. Thus $Q$ consists of some structures $\mathbf{A} = (A, R)$, where $R \subseteq A$. Then

$$\sigma(\mathbf{A}) = (|R|, |A \Leftrightarrow R|).$$

*Example 25.* Suppose $Q$ is of type $(1;2)$. Then $Q$ consists of some structures $\mathbf{A} = (A, R_1, R_2)$, where $R_1 \subseteq A$ and $R_2 \subseteq A$. In this case

$$\sigma(\mathbf{A}) = (|R_1 \cap R_2|, |R_2 \Leftrightarrow R_1|, |R_1 \Leftrightarrow R_2|, |A \Leftrightarrow (R_1 \cup R_2)|).$$

For example, $\sigma(\mathrm{I}_f) = \{(n_2, n_3, n_4, n_1) : n_1 + n_3 = f(n_1 + n_2)\}$.

The non-simple quantifiers are obviously more challenging than the simple ones. We cannot in general visualize non-simple quantifiers by means of the number-triangle. It is much more difficult to find useful invariants like $\mu_{r,\mathbf{F}}$ and $\beta_{\mathbf{F}}$ for deciding definability issues. However, interesting quantifiers tend to be non-simple, especially in the area of natural language semantics (see, e.g., Hella et al., 1997).

The extension FO$(Q)$ of first-order logic by the quantifier $Q$ of type $(1;k)$ is obtained by adding to FO the new logical operation

$$(\phi_1(x,\mathbf{y}),\dots,\phi_k(x,\mathbf{y})) \mapsto \mathrm{Q}x\phi_1(x,\mathbf{y}),\dots,\phi_k(x,\mathbf{y})$$

with

$$\mathbf{A} \models \mathrm{Q}x\phi_1(x,\mathbf{a}),\dots,\phi_k(x,\mathbf{a}) \iff (A,\phi_1^{\mathbf{A}}(\cdot,\mathbf{a})\dots\phi_k^{\mathbf{A}}(\cdot,\mathbf{a})) \in Q.$$

The extension of FO by a set $\mathbf{Q}$ of quantifiers is denoted by FO$(\mathbf{Q})$.

Let $Q$ be a unary quantifier of type $(1;k)$ and $Q'$ a quantifier of type $(1,k')$. We say that $Q$ *sums over* $Q'$ if there is a function $f:[2^k]\to[2^{k'}]$ such that

$$(n_1,\dots,n_{2^k}) \in \sigma(Q) \iff \left(\sum_{f(i)=1} n_i,\dots,\sum_{f(i)=2^{k'}} n_i\right) \in \sigma(Q')$$

for all $n_1,\dots,n_{2^k}$. Here it is thought that a sum over the empty set is 0. If $r\in\mathbf{N}$ and $k=k'$, we say that $Q$ is an $r$-*translate* of $Q'$ provided that there are $a_1,\dots,a_{2^k}\in\mathbf{Z}$ so that

$$\sum_{i=1}^{2^k} a_i = 0, \qquad \sum_{i=1}^{2^k} |a_i| < 2r$$

and

$$(n_1,\dots,n_{2^k}) \in \sigma(Q) \iff (n_1+a_1,\dots,n_{2^k}+a_{2^k}) \in \sigma(Q')$$

for all $n_1,\dots,n_{2^k}$. The point of summing and translates is that they represent in arithmetic form two basic methods of defining one quantifier from another. As it turns out, even these basic methods are very powerful.

*Example 26.* Summing over is related to definability by Boolean operations. If

$$\models \mathrm{Q}xP_1(x)P_2(x) \leftrightarrow \mathrm{Q}'x(P_1(x)\vee P_2(x)),$$

then $Q$ sums over $Q'$ since

$$(n_1,n_2,n_3,n_4) \in \sigma(Q) \iff (n_1+n_2+n_3,n_4) \in \sigma(Q').$$

*Example 27.* Translates are related to definability with the help of first-order quantifiers $\exists$ and $\forall$. If

$$\models \mathrm{Q}xP(x) \leftrightarrow \forall y(\neg P(y) \to \mathrm{Q}'x(P(x)\vee x=y)),$$

then $Q$ is a 2-translate of $Q'$ since

$$(n_1,n_2) \in \sigma(Q) \Leftrightarrow (n_1+1,n_2 \Leftrightarrow 1) \in \sigma(Q').$$

LEMMA 28. *Let $Q$ be a quantifier of type $(1;k)$ that sums over an $r$-translate of a quantifier $Q'$ of type $(1;k')$. Then $Q$ is definable in* $\mathrm{FO}^r(Q')$.

