An extension of a theorem of Zermelo
Or
Between second and first order logic

Jouko Väänänen

Department of Mathematics and Statistics, University of Helsinki

ILLC, University of Amsterdam

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100th anniversary of the death of Cantor

Georg Cantor 1845 (Saint Petersburg) — 1918 (Halle).

- Should we think of second order logic or first order set theory as the foundation of classical mathematics?

- Early researchers (Dedekind, Frege, Russell, Hilbert, Zermelo, Gödel, Mostowski) axiomatized mathematics using second order logic or its extension simple theory of types.
- Then ZFC emerged as a first order theory.
- Later philosophers (e.g. S. Shapiro) claimed second order logic would be better (can characterize mathematical structures) and first order logic is flawed (cannot characterize mathematical structures).
- I argue that this view is wrong.

I claim:

- Zermelo's and Dedekind's second order categoricity results are actually first order at heart.
- The difference betwqeen second order logic or first order set theory is not as clear as what was previously thought.

- Second order logic has great power in characterizing categorically mathematical structures.
- Which structures are second order characterizable?
- For which $\mathcal{A}$ is there a second order θ such that for all $\mathcal{B}$:

$$\mathcal{A} \cong \mathcal{B} \iff \mathcal{B} \models \theta$$

- By early results (Dedekind, Hilbert, Zermelo, et.al.): natural numbers, real numbers, cumulative hierarchy of sets up to the first inaccessible, etc.
- In model theory second order logic has seemed all too strong to develop any interesting theory.

Sometimes infinitary second order logic can characterize "all" models.

Theorem (Hyttinen-Kangas-V. 2013)

Let T be a countable complete first order theory and κ an uncountable cardinal with certain not too uncommon properties[1]*. Then the following are equivalent:*

1. *Every model of T of size κ is $L^2_{\kappa\omega}$-characterizable.*
2. *T is superstable, shallow, without DOP or OTOP.*

[1] A regular cardinal such that $\kappa = \aleph_\alpha$, $\beth_{\omega_1}(|\alpha| + \omega) \leq \kappa$ and $2^\lambda < 2^\kappa$ for all $\lambda < \kappa$.

Theorem (V. 2011)

1. *If a model is second order characterizable, its isomorphism class is Δ_2-definable in set theory.*
2. *A model class is second order definable*[2] *if and only if it is Δ_2-definable in set theory.*

[2] More exactly, second order Δ-definable.

Theorem (V. 2011)

1. *Second order validity is Π_2-complete in set theory.*
2. *The second order theory of a second order characterizable structure is always Δ_2 in set theory.*

Corollary

Second order validity cannot be second order defined in any second order characterizable structure.

- Second order logic is praised for its categoricity results, i.e. its ability to characterize structures.
- But what is universal second order truth — a problem!
- Best understood in terms of provability i.e. truth in all Henkin (rather than "full") models.
- But Henkin models seem to ruin the categoricity results.
- We show that categoricity can be proved for Henkin models, too, in the form of internal categoricity, which implies full categoricity in full models.

- We now demonstrate this in the case of Zermelo's result (1930) to the effect that second order ZFC is κ-categorical for all κ.
- It is (of course) not true that any two Henkin models of second order ZFC of the same cardinality are isomorphic. E.g. one can be well-founded and the other non-well-founded.

- Let us consider the vocabulary $\{\in_1, \in_2\}$, where both $\in_1$ and $\in_2$ are binary predicate symbols.
- $ZFC(\in_1)$ is the first order Zermelo-Fraenkel axioms of set theory when $\in_1$ is the membership relation and formulas are allowed to contain $\in_2$, too.
- $ZFC(\in_2)$ is the first order Zermelo-Fraenkel axioms of set theory when $\in_2$ is the membership relation and formulas are allowed to contain $\in_1$, too.

