On Löwenheim-Skolem-Tarski numbers for extensions of first order logic*

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Abstract

We show that, assuming the consistency of a supercompact cardinal, the first (weakly) inaccessible cardinal can satisfy a strong form of a Löwenheim-Skolem-Tarski theorem for the equicardinality logic $L(I)$, a logic introduced in [5] strictly between first order logic and second order logic. On the other hand we show that in the light of present day inner model technology, nothing short of a supercompact cardinal suffices for this result. In particular, we show that the Löwenheim-Skolem-Tarski theorem for the equicardinality logic at κ implies the Singular Cardinals Hypothesis above κ as well as Projective Determinacy.

*The authors finalized the paper during the special program "Mathematical Logic: Set theory and model theory" at the Mittag-Leffler Institute, Fall 2009. The authors would like to thank the Institute for its support.

†Research partially supported by grant 40734 of the Academy of Finland and by the EUROCORES LogICCC LINT programme.

1 Introduction

The Löwenheim-Skolem Theorem is perhaps the most quoted result about first order logic. It shows the "local" character of first order formulas. The truth of a first order sentence depends only on a small part of the set theoretical universe. For many purposes first order logic is ideal, but there are also interesting and useful extensions of first order logic.

Example 1

- *Second order logic L^2 extends first order logic with quantifiers of the form $\exists R\phi(R, x_0, \ldots, x_{n-1})$, where the second order variable R ranges over n-ary relations on the universe for some fixed n.*
- *The logic $L(Q_1)$ extends first order logic with a new quantifier Q_1 binding one variable. The formula $Q_1 x_0 \phi(x_0, \ldots, x_{n-1})$ has the meaning "there are uncountably many elements x_0 satisfying $\phi(x_0, \ldots, x_{n-1})$".*
- *The logic $L(Q_1^{MM})$ extends first order logic with a new quantifier Q_1^{MM} binding two variables. The formula $Q_1^{MM} x_0 x_1 \phi(x_0, \ldots, x_{n-1})$ has the meaning "there is an uncountable set X such that any two elements x_0 and x_1 from X satisfy $\phi(x_0, \ldots, x_{n-1})$".*

Second order logic is in a sense the opposite of first order logic. It is powerful enough to capture exactly a large part of the set theoretical universe. The logics $L(Q_1)$ and $L(Q_1^{\mathrm{MM}})$ are more close to first order logic. The first is axiomatizable and so is the second, if we assume $\diamond$. In this paper we study the following two, in a sense intermediate, extensions of first order logic:

Example 2

- *Equicardinality logic $L(I)$ [5]. This logic extends first order logic by formulas of the form*

$$I x_0 y_0 \phi(x_0, \ldots, x_{n-1}) \psi(y_0, \ldots, y_{n-1})$$

with the meaning: "for given $a_1, ..., a_{n-1}$ and $b_1, ..., b_{n-1}$ the cardinality of the set of elements x_0 satisfying $\phi(x_0, a_1, ..., a_{n-1})$ is the same as the cardinality of the set of elements y_0 satisfying $\psi(y_0, b_1, ..., b_{n-1})$".

- *Equicofinality logic $L(Q^{ec})$ [11]. This logic extends first order logic by formulas of the form*

$$Q^{ec} x_0 x_1 y_0 y_1 \phi(x_0, \ldots, x_{n-1}) \psi(y_0, \ldots, y_{n-1})$$

with the meaning: "for given $a_2, ..., a_{n-1}$ and $b_2, ..., b_{n-1}$, both the set of pairs of elements x_0 and x_1 satisfying $\phi(x_0, x_1, a_2, ..., a_{n-1})$ and the set of pairs of elements y_0 and y_1 satisfying $\phi(y_0, y_1, b_2, ..., b_{n-1})$ are linear orders, and moreover these linear orders have the same cofinality."

The logics $L(I)$ and $L(Q^{ec})$ are in a clear sense between first order logic and second order logic. The results of this paper show that on the basis of ZFC alone there is mixed information as to whether $L(I)$ and $L(Q^{ec})$ are closer to first order logic or to second order logic.

Very little is known about the logic $L(Q^{ec})$. Shelah [11] conjectures that this logic is compact and axiomatizable. The hidden power of this logic is revealed in models with a wellordering. There the quantifier Q^{ec} can be used to pick elements of the well-ordering corresponding to regular cardinals. This puts severe limitations e.g. to the existence of small elementary submodels. In a sense, the stronger logic $L(I, Q^{ec})$ is better understood. At least we know that this logic is very far from compact and axiomatizable, because $L(I)$ is.