*Proof.* Suppose first $Q$ is an $r$-translate of $Q'$. Let us first consider a special case. Assume that $k = k' = 2, r = 3$ and

$$(n_1, n_2, n_3, n_4) \in \sigma(Q) \Leftrightarrow (n_1 + 2, n_2 \Leftrightarrow 1, n_3 \Leftrightarrow 1, n_4) \in \sigma(Q').$$

Thus $\mathrm{Q}xP_1(x)P_2(x)$ says that if we modify the partition generated by $P_1$ and $P_2$ by taking one element from $P_1 \Leftrightarrow P_2$ and one element from $P_2 \Leftrightarrow P_1$ and putting the two elements into the intersection, then the new partition is the partition of a model in $Q'$. Let us use the universal quantifier $\forall$ to formalize this:

$$\models \mathrm{Q}xP_1(x)P_2(x) \leftrightarrow \forall y_1 \forall y_2((P_1(y_1) \wedge \neg P_2(y_1) \wedge P_2(y_2) \wedge \neg P_1(y_2)) \to$$

$$\to \mathrm{Q}'x(P_1(x) \vee x = y_2)(P_2(x) \vee x = y_1).$$

In the general case there are $a_1, \ldots, a_{2^k} \in \mathbf{Z}$ so that $\sum a_i = 0, \sum |a_i| < 2r$ and

$$(n_1, \ldots, n_{2^k}) \in \sigma(Q) \Leftrightarrow (n_1 + a_1, \ldots, n_{2^k} + a_{2^k}) \in \sigma(Q').$$

Note that $\sum |a_i|$ is necessarily even, say $2\alpha$, where $\alpha < r$. To define the sentence $\mathrm{Q}xP_1(x) \ldots P_k(x)$ in terms of $Q'$ one universally quantifies over $y_1, \ldots, y_\alpha$, distributes these elements into the partition generated by $P_1, \ldots, P_k$, builds new predicates $R_1, \ldots, R_k$ which generate the new partition, and finally demands that $\mathrm{Q}'xR_1(x) \ldots R_k(x)$. We leave the details to the reader.

Let then $Q$ be a quantifier of type $(1;k)$ that sums over a quantifier $Q'$ of type $(1;k')$. Let first $k = 2, k' = 1$ and

$$(n_1, n_2, n_3, n_4) \in \sigma(Q) \Leftrightarrow (n_1 + n_4, n_2 + n_3) \in \sigma(Q').$$

Looking at the definition of coding, we see that $\mathrm{Q}xP_1(x)P_2(x)$ is equivalent to $(i,j) \in \sigma(Q')$, where $i = |(P_1 \cap P_2) \cup (A \Leftrightarrow (P_1 \cup P_2))|$ and $j = |P_2 \Leftrightarrow P_1| + |P_1 \Leftrightarrow P_2|$. Thus,

$$\models \mathrm{Q}xP_1(x)P_2(x) \leftrightarrow \mathrm{Q}'x((P_1(x) \wedge P_2(x)) \vee \neg(P_1(x) \vee P_2(x))).$$

We leave the general case to the reader. □

We define

$$\sigma_{r,k}(\mathbf{Q})$$

as the (finite) set of codes of unary quantifiers of type $(1;k)$ that sum over $r$-translates of elements of $\mathbf{Q} \cup \{\exists\}$. The coloring $\chi_{r,k,\mathbf{Q}}$ is defined now as follows: Let $\mathbf{A} = (A, P_1, \ldots, P_k)$ be an $L_k$-structure. The $(r, k, \mathbf{Q})$-color of $\mathbf{A}$ is the set

$$\chi_{r,k,\mathbf{Q}}(\mathbf{A}) = \{S \in \sigma_{r,k}(\mathbf{Q}) : \sigma(\mathbf{A}) \in S\}.$$

Since $\sigma_{r,k}(\mathbf{Q})$ is finite, there are for any fixed $r$ and $k$ only finitely many $(r, k, \mathbf{Q})$-colors. Let this finite set of colors be $C_{r,k}(\mathbf{Q})$.

