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# REMARKS ON GENERALIZED QUANTIFIERS AND SECOND ORDER LOGICS

Our first remark makes explicite a simple folklore characterization for the logics which are definable by a set of generalized quantifiers.

We shall restrict ourselves to finitary generalized quantifiers in the sense of Lindström [2] . Our abstract logics and systems of abstract logics are defined as in Barwise [1], and moreover closed under operations of $L_{\omega\omega}$ .

Suppose we have a system of abstract logics $L^*$ which in some sense is generated by a set of formation rules. Then the logic is usually closed under substitution in the sense that atomic formulas in a given formula can be substituted semantically. Also it is clear that the logic should become finitary in the sense that each formula depends only on a finite nonlogical language (signature). In fact, these two properties are sufficient to give the logic a kind of a set of formation rules, in the semantical sense.

## 1.

The following two conditions are equivalent for any system of logics $L^*$:

(i) There are generalized quantifiers $Q_i$, $i \in I$, s.t. ( for all L )
$L^* = L(Q_i)\ i \in I$,

(ii) $L^*$ is finitary and satisfies substitution,

Proof

It suffices to look at the implication from (ii) to (i). Let us take a language $L_0$ which contains a countably infinite amount of n-ary relation symbols for any arity n (we are omitting function and constant symbols throughout).

Assuming

$$L_o^* = \{ \varphi_i \mid i \in I \}$$

let

$$Q_i = \text{Mod}(\varphi_i), \quad i \in I.$$

Each $Q_i$ can be considered as consisting of structures over a fixed finite signature, namely over the signature $\varphi_i$, on which it essentially depends. Necessarily each $Q_i$ is closed under isomorphism. The relation $L(Q_i)_{i \in I} \leqslant L^*$ is now proved by induction, using the substitution property. For the converse, assume $\varphi \in L^*$ and $\varphi$ depends only on a finite $L_1 \subset L$. Suppose $f$ is an arity-preserving injection of $L_1$ into $L_o$ and $\varphi_f$ is the result of replacing R by $fR$ in $\varphi$ for $R \in L_1$. Then $\varphi_f$ is some $\varphi_i$ and hence, assuming $L_1 = \{ R_j \mid j \leqslant n \}$,

$$\models \varphi \longleftrightarrow Q_i \underset{\sim}{x}_o \ldots \underset{\sim}{x}_n ; R_o(\underset{\sim}{x}_o), \ldots, R(\underset{\sim}{x}_n).$$

In the following $L^{II}$ denotes the usual second order logic with set variables ranging over unary predicates.

$L_\alpha^{II}$ differs from $L^{II}$ in having set variables over unary predicates of power $< \delta_\alpha$. An immediate application of 1 is

2.

There are $Q_i, i < \omega$, s.t. (for any L) $L^{II} = L(Q_i)_{i<\omega}$.
There are $Q_i, i < \omega$, s.t. (for any L) $L_\alpha^{II} = L(Q_i)_{i<\omega}$.

In the following we answer the question whether the quantifiers $Q_i$ above can be chosen among the monadic ones.

3.

There are no monadic $Q_i$ $i < \omega$, s.t. $L_\alpha^{II} \leqslant L(Q_i)_{i<\omega}$.

<u>Proof</u>

Let us assume that $Q_i, i < \omega$ are monadic quantifiers. The proof is restricted to the case; each $Q_i$ is of type $\langle 1 \rangle$, but it readily generalizes to type $\langle 1, \ldots, 1 \rangle$ quantifiers.

In the following $L = \{ R \}$, where R is binary, and K is a class of L-structures s.t. K is closed under isomorphism and if $\langle A, E \rangle \in K$ then E is an equivalence relation on A with at least two equivalence classes each being of the power of A. We shall write $\models_K \varphi$ for $\forall \mathfrak{A} \in K : \mathfrak{A} \models \varphi$.