There is a quite general concept of a logic, that the above examples are special cases of. We define it as follows:[1]

Definition 3 *Let τ be a fixed vocabulary. A* logic *L consists of*

1. *A set, also denoted by L, of "formulas" of L. If $\phi \in L$, then there is a natural number n_ϕ, called the length of the sequence of free variables,*
2. *A relation*
$$\mathcal{A} \models \phi[a_0, \ldots, a_{n_\phi - 1}]$$
between models of vocabulary τ, sequences $(a_0, \ldots, a_{n_\phi-1})$ of elements of A and formulas $\phi \in L$. It is assumed that this relation satisfies the isomorphism axiom, *that is, if $\pi : \mathcal{A} \cong \mathcal{B}$, then $\mathcal{A} \models \phi[a_0, \ldots, a_{n_\phi-1}]$ and $\mathcal{B} \models \phi[\pi a_0, \ldots, \pi a_{n_\phi-1}]$ are equivalent.*

We call τ the vocabulary *of the logic L.*

Note that no syntax is a priori assumed of a logic. The meaning of "ϕ has a model", and "the theory $T \subset L$ has a model" is obvious. We write $\mathcal{A} \equiv_L \mathcal{B}$ if $\mathcal{A} \models \phi$ and $\mathcal{B} \models \phi$ are equivalent for all $\phi \in L$ with $n_\phi = 0$. We write

[1] This is a little different than usual (e.g. [1, 7]) in that our logics have a fixed vocabulary.

$\mathcal{A} \prec_L \mathcal{B}$ if $\mathcal{A} \models \phi[a_0, \ldots, a_{n_\phi-1}]$ and $\mathcal{B} \models \phi[a_0, \ldots, a_{n_\phi-1}]$ are equivalent for all $\phi \in L$ and all $a_0, \ldots, a_{n_\phi-1} \in A$.

We now define two natural invariants for any logic L:

Definition 4 *The* Löwenheim-Skolem number $\mathrm{LS}(L)$ *of* L *is the smallest cardinal* κ *such that if a theory* $T \subset L$ *has a model, it has a model of cardinality* $< \max(\kappa, |T|)$. *The* Löwenheim-Skolem-Tarski number $\mathrm{LST}(L)$ *of* L *is the smallest cardinal* κ *such that if* $\mathcal{A}$ *is any* τ*-structure, then there is a substructure* $\mathcal{A}'$ *of* $\mathcal{A}$ *of cardinality* $< \kappa$ *such that* $\mathcal{A}' \prec_L \mathcal{A}$.

Note that $\mathrm{LS}(L)$ always exists, because L is a set. In general there is no guarantee that $\mathrm{LST}(L)$ exists, but if it exists, it is at least as big as $\mathrm{LS}(L)$. We can think of the sizes of $\mathrm{LS}(L)$ and $\mathrm{LST}(L)$ as a "test" of how close the logic is to being first order. For first order logic these numbers are both $\aleph_1$, and for $L(Q_1)$ and $L(Q_1^{\mathrm{MM}})$ they are $\aleph_2$. If κ is strongly inaccessible, then $\mathrm{LST}(L_{\kappa\kappa}) = \kappa$.

The Löwenheim-Skolem numbers of $L(I)$ and $L(Q^{ec})$ are quite high in the hierarchy of cardinal numbers, certainly both cardinals are fixed points of the function $\alpha \mapsto \aleph_\alpha$. Whether the Löwenheim-Skolem number of $L(I)$ can be below the first weakly inaccessible, was asked in [16] and has been an open question ever since, but will be settled positively in this paper (Theorem 21). On the other hand, in the inner model L^μ it is easy to see that $\mathrm{LS}(L(I))$ is above the measurable cardinal.

For second order logic, $\mathrm{LS}(L^2)$ is the supremum of Π_2-definable ordinals ([15]), which means that it exceeds the first measurable, the first κ^+-supercompact κ, and the first huge cardinal if they exist.

Theorem 5 ([8]) *1. Suppose* κ *is strong, then* $\mathrm{LS}(L^2) < \kappa$.