LEMMA 29. *For each* $c \in C_{r,k}(\mathbf{Q})$ *there is a sentence* $\phi_c$ *of* $\mathrm{FO}^r(\mathbf{Q})$ *of vocabulary* $L_k$ *so that for all* $L_k$*-structures* $\mathbf{A}$*:*

$$\mathbf{A} \models \phi_c \iff \chi_{r,k,\mathbf{Q}}(\mathbf{A}) = c.$$

*Proof*. The claim follows from Lemma 28. □

The sentences $\phi_c$ of Lemma 29, characterizing $\chi_{r,k}(\mathbf{Q})$-color classes, have the property of *flatness*, that is, no *nesting* of the quantifiers in $\mathbf{Q}$ occurs. In fact, the sentences $\phi_c$ are Boolean combinations of sentences of the form

$$\exists \mathbf{x}(\psi(\mathbf{x}) \wedge \mathrm{Q}y\theta_1(y, \mathbf{x}) \ldots \theta_{k'}(y, \mathbf{x})),$$

where $\psi(\mathbf{x}), \theta_1(y, \mathbf{x}), \ldots, \theta_{k'}(y, \mathbf{x})$ are quantifier-free and $Q \in \mathbf{Q} \cup \{\exists\}$.

LEMMA 30. *Suppose* $\mathbf{A}$ *and* $\mathbf{B}$ *are* $L_k$*-structures. Then the following conditions are equivalent:*

*(i)* $\mathbf{A} \equiv_{\mathrm{FO}^r(\mathbf{Q})} \mathbf{B}$,
*(ii)* $\chi_{r,k,\mathbf{Q}}(\mathbf{A}) = \chi_{r,k,\mathbf{Q}}(\mathbf{B})$.

*Proof*. Lemma 29 gives (i) ⇒ (ii). So we have to prove (ii) ⇒ (i). To this end we have to modify the concept of $(r, \mathbf{F})$-Ehrenfeucht–Fraïssé game from Section 2 so as to make it appropriate for non-simple quantifiers.

Given two structures $\mathbf{A}$ and $\mathbf{B}$ of the same vocabulary, the $(r, \mathbf{Q})$-*Ehrenfeucht–Fraïssé game* on $\mathbf{A}$ and $\mathbf{B}$ is defined as follows: The game has two players: I and II. The game starts with a move of Player I. He chooses one of the models, say $\mathbf{A}$, one of the quantifiers of $\mathbf{Q} \cup \{\exists\}$, say the type $(1; k)$ quantifier $Q$, and $k$ subsets $X_1, \ldots, X_k$ of $A$. Then Player II chooses $k$ subsets $Y_1, \ldots, Y_k$ of $B$. Next, Player I chooses an element $y$ of $B$. Finally, Player II chooses an element $x$ of $A$. This sequence of moves is repeated $r$ times. The element $x$ played by II has to satisfy

$$x \in X_i \iff y \in Y_i$$

for all $i = 1, \ldots, k$. It is also required that

$$(A, X_1, \ldots, X_k) \in Q \text{ and } (B, Y_1, \ldots Y_k) \in Q.$$

Suppose the players have played the elements $x_i$ of $A$ and $y_i$ of $B$ on round $i$. Player II wins if the relation $\{(x_i, y_i) : i = 1, \ldots, r\}$ is a partial isomorphism between $\mathbf{A}$ and $\mathbf{B}$. Otherwise Player I wins. Again, it makes no difference to Player II's chances for winning the game, if Player I is required to play sets that are invariant under automorphisms of the model that fix the elements that have been chosen so far.

LEMMA 31. *If Player II has a winning strategy in the* $(r, \mathbf{Q})$*-Ehrenfeucht-Fraïssé game on* $\mathbf{A}$ *and* $\mathbf{B}$*, then* $\mathbf{A} \equiv_{\mathrm{FO}^r(\mathbf{Q})} \mathbf{B}$.