Theorem (V. 2018, extending Zermelo 1930 and D. Martin (EFI-paper, draft) 2018)

If $(M, \in_1, \in_2) \models ZFC(\in_1) \cup ZFC(\in_2)$, *then* $(M, \in_1) \cong (M, \in_2)$.

Proof of the Theorem

- We work in $ZFC(\in_1) \cup ZFC(\in_2)$.
- We alternate between $\in_1$-set theory and $\in_2$-set theory[3].

[3] It is not clear whether $\forall x \exists y \forall z (z \in_1 x \leftrightarrow z \in_2 y)$ is true, but we do not need this either.

Proof of the Theorem

- Let $\mathrm{tr}_i(x)$ be the formula $\forall t \in_i x \forall w \in_i t(w \in_i x)$. It says that x is transitive in $\in_i$-set theory.
- Let $\mathrm{TC}_i(x)$ be the unique u such that $\mathrm{tr}_i(u) \wedge x \in_i u \wedge \forall v((\mathrm{tr}_i(v) \wedge x \in_i v) \rightarrow \forall w \in_i u(w \in_i v))$ (i.e. "u is the $\in_i$-transitive closure of x").
- Let $\varphi(x, y)$ be the formula $\exists f \psi(x, y, f)$, where $\psi(x, y, f)$ is the conjunction of the following formulas (where $f(t)$ and $f(w)$ are understood in the sense of $\in_1$):

Proof of the Theorem

$$\psi(x, y, f):$$

(1) In the sense of $\in_1$, the set f is a function with $\mathrm{TC}_1(x)$ as its domain.

(2) $\forall t \in_1 \mathrm{TC}_1(x)(f(t) \in_2 \mathrm{TC}_2(y))$

(3) $\forall t \in_2 \mathrm{TC}_2(y) \exists w \in_1 \mathrm{TC}_1(x)(t = f(w))$

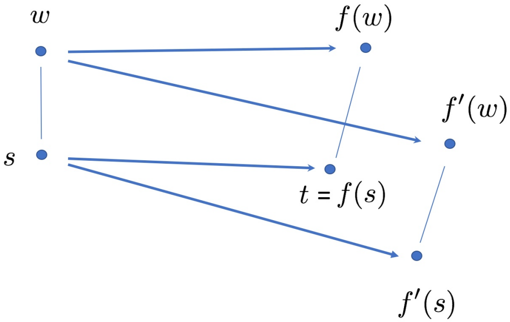
(4) $\forall t \in_1 \mathrm{TC}_1(x) \forall w \in_1 \mathrm{TC}_1(x)(t \in_1 w \leftrightarrow f(t) \in_2 f(w))$

(5) $f(x) = y$

Lemma

If $\psi(x, y, f)$ and $\psi(x, y, f')$, then $f = f'$.

Proof:

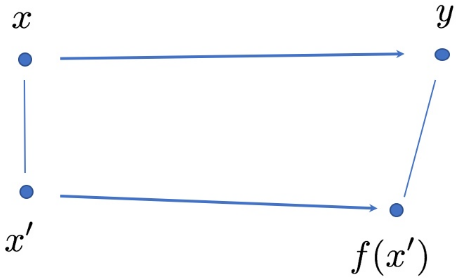


Proof of the Theorem

Lemma

If $\psi(x, y, f)$ and $x' \in_1 x$, then $\varphi(x', f(x'))$.

Proof:

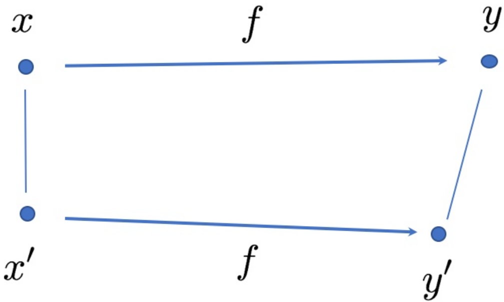


Proof of the Theorem

Lemma

If $\psi(x, y, f)$ and $y' \in_2 y$, then there is $x' \in_1 x$ such that $f(x') = y'$ and $\varphi(x', y')$.