2. $\mathrm{LST}(L^2)$ *exists if and only if supercompact cardinals exist, and then* $\mathrm{LST}(L^2)$ *is the first of them.*

Proof. For the first claim, suppose T is a theory in L^2 and T has a model $\mathcal{A}$. We may assume that the universe of $\mathcal{A}$ is an ordinal δ. Let i be an embedding into M with critical point κ such that $T, \mathcal{A}, \mathcal{P}(\delta) \in M$. It is easy to prove by induction on formulas $\phi \in L^2$ that for all $\vec{a} \in A^n$ and $\vec{X} \in \mathcal{P}(A^{n_1}) \times \ldots \times \mathcal{P}(A^{n_k})$ we have

$$\mathcal{A} \models \phi(\vec{a}, \vec{X}) \iff M \models \text{“}\mathcal{A} \models \phi(\vec{a}, \vec{X})\text{”}.$$

The point is that all subsets of A are in M. Thus $M \models \exists x(x \models T$ and $|x| < i(\kappa))$. Hence there is in V a model of T of cardinality $< \kappa$. For the second claim we refer to [8] but give the following argument for $\mathrm{LST}(L^2) \leq \kappa$ for supercompact κ since we will use it later: Suppose $\mathcal{A}$ is a model of cardinality λ. Let i be an elementary embedding of V into a transitive M so that ${}^{\lambda}M \subseteq M$ and $i(\kappa) > \lambda$. Let $\mathcal{B}$ be the pointwise image of $\mathcal{A}$ under i. Since ${}^{\lambda}M \subseteq M$, $\mathcal{B} \in M$. It is easy to prove by induction on formulas $\phi \in L^2$ that for all $\vec{a} \in A^n$ and $\vec{X} \in \mathcal{P}(A^{n_1}) \times ... \times \mathcal{P}(A^{n_k})$ we have

$$\begin{aligned} M \models \text{“}\mathcal{B} \models \phi(\vec{a}, \vec{X})\text{”} &\iff \mathcal{A} \models \phi(\vec{a}, \vec{X}) \\ &\iff M \models \text{“}i(\mathcal{A}) \models \phi(i(\vec{a}), i(\vec{X}))\text{”}. \end{aligned}$$

Thus $M \models \exists \mathcal{B}(\mathcal{B} \prec \mathcal{A}$ and $|B| < i(\kappa))$. Hence there is in V a model $\mathcal{C} \prec \mathcal{A}$ of T of cardinality $< \kappa$. $\square$

So second order logic meets the test of being very far from first order in terms of the size of its Löwenheim-Skolem numbers. We show that according to this test, $L(I)$ and $L(Q^{ec})$ can be close to second order logic but can also be, relatively speaking, close to first order logic.

The strongest large cardinal axiom from the point of view of Löwenheim-Skolem theorems is *Vopenka's Principle*, which states that every proper class of structures of the same vocabulary has two members one of which is isomorphic to an elementary substructure of the other. In [13] an equivalent condition is given: Suppose A is a class. Let us call a cardinal κ *A-supercompact* if for all $\eta > \kappa$ there is $\alpha < \kappa$ and an elementary embedding

$$j : (V_\alpha, \in, A \cap V_\alpha) \to (V_\eta, \in, A \cap V_\eta)$$

with a critical point γ such that $j(\gamma) = \kappa$. It is proved in [13] that Vopenka's Principle is equivalent to the statement that for every class A there is an A-supercompact cardinal. From this and the proof of Theorem 5 we get the following unpublished result of J. Stavi:

Theorem 6 *Vopenka's Principle holds if and only if every logic has a Löwenheim-Skolem-Tarski number.*

For the intermediate logics $L(I)$ and $L(I, Q^{ec})$ the analogue of Theorem 5 (2) is the substantially less conclusive:

Theorem 7 ([14]) *1.* $\mathrm{LST}(L(I))$ *exists only if inaccessible cardinals exist, and then* $\mathrm{LST}(L(I))$ *is at least as large as the first of them.*

2. LST($L(I, Q^{ec})$) *exists only if Mahlo cardinals exist, and then the cardinal* LST($L(I, Q^{ec})$) *is at least as large as the first of them.*

Proof. Let $\mathcal{A} = (R(\kappa^+), \epsilon)$, where $\kappa = \mathrm{LST}(L(I))$. By the definition of LST($L(I)$) there is a transitive set M of power $< \kappa$ and a monomorphism $i : (M, \epsilon) \to \mathcal{A}$ which preserves $L(I)$-truth. Moreover, every M-cardinal is a real cardinal. Let λ be the largest cardinal in M. Clearly $i(\lambda) = \kappa > \lambda$. Let γ be the first ordinal moved by i. Trivially, γ is a limit cardinal. Suppose $f \in M$ is a cofinal δ-sequence in γ for some $\delta < \gamma$. Now $i(f)$ is a cofinal δ-sequence in $i(\gamma)$ whence $i(f)(\beta) > \gamma$ for some $\beta < \delta$. But $i(f)(\beta) = i(f(\beta)) = f(\beta) < \gamma$. Thus γ is weakly inaccessible in M, and therefore, $i(\gamma)$ is weakly inaccessible in V. The second claim is proved similarly. □