*Proof.* We use induction on the quantifier-rank ($\leq r \Leftrightarrow s$) of $\phi(z_1, \dots, z_s)$ to prove that

$$\mathbf{A} \models \phi(a_1, \dots, a_s) \iff \mathbf{B} \models \phi(b_1, \dots, b_s) \tag{10}$$

whenever the sequence $\pi = \{(a_1, b_1) \dots, (a_s, b_s)\}$ $(s \leq r)$ represents a position of pairs of chosen elements in the $(r, \mathbf{Q})$-Ehrenfeucht–Fraïssé game on $\mathbf{A}$ and $\mathbf{B}$, when Player II is playing his winning strategy.

If the quantifier-rank of $\phi(z_1, \dots, z_s)$ is 0, then (10) follows from the rules of the game. Let us then assume $\phi(z_1, \dots, z_s)$ is the formula

$$\mathrm{Q}x\Psi_1(x, \mathbf{z}) \dots \Psi_k(x, \mathbf{z})$$

of quantifier-rank $q + 1 \leq r \Leftrightarrow s$, where $Q \in \mathbf{Q} \cup \{\exists\}$. Suppose

$$\pi = \{(a_1, b_1) \dots, (a_s, b_s)\}$$

$(s \leq r)$ represents a position of pairs of chosen elements in the game, when Player II is playing his winning strategy. For each $j \in [k]$, let

$$A_j = \{a \in A : \mathbf{A} \models \Psi_j(a, a_1, \dots, a_s)\}$$

$$B_j = \{b \in B : \mathbf{B} \models \Psi_j(b, b_1, \dots, b_s)\}.$$

Assume $\mathbf{A} \models \phi(a_1, \dots, a_s)$. Since $s < r$, we can let Player I play $A_1, \dots, A_k$ as his next move after position $\pi$. The winning strategy of Player II gives him subsets $Y_1, \dots, Y_k$ of $B$. We claim that $Y_j = B_j$ for $j \in [k]$. Take any $b \in B$ and let Player I choose element $b$. The winning strategy of Player II gives him an element $a$ of $A$. By the rules of the game,

$$a \in A_j \iff b \in Y_j.$$

and by the induction hypothesis

$$a \in A_j \iff b \in B_j.$$

Thus $Y_j = B_j$. Since II plays a winning strategy, we have $(B, B_1, \dots, B_k) \in Q$, whence $\mathbf{B} \models \phi(b_1, \dots, b_s)$. □

*Proof of Lemma 30* continued. To prove (ii) ⇒ (i) it now suffices to use (ii) to describe a winning strategy of Player II in the $(r, \mathbf{Q})$-Ehrenfeucht–Fraïssé game on $\mathbf{A}$ and $\mathbf{B}$. Let us assume we are in the middle of the game with the chosen elements $a_1, \dots, a_t$ of $A$ and the corresponding elements $b_1, \dots, b_t$ of $B$, where $t < r$. Suppose Player I has chosen $Q \in \mathbf{Q} \cup \{\exists\}$ of type $(1; k')$ and plays subsets $X_1, \dots, X_{k'}$ of, say, $A$ so that $\mathbf{A}' = (A, X_1, \dots, X_{k'}) \in Q$ and the subsets $X_j$ are invariant under automorphisms that fix $a_1, \dots, a_t$. Let $A_i = \psi^k_{i,\mathbf{A}}$ for $i \in [2^k]$ and $A'_i = \psi^{k'}_{i,\mathbf{A}'}$ for $i \in [2^{k'}]$. By automorphism invariance, each $A'_i$ is a union of some sets $A_j$, up to some elements in $\{a_1, \ldots, a_t\}$. Thus there is $f : [2^k] \to [2^{k'}]$ and $\alpha_1, \ldots, \alpha_{2^{k'}}$ so that

$$\sum \alpha_i = 0, \qquad \sum |\alpha_i| < 2r$$

and

$$|A'_i| = \sum_{f(j)=i} |A_j| + \alpha_i.$$

Let $S$ consist of $(n_1, \ldots, n_{2^{k'}})$ so that

$$(n_1 + \alpha_1, \ldots, n_{2^{k'}} + \alpha_{2^{k'}}) \in \sigma(Q)$$

and let $S_1$ consist of all $(n_1, \ldots, n_{2^k})$ so that

$$\left( \sum_{f(j)=1} n_j, \ldots, \sum_{f(j)=2^{k'}} n_j \right) \in S.$$

Thus $\sigma(\mathbf{A}) \in S_1$. Since $\chi_{r,k,\mathbf{Q}}(\mathbf{A}) = \chi_{r,k,\mathbf{Q}}(\mathbf{B})$ and $S_1 \in \sigma_{r,k}(\mathbf{Q})$, we have $\sigma(\mathbf{B}) \in S_1$. We let Player II play such subsets $Y_1, \ldots, Y_{k'}$ of $B$ that if we denote $(B, Y_1, \ldots, Y_{k'})$ by $\mathbf{B}'$, then