For $i < \omega$ let

$Q_i(n)$ be $\exists x_1,\ldots,x_n(\bigwedge_{1\leq i<j\leq n} x_i \neq x_j \;\&\; Q_i$ [illegible]

$Q_i(\check{n})$ be $\exists x_1,\ldots,x_n(\bigwedge_{1\leq i<j<n} x_i \neq x_j \;\&\; Q_i x \bigwedge_{i\leq n} x \neq x_i)$

$Q_i(I)$ be $\exists y Q_i x R(x,y)$

$e_n$ be $\exists x_1,\ldots,x_n(\bigwedge_{1\leq i<j\leq n} \neg R(x_i,x_j))$,

$\mathbb{B} = \{ \varphi \in L(Q_i)_{i<\omega} \mid\; \models_K \varphi \longleftrightarrow \Psi$ for some Boolean combination $\Psi$ of formulae $x = y$, $R(x,y)$, $Q_i(n)$, $Q_i(n)$, $Q_i(I)$ and $e_n \}$.

$$IF_x = \{ \varphi \in B \mid \text{for all } i<\omega : Q_i x\, \varphi \in B \}.$$

We shall aim at showing $IF_x = IB$. At first we easily see that

3.1 $F_x$ contains atomic formulae and all formulae without $x$ free.

3.2 If $\varphi \in F_x$ and $x$ not free in $A$, then $\varphi \vee A \in F_x$.

3.3 If $\varphi$, $\Psi$ and $\varphi \vee \Psi$ are in $F_x$ and $x$ is not free in $A$, then $(A \,\&\, \varphi) \vee \Psi$ is in $F_x$.

3.4 If $\varphi_1(x,\underset{\sim}{y}),\ldots,\varphi_n(x,y) \in F_x$ and $\models_K \bigvee!_{i\leq n} \exists x \varphi_i(x,\underset{\sim}{y})$ then $\bigvee_{i\leq n} \varphi_i \in F_x$. Here $\bigvee!_{i\leq n}$ means "there is exactly one $i\leq n$ s.t."

Let $U$, $V$ and $W$ be sets of variables. We write

$Max(U,V,\varphi(x'))$ for $\bigwedge_{u\in U} (x'/u) \;\&\; \bigwedge_{U \subsetneq W \subseteq V} \bigvee_{w\in W} \neg\varphi(x'/w)$

$E_x(U,V,W)$ for $\bigwedge_{v\in V} x \neq v \;\&\; (\bigvee_{u\in U} R(x,u) \vee \bigvee_{w\in W} x = w)$

$N_x(U,V)$ for $\bigwedge_{u\in U} \neg R(x,u) \;\&\; \bigwedge_{v\in V} x \neq v$.

A direct calculation shows that

3.5. If the finite $U,V,W,S$ and $T$ do not contain $x$, then $E_x(U,V,W)$, $N_x(U,V)$ and $E_x(U,V,W) \vee N_x(T,S)$ are in $F_x$.

Furthermore we notice that

3.6. $\models_K x = y \vee E_x(U,V,W) \longleftrightarrow \bigvee!_{V_0 \in V} Max(V_0, V x' = y) \;\&\; E_x(U, V-V_0, W\cup\{y\})$

3.7. $\models_K (R(x,y) \;\&\; \bigwedge_{v\in V_0} x \neq v) \vee E_x(U,V,W) \longleftrightarrow$

$\bigvee!_{V_1 \subseteq V} \bigvee!_{V_2 \subseteq V_0} Max(V_1, V, R(x',y)) \;\&$

$Max(V_2, V_0, R(x',y)) \;\&\; E_x(U\cup\{y\}, (V-V_1)\cup(V_2,W))$.

3.8. $\models_K N_x(U_1,V_1) \vee N_x(U_2,V_2) \leftrightarrow \bigvee_{U \subseteq U_1} \bigvee_{V \subseteq V_1}$

$Max(U,U_1, \bigvee_{u \in U_2} R(x',u))$ & $Max(V,V_1, \bigvee_{v \in V_2} (x' = v))$ &

$N_x(U,V))$.

Now we are ready to prove.

3.9. $F_x = B$.