Proof:

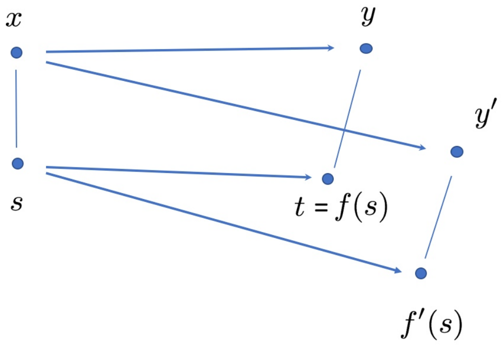


Proof of the Theorem

Lemma

If $\varphi(x, y)$ and $\varphi(x, y')$, then $y = y'$.

Proof:

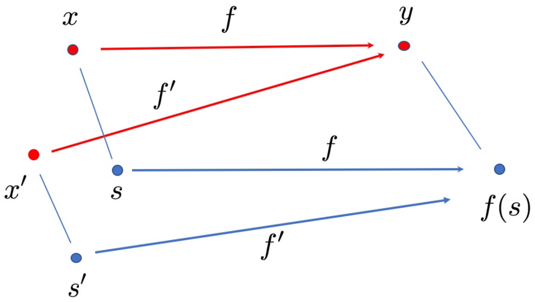


Proof of the Theorem

Lemma

If $\varphi(x, y)$ and $\varphi(x', y)$, then $x = x'$.

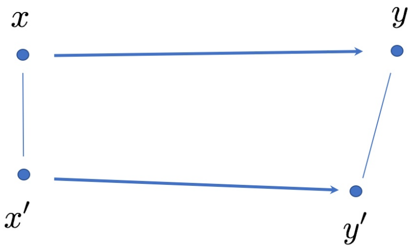
Proof:



Proof of the Theorem

Lemma

If $\varphi(x, y)$ and $\varphi(x', y')$, then $x' \in_1 x \leftrightarrow y' \in_2 y$.



Proof of the Theorem

- Let $\mathrm{On}_1(x)$ be the $\in_1$-formula saying that x is an ordinal i.e. a transitive set of transitive sets, and similarly $\mathrm{On}_2(x)$.
- For $\mathrm{On}_1(\alpha)$ let V^1_α be the α^{th} level of the cumulative hierarchy in the sense of $\in_1$, and similarly V^2_a.

Proof of the Theorem

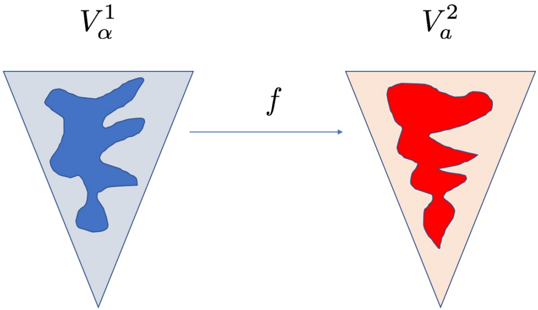
Lemma

1. *If* $\varphi(\alpha, y)$*, then* $\mathrm{On}_1(\alpha)$ *if and only if* $\mathrm{On}_2(y)$*.*
2. *If* α *is a limit ordinal then so is* y *i.e. if* $\forall u \in_1 \alpha \exists v \in_1 \alpha (u \in_1 v)$*, then* $\forall u \in_2 y \exists v \in_2 y (u \in_2 v)$*.*
3. *Also vice versa.*

Proof of the Theorem

Lemma

Suppose $\psi(\alpha, y, f)$. *If* $\mathrm{On}_1(\alpha)$ *(or equivalently* $\mathrm{On}_2(y)$*), then there is* $\bar{f} \supseteq f$ *such that* $\psi(V^1_\alpha, V^2_y, \bar{f})$.