The results of this paper explain why Theorem 7 is weaker than Theorem 5. The proof theoretic strength of the existence of either LST($L(I)$) or LST($L(I, Q^{ec})$) exceeds substantially what follows from the mere size of these cardinals. Accordingly, and unlike LST(L^2), the numbers LST($L(I)$) and LST($L(I, Q^{ec})$) do not have to be very high in the scale of large cardinals. We will show in this paper that LST($L(I)$) can be the first weakly inaccessible cardinal and LST($L(I, Q^{ec})$) can respectively be the first Mahlo cardinal. Also they can be of continuum size:

Theorem 8 ([14]) *Suppose κ is a supercompact cardinal and $\mathcal{P}$ is the notion of forcing C_κ. Let L be a provably $\mathcal{C}$-absolute logic which is provably a sublogic of L^2. Then*

$$\Vdash_{\mathcal{P}} \mathrm{LST}(L(I,R)) \leq 2^\omega.$$

Proof. We give an outline of the proof for completeness. Suppose $\mathcal{A}$ is a name for a finitary structure with universe λ in the $\mathcal{P}$-forcing language. Let $i : V \to M$ be an elementary embedding of the universe such that $i(\kappa) > \lambda, {}^\lambda M \subseteq M$ and $i``\kappa = \kappa$. Let $\mathcal{B}$ be the point-wise image of $\mathcal{A}$ under i. Using the fact that $\mathcal{P}$ preserve cardinals and cofinalities it is possible to show

$$M \models \text{``} \Vdash_{i(\mathcal{P})} i : \mathcal{B} \to_{L(I,R)} i(\mathcal{A})\text{''}. \quad (1)$$

It follows from this that

$$M \models \text{``} \Vdash_{i(\mathcal{P})} i(\mathcal{A}) \text{ has an } L\text{-elementary substructure of power } < i(\kappa)\text{''}.$$

Therefore

$$\Vdash_{\mathcal{P}} \mathcal{A} \text{ has an } L\text{-elementary substructure of power } < \kappa.$$

□

To see some of the strength of the Löwenheim-Skolem-Tarski Theorem for the equicardinality quantifier, let us recall the following observation from [14]: Let $\mathcal{A}$ be the structure $(R(\kappa^+), \in)$. Let $\pi : (M, \in) \to (R(\kappa^+, \in)$ be an elementary embedding with M transitive and $|M| < \kappa$. If $\delta = M \cap On$, then $\pi \restriction L_\delta : (L_\delta, \in) \to (L_{\kappa^+}, \in)$. Thus $0^\#$ exists. Obviously this argument can be considerably strengthened. We show in this paper that the existence of $\mathrm{LST}(L(I))$ has enough combinatorial power to imply, when combined with current state of the inner model technology, Projective Determinacy.

2 The Failure of Squares

We have already alluded to the fact that the existence of $\mathrm{LST}(L(I))$ has non-trivial consistency strength, for example, it implies $0^\#$. In this section we show that the existence of $\mathrm{LST}(L(I))$ has a much stronger consistency strength, probably at the level of a supercompact cardinal.

We shall show that the existence of $\mathrm{LST}(L(I))$ implies that the combinatorial principle $\Box_\lambda$ fails for *every* λ above it. For λ singular of cofinality ω we can do better than that and show that any reasonable version of $\Box_\lambda$ fails, in particular a consequence of any reasonable weakening of $\Box_\lambda$ (for $\mathrm{cof}(\lambda) = \omega$) fails globally above $\mathrm{LST}(L(I))$. The consequence we allude to is the existence of "good" scales.

We shall conclude this section by showing that assuming the consistency of a supercompact cardinal it is consistent that the first $\mathrm{LST}(L(I))$ cardinal is the same as the first supercompact cardinal.

Definition 9 *The* square principle $\Box_\lambda$ *says: There is a sequence* $\langle C_\alpha : \alpha < \lambda^+ \text{ a limit ordinal } \rangle$*, such that:*

1. *C_α is closed unbounded subset of α.*
2. *The order type of C_α is always $\leq \lambda$.*
3. *If β is a limit point of C_α, then $C_\beta = C_\alpha \cap \beta$.*

Theorem 10 *If κ is an* $\mathrm{LST}(L(I))$ *number and $\lambda \geq \kappa$, then $\Box_\lambda$ fails.*