For, let $\varphi \in IB$. Then $\models_K \varphi \leftrightarrow \Psi_1 \vee \ldots \vee \Psi_n$, where each $\Psi_i$ is a conjunctionof the generators of IB. By 3.2 and 3.3 we may assume each conjunct of each $\Psi_i$ contains x free. Using well known properties of identity and equivalence relation we mayfinally reduce all conjunctions to one of the following forms: $x = y$, $R(x,y)$ & $\bigwedge_{u \in U} x \neq u$, $\bigwedge_{u \in U} \neg R(x,u)$ & $\bigwedge_{v \in V} x \neq v$. All these are of the form of an $E_x(U,V,W)$ or a $N_x(U,V)$. By 3.6 – 3.8 $\varphi$ can be built from formulae of the form $E_x(U,V,W)$, $N_x(U,V)$ or $E_x(U,V,W) \vee N_x(T,S)$, which all are in $F_x$, using the restricted Boolean operations under which $F_x$ is closed by 3.2 – 3.4. So finally $\varphi$ is in $F_x$.

A direct consequence of 3.9 is

3.10. $L(Q_i)_{i<\omega} = B$.

Using 3.10 we can easily prove

3.11. For any $\varphi$ in $L(Q_i)_{i<\omega}$ there is an $n<\omega$ s.t. is $\mathfrak{A} \models e_n$ & $\varphi$ for some $\mathfrak{A} \in K$, then $\mathfrak{B} \models e_n \rightarrow \varphi$ for any $\mathfrak{B} \in K$ s.t. $|\mathfrak{B}| = |\mathfrak{A}|$.

<u>Proof of 3.</u>

Let $\varphi$ be the following $L^{II}_{\alpha}$ - sentence :

$$\forall x R(x,x) \ \& \ \forall xy(R(x,y) \rightarrow R(y,x)) \& \forall xyz(R(x,y) \& R(y,z) \rightarrow R(x,z)) \&$$
$$\neg \exists X \forall x \exists y(X(y) \ \& \ R(x,y)).$$

Suppose $\varphi$ is equivalent to an $L(Q_i)_{i<\omega}$-formula, and 3.11 gives the natural number n to this formula. Let E be an equivalence relation on $\aleph_\alpha$ with $\aleph_\alpha$ many equivalence classes each of the power $\aleph_\alpha$, and E' an eqivalence relation on $\aleph_\alpha$ like E but with n classes. Now $\langle \aleph_\alpha, E \rangle \models \varphi$ Hence, by 3.11 $\langle \aleph, E' \rangle \models \varphi$ , a contradiction.

We have studied logics with an infinite number of generalized quantifiers. In the following we shall study the extent to which 2 holds for finite I.

A signature consisting of n-ary relation symbols only is called below n-ary.

4.

Assume $n<\omega$ . There is a Q s.t. for any n-ary L $L^{II} \leqslant LQ$. There is

also a Q s.t. $L_o^{II} < LQ < L_{(\omega)}$ [illegible]
of course.

<u>Proof</u>

Let $R_j^i$ be i-ary, $L_o = \{ R_j^i \mid 0 < i \leqslant n, j < \omega \}$ and $(L_o)^{II} = \{ \varphi_k \mid k < \omega \}$, for $K < \omega$ let $t_k < \omega$ s.t.

$$\varphi_k = \varphi_k(R_{j_1}^{i_1}, \ldots, R_{j_{t_k}}^{i_{t_k}}).$$

Let $\Psi_k(x_1, \ldots, x_{t_k})$ be the result of replacing $R_{j_l}^{i_l}(u_1, \ldots, u_{i_l})$ in $\varphi_k$ by $B(x_l, u_1, \ldots, u_{i_l}, u_{i_l}, \ldots, u_{i_l})$, where B is a new (n+1)-ary predicate. Let Q be the type $\langle 2, n+1 \rangle$ quantifier consisting of those structures $\langle M, A, B \rangle$ which satisfy the following sentence for some $m, k < \omega$ s.t. $t_k = m$:

$$\exists x_1, \ldots, x_m \, (\forall x (\exists y A(x,y) \leftrightarrow \bigvee_{l \leqslant m} x_l = x) \;\&\; \bigwedge_{l \leqslant m} \exists !^k x A(x_l, x) \;\&$$
$$\Psi_k(x_1, \ldots, x_m)),$$

where $\exists !^k$ is the quantifier " there exists exactly k x s.t."
It remains to prove $L^{II} \leqslant LQ$. So assume we are given an n-ary language L and $\varphi \in L^{II}$. Then there are $m, k < \omega$, $m = t_k$, and $R_1, \ldots, R_m$ $n_i$-ary s.t. $\varphi$ is equivalent to

$$\exists x_1^1 \ldots x_k^1 \ldots x_1^m \ldots x_k^m \, (\wedge \{ x_j^i \neq x_l^i \mid i \leqslant m, \; j \leqslant k, \; l \leqslant j \} \;\&\; Qxy, tu_1 \ldots u_n \,($$

$$\vee \{ x = x_1^i \;\&\; y = x_j^i \mid i \leqslant m, \; j \leqslant k \}, \; (\vee \{ t = x_1^i \;\&\; R_i(u_1, \ldots, u_{n_i}) \mid i \leqslant m \})).$$

It is obvious that the Q above is not definable in $L^{II}$. In fact we have

5.