$$\psi^{k'}_{i,\mathbf{B}'} = (\cup\{\psi^k_{j,\mathbf{B}} : f(j) = i\} \cup \{b_j : a_j \in A'_i\}) \Leftrightarrow \{b_j : a_j \notin A'_i\}.$$

Since II has not lost yet, we may assume that

$$a_j \in A'_i \iff b_j \in \psi^{k'}_{i,\mathbf{B}'}$$

and

$$a_j = a_{j'} \iff b_j = b_{j'}.$$

Thus,

$$|\psi^{k'}_{i,\mathbf{B}'}| = \sum_{f(j)=i} |\psi^k_{j,\mathbf{B}}| + \alpha_i.$$

Since $\sigma(\mathbf{B}) \in S_1$, we may conclude that $\mathbf{B}' \in Q$. Now comes the part of the game where Player I chooses an element $y$ of $B$ and Player II responds with an element $x$ of $A$ so that the partial isomorphism is preserved. This part is trivial in view of the way Player II has chosen the sets $Y_i$. The non-emptiness of the sets, where Player II has to choose, follows from the fact that the set $\sigma_{r,k,\mathbf{Q}}$ contains codes of quantifiers that sum over $\{\exists\}$, too. We have described the required winning strategy. □

THEOREM 32. *Suppose* $\mathbf{Q}$ *is a finite set of unary quantifiers and* $Q$ *is a unary quantifier of type* $(1; k)$. *Then the following conditions are equivalent:*

*(i)* $Q$ *is* $\mathrm{FO}^r(\mathbf{Q})$*-definable,*
*(ii)* $Q$ *is closed under the equivalence relation*

$$\mathbf{A} \sim \mathbf{B} \iff \chi_{r,k,\mathbf{Q}}(\mathbf{A}) = \chi_{r,k,\mathbf{Q}}(\mathbf{B}),$$

*(iii) There are* $c_1, \ldots, c_n \in C_{r,k}(\mathbf{Q})$ *so that*

$$\models \mathrm{Q}xP_1(x) \ldots P_k(x) \leftrightarrow (\phi_{c_1} \vee \ldots \vee \phi_{c_n}),$$

*(iv)* $Q$ *is definable by a Boolean combination of* $\mathrm{FO}^r(Q)$*-sentences of the form*

$$\exists \mathbf{x}(\psi(\mathbf{x}) \wedge \mathrm{Q}y\theta_1(y,\mathbf{x}) \ldots \theta_{k'}(y,\mathbf{x}))$$

*where* $\psi(\mathbf{x}), \theta_1(y,\mathbf{x}), \ldots, \theta_{k'}(y,\mathbf{x})$ *are quantifier-free and* $Q \in \mathbf{Q} \cup \{\exists\}$.

*Proof.* We omit the proof as it is similar to the proof of Corollary 7. □

We can formulate Theorem 32 in a more algebraic manner as follows: It is clear, in view of the above, what it means for $S \subseteq \mathbf{N}^{2^k}$ to be an $r$-translate of $S' \subseteq \mathbf{N}^{2^k}$, and what it means for $S \subseteq \mathbf{N}^{2^k}$ to sum over $S' \subseteq \mathbf{N}^{2^{k'}}$. Suppose $\mathcal{S}$ is a set of sets each of which is a subset of $\mathbf{N}^{2^l}$ for some $l$. We can think of $\mathcal{P}(\mathbf{N}^{2^k})$ as a Boolean algebra with the usual set-theoretic operations $\cap, \cup$ and $\Leftrightarrow$. Let $B_k(\mathcal{S})$ be the subalgebra of $\mathcal{P}(\mathbf{N}^{2^k})$ generated by sets $\subseteq \mathbf{N}^{2^k}$ that sum over $r$-translates of elements of $\mathcal{S} \cup \{\sigma(\exists)\}$ for some $r \geq 0$.