Proof. Suppose $\langle C_\alpha : \alpha < \lambda^+ \text{ a limit ordinal } \rangle$ is a $\square_\lambda$ sequence. Consider the structure

$$\mathcal{A} = \langle \lambda^+, \lambda, T, C \rangle,$$

where T is a function defined on the limit ordinals in λ^+ such that $T(\alpha)$ is the order-type of C_α, and C is a ternary relation such that $C(\alpha, \gamma, \eta)$ holds if and only if "η is the γ-th member of C_α". Let $\mathcal{B}$ be an $L(I)$-elementary substructure of $\mathcal{A}$ of cardinality $< \kappa$. It is easily verified that the order-type of the universe B of $\mathcal{B}$ is a successor cardinal μ^+, where μ is the order-type of $B \cap \lambda$. Let $\mathcal{B}^*$ be the transitive collapse of $\mathcal{B}$. It is easily seen that $\mathcal{B}^*$ has the form $\langle \mu^+, \mu, T^*, C^* \rangle$, where for some $\square_\mu$-sequence $\langle C^*_\alpha : \alpha < \mu^+ \text{ a limit ordinal } \rangle$, T^* is a function defined on the limit ordinals in μ^+ such that $T^*(\alpha)$ is the order-type of C^*_α, and C^* is a ternary relation such that $C^*(\alpha, \gamma, \eta)$ holds if and only if "η is the γ-th member of C^*_α". Let $\pi : B^* \to B$ be the inverse of the transitive collapse of $\mathcal{N}$. Let $\delta = \sup(B) = \sup(\pi'' B^*)$. Note that $\delta < \lambda^+$ and $\mathrm{cof}(\delta) = \mu^+$. C_δ is a closed unbounded subset of δ, $B = \{\pi(\alpha) : \alpha < \mu^+\} = \pi'' B^*$ is cofinal in δ. So the set

$$A' = \{\eta < \delta : \eta \text{ limit point of } C_\delta \text{ and a limit point of } B\}$$

is closed unbounded in B_δ.

For $\eta \in A'$ let $\bar{\eta}$ be the minimal element of B^* such that $\pi(\bar{\eta}) \geq \eta$. Obviously, $\bar{\eta}$ is always defined because $\sup(A') = \sup \pi'' B^* = \delta$. And if $\eta_1 < \eta_2$ in A', then $\bar{\eta}_1 < \bar{\eta}_2$.

Claim: For $\eta \in A'$, η is a limit point of $C_{\pi(\bar{\eta})}$.

Proof. Otherwise, let ρ be $\sup(C_{\pi(\bar{\eta})} \cap \eta)$. So our assumption is $\rho < \eta$. As $\eta \in A'$, the range of π is cofinal in η, so there is ρ' such that

$$\rho < \pi(\rho') < \eta \leq \pi(\bar{\eta}).$$

By elementarity there is ρ'' such that

$$\pi(\rho') < \pi(\rho'') \in C_{\pi(\bar{\eta})}.$$

But clearly $\pi(\rho'') < \eta$, so we get a contradiction. $\square$ (Claim.)

Claim: If $\eta_1 < \eta_2$ are in A', then $\bar{\eta}_1$ is a limit point of $C_{\bar{\eta}_2}$.

Proof. Otherwise, let ρ be $\sup(C^*_{\pi(\bar{\eta}_2)} \cap \bar{\eta}_1)$. We assume $\rho < \bar{\eta}_1$, which means by definition of $\bar{\eta}_1$ that $\pi(\rho) < \eta_1$. By the previous claim η_2 is a limit point of $C_{\pi(\bar{\eta}_2)}$. So by the definition of the square principle

$$C_{\eta_2} = C_{\pi(\bar{\eta}_2)} \cap \eta_2.$$

Note that $\eta_2 \geq \pi(\bar{\eta}_1)$. By elementarity of π

$$\pi(\rho) = \sup(C_{\pi(\bar{\eta}_2)} \cap \pi(\bar{\eta}_1)) = \sup(C_{\eta_2} \cap \pi(\bar{\eta}_1)). \tag{2}$$

On the other hand η_1 and η_2 are in A', hence they are limit points of C_δ, so $C_{\eta_2} = C_\delta \cap \eta_2$, so η_1 is a limit point of C_{η_2}. This contradicts (2). $\Box$ (Claim.)

It follows from the previous claim that if $\eta_1, \eta_2 \in A'$, then the order-type of $C^*_{\bar{\eta}_2}$ exceeds the order-type of $C^*_{\bar{\eta}_1}$. So $T^*(\bar{\eta}_2) > T^*(\bar{\eta}_1)$. The set A' being cofinal in δ, it has order-type at least μ^+, so T^* is a monotone function from a set of ordinals of order-type $\geq \mu^+$ into μ, which is clearly a contradiction. $\Box$ (Theorem)

By varying κ we get the following list of weaker and weaker principles.