There are no $Q_1 \ldots Q_n$ s.t. for all binary L, $L^{II} = LQ_1 \ldots Q_n$. The same applies to $L_{(\omega)^+}$ (with finite L) and to any $L_\alpha^{II}$, $\alpha > 0$.

<u>Proof</u>

The claim concerning $L_{(\omega)^+}$ follows from theorem 5 of Lindström[3]. For $L^{II}$ it suffices to notice that the formula

$\varphi(x) \leftrightarrow_{df} \exists X$ "x is an $LQ_1 \ldots Q_n$-sentence, X is a satisfaction relation for $LQ_1 \ldots Q_n$ and the empty sequence satisfies x" can be arranged to define the true $LQ_1 \ldots Q_n$-sentences in a suitable expansion of $\omega$. Now assuming $LQ_1 \ldots Q_n \leq L^{II}$, $\varphi(x)$ is $L^{II}$-definable but, by an undefinability of truth argument, not $LQ_1 \ldots Q_n$-definable.

Our final remark shows that 5 fails for monadic L. The quantifier $Q_\alpha$ says "there exists at least $\aleph_\alpha$ many x s.t."

6.

For monadic L, $L^{II}_\alpha = LQ_\alpha$ (and $L^{II} = L_{\omega\omega}$, of course).

<u>Proof.</u>

We call atomic formulae and their negations basic. We have identity, of course. It can be proved that every $LQ_\alpha$-formula is equivalent to a formula of the following type, which we call reduced formulae,

$$\exists x_1 \ldots x_n \bigvee_{i \leq i_0} \bigwedge_{j \leq j_0} \varphi_{ij} ,$$

where each $\varphi_{ij}$ is of the form

$$Q_x(A_1(x) \vee \ldots \vee A_n(x))$$

or of the form

$$Q_\alpha x(A_1(x) \,\&\, \ldots \,\&\, A_n(x))$$

with $A_1, \ldots, A_n$ basic and Q being $\forall$ or $\check{Q}_\alpha$, the dual of $Q_\alpha$. Now we prove that $L^{II}$-formula is equivalent to a reduced formula. By the above it suffices to show that $\exists X \varphi$ has this property whenever $\varphi$ does. So assume $\varphi$ is equivalent to a reduced formula. As $\exists X$ commutes with $\exists x_i$ and disjunction, it suffices to reduce the formulae $\exists X \varphi_i$, where $\varphi_i$ is $\bigwedge_{j \leq j_0} \varphi_{ij}$ as above. After eliminating some trivial cases, $\varphi_i$ can be written in the form

$$\forall x((X(x) \vee A(x) \,\&\, (\neg X(x) \vee B(x))) \,\&\, \check{Q}_\alpha x\,(X(x) \vee C(x)) \,\&\, Q_\alpha x\,(\neg X(x) \,\&\, D_1(x)) \,\&\, \ldots \,\&\, Q_\alpha x(\neg X(x) \,\&\, D_m(x)),$$

where $A, B, C, D_1, \ldots, D_m$ are conjunctions or disjunctions of basic formulae not containing X. A direct calculation shows now that $\exists X \varphi_i$ is equivalent to

$$\forall x(A(x) \vee B(x) \,\&\, \check{Q}_\alpha xC(x) \,\&\, \check{Q}_\alpha xA(x) \,\&\, Q_\alpha x(A(x) \,\&\, D_1(x)) \,\&\, \ldots \,\&\, Q_\alpha x(A(x) \,\&\, D_m(x)).$$

By the above this can be brought to a reduced form. ∎

So we have proved that $L^{II}_\alpha \leq LQ_\alpha$. The converse inclusion is trivial.

## References

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