COROLLARY 33. *Suppose* $\mathbf{Q}$ *is a finite set of unary quantifiers containing* $\exists$*, and* $Q'$ *is a unary quantifier of type* $(1;k)$*. Then* $Q'$ *is* $\mathrm{FO}(\mathbf{Q})$*-definable iff* $\sigma(Q') \in B_k(\sigma(\mathbf{Q}))$*, where*

$$\sigma(\mathbf{Q}) = \{\sigma(Q) : Q \in \mathbf{Q}\}.$$

We shall now apply the general framework to prove the undefinability of the generalized Rescher quantifier in terms of simple unary quantifiers, except in trivial cases. To this end, let $\mathbf{Q}_1$ denote the family of all simple unary quantifiers, that is, the quantifiers of type $(1;1)$.

THEOREM 34. *Suppose* $f : \mathbf{N} \to \mathbf{N}$*. Then the following conditions are equivalent:*

*(i)* $\mathrm{MORE}_f$ *is* $\mathrm{FO}(\mathbf{Q}_1)$*-definable.*
*(ii)* $\exists m \forall n (f(n) \leq m)$.

*Proof.* (ii) ⇒ (i). Let $m$ be given by (ii). Let for each $i \leq m$: $Q_i = \{(A, P) : f(|P|) = i\}$. Now $(A, P_1, P_2) \in \text{MORE}_f$ if and only if $|P_1| > f(|P_2|)$ if and only if

$$(A, P_1, P_2) \models \bigvee_{i \leq m} [\text{Q}_i x P_2(x) \wedge \exists x_0 \ldots \exists x_i \bigwedge_{0 \leq j < k \leq i} (x_j \neq x_k \wedge P_1(x_j))].$$

(i) ⇒ (ii). Suppose $\text{MORE}_f$ is $\text{FO}^r(\mathbf{Q})$-definable, where $\mathbf{Q} \subset \mathbf{Q}_1$ is finite. Let $\alpha = |C_{r,1}(\mathbf{Q})| + 1$. Towards a contradiction, we assume $\forall m \exists n(f(n) > m)$. Let $x_1 < \ldots < x_\alpha$ so that $x_{i+1} > \max(f(x_1), \ldots, f(x_i))$ and $f(x_{i+1}) > \max(x_1, \ldots, x_i)$. Let $i < j \leq \alpha$ so that

$$\chi_{r,1,\mathbf{Q}}((A, [x_i])) = \chi_{r,1,\mathbf{Q}}((A, [x_j])), \tag{11}$$

where $A = [n]$ and $n = 2x_\alpha$. Let $\mathbf{A} = (A, P_1, P_2)$ and $\mathbf{B} = (A, P_2, P_1)$, where $P_1 = [x_j]$ and $P_2 = \{x_j + 1, \ldots, x_j + x_i\}$. Note that $\mathbf{A} \in \text{MORE}_f$, but $\mathbf{B} \notin \text{MORE}_f$. Thus we get a contradiction with Theorem 32 if we show that

$$\chi_{r,2,\mathbf{Q}}(\mathbf{A}) = \chi_{r,2,\mathbf{Q}}(\mathbf{B}). \tag{12}$$

We have $\sigma(\mathbf{A}) = (0, x_i, x_j, n \Leftrightarrow x_i \Leftrightarrow x_j)$ and $\sigma(\mathbf{B}) = (0, x_j, x_i, n \Leftrightarrow x_i \Leftrightarrow x_j)$. Suppose $S \in \sigma_{r,2}(\mathbf{Q})$. By looking at the definition of of $\sigma_{r,2}(\mathbf{Q})$ and applying (11) systematically in different cases, it is not hard to see that

$$\sigma(\mathbf{A}) \in S \iff \sigma(\mathbf{B}) \in S.$$

Thus (12) follows. □

A closely related quantifier is

$$\text{MOST}_f = \{(A, P_1, P_2) : |P_1 \cap P_2| > f(|P_1|)\}. \tag{13}$$

In fact

$$\text{MOST}_f x R_1(x) R_2(x) \leftrightarrow \text{MORE}_f x(R_1(x) \wedge R_2(x)) R_1(x).$$

THEOREM 35. *Suppose* $f : \mathbf{N} \to \mathbf{N}$.