Definition 11 *The* weak square principle $\Box_{\kappa,\lambda}$ *says: There is a sequence* $\langle \mathcal{C}_\alpha : \alpha < \kappa$ *a limit ordinal*$\rangle$*, such that:*

1. $\mathcal{C}_\alpha$ *is a set of closed unbounded subsets of* α*.*
2. $|\mathcal{C}_\alpha| \leq \lambda$
3. *The order type* $\mathrm{otp}(C)$ *of each member* C *of* $\mathcal{C}_\alpha$ *is* $\leq \kappa$*.*
4. *If* $C \in \mathcal{C}_\alpha$ *and* $\beta \in \lim(C)$*, i.e.* β *is a limit point of* C*, then* $\mathcal{C} \cap \beta \in \mathcal{C}_\beta$*.*

The principle $\Box_{\lambda,\lambda^+}$, the so-called "silly square" is actually provable (see the proof of Lemma 17), so the weakest reasonable principle is $\Box_{\lambda,\lambda}$. Our goal is now to show that if λ is singular of cofinality ω and above $\mathrm{LST}(L(I))$, then $\Box_{\lambda,\lambda}$ fails. This fact by itself indicates that the assumption of the existence of a $\mathrm{LST}(L(I))$ cardinal has a large consistency strength. At the present it is not known how to get a model in which $\Box_{\lambda,\lambda}$ fails even for a single singular λ without assuming a supercompact cardinal.

The way we shall prove the failure of $\Box_{\lambda,\lambda}$ is by refuting an even weaker property: "The existence of a good sequence in λ^ω/FIN of length λ^+". The definitions and facts about "good sequences in λ^ω/FIN" are due to Shelah and based on his pcf theory ([12]). Since we shall need a much simpler version of the notions and the basic lemmas, we include them for the sake of completeness.

We consider elements of On^ω ordered by eventual domination, i.e. for $f, g \in \mathrm{On}^\omega$

$$f <^* g \text{ if } f(n) < g(n) \text{ for all but finitely many } n < \omega.$$

Definition 12 *Suppose $\langle f_\alpha : \alpha < \mu \rangle$ is a $<^*$-increasing sequence in* On^ω.

(i) A point $\delta \in \mu$ is called a good *point for the sequence if there is a cofinal set $C \subseteq \delta$ and a function $\alpha \mapsto n_\alpha$ from C to ω such that if $\alpha < \beta$ in C and $k > \max(n_\alpha, n_\beta)$, then $f_\alpha(k) < f_\beta(k)$.*

(ii) The sequence is good*, if there is a closed unbounded subset D of μ such that $\delta \in D$ implies that δ is a good point of the sequence.*

Lemma 13 *Suppose δ is a good point for the sequence $\langle f_\alpha : \alpha < \mu \rangle$ and D is any cofinal subset of δ. Then there is $E \subseteq D$ witnessing the goodness of δ.*

Proof. Let C and $\alpha \mapsto n_\alpha$ witness the goodness of δ. W.l.o.g. $\mathrm{otp}(C) = \mathrm{otp}(D) = \mathrm{cof}(\delta)$. Let $E \subseteq D$ be chosen so that for every $\gamma \in E$ there are $\gamma^-, \gamma^+ \in C$ in such a way that $\gamma^- < \gamma < \gamma^+$ and if $\gamma < \eta \in E$, then $\gamma^+ \leq \eta^-$. Let $m_\gamma \in \omega$ (for $\gamma \in E$) be such that if $i > m_\gamma$, then $f_{\gamma^-}(i) < f_\gamma(i) < f_{\gamma^+}(i)$. Let $n^*_\gamma = \max(n_{\gamma^-}, n_{\gamma^+}, m_\gamma)$. Now, if $i \geq \max(n^*_\gamma, n^*_\eta)$, then

$$f_\gamma(i) < f_{\gamma^+}(i) \leq f_{\eta^-}(i) < f_\eta(i).$$

$\Box$ (Lemma)

Theorem 14 (Shelah [12], see also [2] p.18) *If $\mathrm{cof}(\lambda) = \omega$ and $\Box_{\lambda,\lambda}$ holds, then there is a good sequence in λ^ω of length λ^+.*

Proof. Fix a sequence of regular cardinals λ_n cofinal in λ. We shall actually get our sequence in $\prod_{n<\omega} \lambda_n \subseteq \lambda^\omega$. Note that every sequence of functions in $\prod_{n<\omega} \lambda_n$ of size λ has a $<^*$-upper bound in $\prod_{n<\omega} \lambda_n$ (By taking $g(n) =$ the supremum of $f_n(n)$ for the first λ_{n-1} of our functions).