Proof of the Theorem

Lemma

$\forall x \exists y \varphi(x, y)$ and $\forall y \exists x \varphi(x, y)$.

Proof: Consider

$$\forall \alpha(\mathrm{On}_1(\alpha) \to \exists y \varphi(\alpha, y)) \tag{1}$$

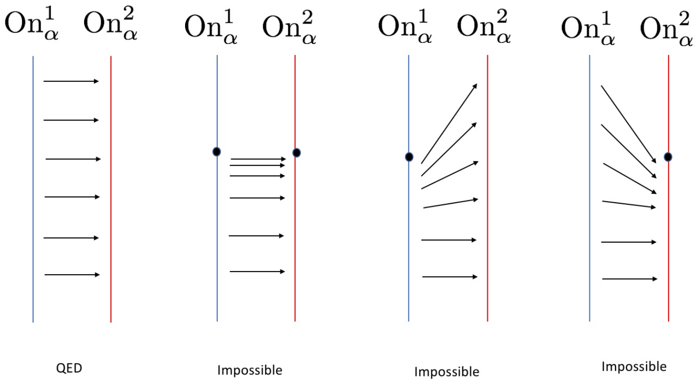
$$\forall y(\mathrm{On}_2(y) \to \exists \alpha \varphi(\alpha, y)). \tag{2}$$

Case 1: (1)$\wedge$(2). The claim can be proved.

Case 2: $\neg$(1)$\wedge\neg$(2). Impossible!

Case 3: (1)$\wedge\neg$(2). Impossible!

Case 4: $\neg$(1)$\wedge$(2). Impossible!


On_α^1
On_α^2
QED
Impossible
Impossible
Impossible

Proof of the Theorem

Lemma

The class defined by $\varphi(x, y)$ is an isomorphism between the $\in_1$-reduct and the $\in_2$-reduct.

Proof.

By the previous Lemmas. □

- Zermelo (1930) showed that if $(M, \in_1)$ and $(M, \in_2)$ both satisfy the second order Zermelo-Fraenkel axioms, then $(M, \in_1) \cong (M, \in_2)$.
- Zermelo's result follows from our theorem.
- Note: $ZFC(\in_1)$ and $ZFC(\in_2)$ are first order theories.
- We allow in these axiom systems formulas from the extended vocabulary $\{\in_1, \in_2\}$.
- Without this the result is false: there are[4] countable non-isomorphic models of *ZFC*.

[4] Assuming there are models of *ZFC* at all.

- Note that $(M, \in_1)$ and $(M, \in_2)$ can be models of $V = L$, $V \neq L$, CH, $\neg CH$, even of $\neg Con(ZF)$.
- It is easy to construct such pairs of models using classical methods of Gödel and Cohen.
- Not all of them can be models of second order set theory.

- An internal categoricity result.
- A strong robustness result for set theory.
- The model cannot be changed “internally”.
- To get non-isomorphic models one has to go “outside” the model.
- But going “outside” raises the potential of an infinite regress of meta theories.

Continuum Hypothesis (CH)

- What if $(M, \in_1) \models CH$ and $(M, \in_2) \models \neg CH$?
- Then either $(M, \in_1)$ or $(M, \in_2)$ does not satisfy the Separation Schema or the Replacement Schema if formulas are allowed to mention the other membership-relation.

- A similar result holds for *first order* Peano arithmetic: If

$$(M, +_1, \times_1 +_2, \times_2) \models P(+_1, \times_1) \cup P(+_2, \times_2),$$

  then

$$(M, +_1, \times_1) \cong (M, +_2, \times_2).$$

- This extends (and implies) Dedekind's (1888) categoricity result for *second order* Peano axioms.

- Should we think of second order logic or first order set theory as the foundation of classical mathematics?
- The answer: We need a new understanding of the difference between the two. The difference is not as clear as what was previously thought.
- The nice categoricity results of second order logic can be seen already on the first order level, revealing their inherent limitations.

Thank you!