(i) *If* $\exists m \forall n(f(n) \leq m \vee f(n) \geq n \Leftrightarrow m)$, *then* $\text{MOST}_f$ *is* $\text{FO}(\mathbf{Q}_1)$*-definable.*
(ii) *If* $\forall m \exists n(m < f(n) < n \Leftrightarrow m)$ *and* $\forall n \forall m(n \leq m \Rightarrow f(n) \leq f(m))$, *then* $\text{MOST}_f$ *is not* $\text{FO}(\mathbf{Q}_1)$*-definable.*

*Proof.* (i) Let $m$ be such that $f(n) \leq m \vee f(n) \geq n \Leftrightarrow m$ holds for all $n$. For $i \leq m$ let

$$Q_i = \{(A, P) : f(|P|) = i\} \quad \text{and} \quad Q'_i = \{(A, P) : f(|P|) = |P| \Leftrightarrow i\}.$$

Then

$$(A, P_1, P_2) \in \mathrm{MOST}_f \iff$$
$$|P_1 \cap P_2| > f(|P_1|) \iff$$
$$(A, P_1, P_2) \models \bigvee_{i \leq m} [\mathrm{Q}_i x P_1(x) \wedge$$
$$\exists x_0 \ldots \exists x_i (\bigwedge_{0 \leq j < k \leq i} x_j \neq x_k \wedge P_1(x_j) \wedge P_2(x_j))] \vee$$
$$\bigvee_{i \leq m} [\mathrm{Q'}_i x P_1(x) \wedge \exists x_2 \ldots \exists x_i \forall x (P_1(x) \rightarrow$$
$$(P_2(x) \vee x = x_2 \vee \ldots \vee x = x_i))].$$

(ii) Suppose $\mathrm{MOST}_f$ is $\mathrm{FO}^r(\mathbf{Q})$-definable with $\mathbf{Q} \subseteq \mathbf{Q}_1$ finite. Let $\alpha = |C_{r,1}(\mathbf{Q})| + 1$. Using the first assumption concerning $f$, we can construct a sequence

$$y_1 < f(y_1 + y_2) < y_2 < f(y_2 + y_3) < y_3 \ldots < y_\alpha.$$

Choose $y_i < y_j$ so that

$$\chi_{r,1,\mathbf{Q}}((A, [y_i])) = \chi_{r,1,\mathbf{Q}}((A, [y_j])),$$

where $A = [n]$ and $n = 2y_\alpha$. Let $\mathbf{A} = (A, P_1, P_2)$ and $\mathbf{B} = (A, P_1, P_3)$ where $P_1 = [y_i + y_j], P_2 = [y_j]$ and $P_3 = [y_i]$. Then by the second assumption, $\mathbf{A} \in \mathrm{MOST}_f$ but $\mathbf{B} \notin \mathrm{MOST}_f$. Now the proof proceeds as in Theorem 34. □

*Conjecture 36.* Suppose $f : \mathbf{N} \rightarrow \mathbf{N}$. Then $\mathrm{MOST}_f$ is $\mathrm{FO}(\mathbf{Q}_1)$-definable iff $\exists m \forall n (f(n) \leq m \vee f(n) \geq n \Leftrightarrow m)$.

The case of the generalized Härtig quantifier (8) seems more difficult. We mention without proof some rather special cases: If $\exists m \forall n (f(n) \leq m)$, then $\mathrm{I}_f$ is $\mathrm{FO}(\mathbf{Q}_1)$-definable, and if $\exists n_0 \forall n \geq n_0 (f(n) = a \cdot n + b)$, where $a, b \in \mathbf{N}$, then $\mathrm{I}_f$ is $\mathrm{FO}(\mathbf{Q}_1)$-definable if and only if $a = 0$.

*Conjecture 37.* Suppose $f : \mathbf{N} \rightarrow \mathbf{N}$. Then $\mathrm{I}_f$ is $\mathrm{FO}(\mathbf{Q}_1)$-definable if and only if $\exists m \forall n (f(n) \leq m)$.

Strong hierarchy results for unary quantifiers have been obtained by Luosto (1996).

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