Fix a $\Box_{\lambda,\lambda}$-sequence $\langle \mathcal{C}_\alpha : \alpha < \kappa \text{ a limit ordinal} \rangle$. Without loss of generality we can assume that $\mathrm{otp}(C) < \lambda$ for each $C \in \mathcal{C}_\alpha$. (Indeed, if $\mathrm{otp}(C) = \lambda$ when $C \in \mathcal{C}_\alpha$, then $\mathrm{cof}(\alpha) = \omega$ and we can replace C by an ω-sequence cofinal in α. Note that this C is never used as an initial segment of $D \in \mathcal{C}_\beta$ for $\alpha < \beta$ because it would imply $\mathrm{otp}(D) > \lambda$).

We define the $<^*$-increasing sequence $\langle f_\alpha : \alpha < \lambda^+ \rangle$ in $\prod_{n<\omega} \lambda_n$ by induction. The successor stage is trivial: $f_{\alpha+1}(n) = f_\alpha(n)$. Suppose then α is limit. For each $C \in \mathcal{C}_\alpha$ we define a function in $\prod_n \lambda_n$ as follows:

$$g_C(i) = \begin{cases} \sup_{\beta \in C} g_\beta(i), & \text{if } \mathrm{otp}(C) < \lambda_i, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\mathcal{C}_\alpha \leq \lambda$, we can find $f_\alpha \in \prod_n \lambda_n$ such that $g_C <^* f_\alpha$ for every $C \in \mathcal{C}_\alpha$. Clearly $f_\beta <^* f_\alpha$ for every $\beta < \alpha$. We claim that $\langle f_\alpha : \alpha < \lambda^+ \rangle$ is a good sequence. Actually the claim is that every limit $\delta < \lambda^+$ is a good point of the sequence. If $\mathrm{cof}(\delta) = \omega$, then we pick a cofinal sequence $\langle \delta_n : n < \omega \rangle$ in δ. Let $n(\delta_n)$ be such that for $i \geq n(\delta_n)$ we have

$$f_{\delta_{n-1}}(i) < f_{\delta_n}(i) < f_{\delta_{n+1}}(i).$$

Clearly the set $\{\delta_n : n < \omega\}$ and the map $\delta_n \mapsto n(\delta_n)$ witness the goodness of δ. If $\mathrm{cof}(\delta) > \omega$, pick $C \in \mathcal{C}_\delta$ and let C^* be the set of limit points of C. Let n be such that $\mathrm{otp}(C) < \lambda_n$ and also $g_C(i) \leq f_\alpha(i)$ for $i \geq n$. If $\beta < \beta' \in C^*$ and if $i \geq \max(n_\beta, n_{\beta'})$, we get $f_\beta(i) < g_{C \cap \beta'}(i) < f_{\beta'}(i)$ (because $\beta \in C \cap \beta'$). So the set C^* and the map $\beta \mapsto n_\beta$ witnesses the goodness of δ. $\Box$ (Theorem)

The result for the existence of $\mathrm{LST}(L(I))$ number follows from

Theorem 15 *Suppose* $\kappa = \mathrm{LST}(L(I))$ *and* $\lambda \geq \kappa$ *with* $\mathrm{cof}(\lambda) = \omega$. *Then there is no no good sequence in* λ^ω *of length* λ^+.

Proof. Suppose $\mathrm{cof}(\lambda) = \omega$. Suppose that $\langle f_\alpha : \alpha < \lambda^+ \rangle$ is a good sequence in λ^ω. Suppose D is a cub on λ^+ such that all points of D of cofinality $> \omega$ are good. Let

$$F = \{(\alpha, \beta, \gamma) \in \lambda^+ \times \omega \times \lambda : f_\alpha(\beta) = \gamma\},$$

and

$$\mathcal{A} = \langle \lambda^+, \lambda, <, F, D \rangle.$$

Since $\kappa = LST(L(I))$ there is

$$\mathcal{B} = \langle B, B \cap \lambda, <, F \cap B^3, D \cap B \rangle \prec_{L(I)} \mathcal{A}$$

such that $|B| < \kappa$. Of course, $\omega \subset B$. Since

$$\forall x \neg Iyz(y < x)(z = z)$$

$$\forall x(x < \lambda \rightarrow \neg Iyz(y < x)(z < \lambda))$$

$$\forall y(\lambda < y \rightarrow Iuv(u < \lambda)(v < y))$$

are true in $\mathcal{A}$, they are true in $\mathcal{B}$ and it follows that for some cardinal $\mu < \kappa$, $\mathrm{otp}(B)$ is μ^+ and $\mathrm{otp}(B \cap \lambda) = \mu$. Let $\delta = \sup(B)$. Note that δ is a limit point of D, and hence $\delta \in D$. Since B is cofinal in δ, $\mathrm{cof}(\delta) = \mu^+$. By elementarity, each function f_α, $\alpha \in B$, maps ω into $B \cap \lambda$. We now argue that δ cannot be a good point of $\langle f_\alpha : \alpha < \lambda^+ \rangle$. Assume otherwise. Then there is a cofinal set $C \subseteq \delta$ and a function $\alpha \mapsto n_\alpha$ from C to ω such that if $\alpha < \beta$ in C and $k \geq \max(n_\alpha, n_\beta)$, then $f_a(k) < f_\beta(k)$. By Lemma 13 we may assume $C \subseteq B$. Let C' be cofinal in C such that n_α is a fixed integer N for all $\alpha \in C'$. Now $\{f_\alpha(N) : \alpha \in C'\}$ is a subset of $B \cap \lambda$ which is of order-type μ^+, a contradiction. □ (Theorem)

Corollary 16 *If $\kappa = \mathrm{LST}(L(I))$, then $\square_{\lambda,\lambda}$ fails for every singular $\lambda \geq \kappa$ of cofinality ω. Hence, in particular, PD holds.*

The existence of $\mathrm{LST}(L(I))$ also implies the Singular Cardinals Hypothesis above κ, i.e. if λ is singular $\geq \kappa$, then

$$\text{(SCH)} \qquad \lambda^{\mathrm{cof}(\lambda)} = \max(\lambda^+, 2^{\mathrm{cof}(\lambda)}).$$

It follows from Silver's singular cardinals theorem that if λ violates the SCH and $\mathrm{cof}(\lambda) > \omega$, then λ is a limit of cardinals that violate the SCH.

Lemma 17 ([12]) *If λ is a singular cardinal of cofinality ω and λ violates the SCH, then there is a good sequence in λ^ω of length λ^+.*

Proof. By Shelah [12], if λ violates SCH and $\mathrm{cof}(\lambda) = \omega$, then there is a sequence $\langle \lambda_n : n < \omega \rangle$ cofinal in λ such that $\prod_n \lambda_n/FIN$ has true cofinality λ^{++}, which implies that every set of functions in $\prod_n \lambda_n$ of cardinality λ^+ has a $<^*$-upper bound in $\prod_n \lambda_n$. Now one can repeat the proof Theorem 14 by replacing in that proof the $\square_{\lambda,\lambda}$-sequence by a $\square_{\lambda,\lambda^+}$-sequence (the "Silly Square") and getting the good sequence in $\prod_n \lambda_n$. The silly square is always true, for if C_α is a cub subset of α of order type $\mathrm{cof}(\alpha)$, we can let $\mathcal{C}_\alpha = \{C_\beta \cap \alpha : \beta < \lambda^+,\ \alpha \text{ limit point of } C_\beta\}$ and then $\langle \mathcal{C}_\alpha : \alpha < \lambda^+ \rangle$ witnesses $\square_{\lambda,\lambda^+}$. The proof works as before using the fact that for every $\alpha < \lambda^+$ $|\mathcal{C}_\alpha| \leq \lambda^+$ and that every set of functions in $\prod_n \lambda_n$ of cardinality λ^+ has a $<^*$-upper bound. □ (Lemma)

Corollary 18 *If $\kappa = \mathrm{LST}(L(I))$, then SCH holds above κ.*

Theorem 19 *If it is consistent to assume the existence of a supercompact cardinal, then it is its consistent to assume that $\mathrm{LST}(L(I))$ is the first supercompact cardinal.*

Proof. We refer to Magidor [9]. In this paper, assuming the existence of a supercompact cardinal, a model is constructed in which the first supercompact is the first strongly compact. It is achieved by forcing over a model in which κ is supercompact and arranging for SCH to fail for unboundedly many λ's below κ, while preserving the supercompactness of κ. In the resulting model $\kappa \geq \mathrm{LST}(L(I))$ (even $\kappa \geq \mathrm{LST}(L^2)$). If $\mu < \kappa$ and $\mu \geq \mathrm{LST}(L(I))$, then pick $\mu < \lambda < \kappa$ violating SCH to get a contradiction with Corollary 18. So $\kappa = \mathrm{LST}(L(I))$ cardinal. $\square$ (Theorem)

In the next section we shall show that $\mathrm{LST}(L(I))$ can be much smaller than the first supercompact cardinal, namely it can be the first inaccessible, so we are in a true "identity crisis" situation.

3 The First Mahlo Cardinal

As we pointed out in Theorem 7, $\mathrm{LST}(L(I,Q^{ec}))$ is, if it exists at all, at least as big as the first Mahlo cardinal. We now prove the consistency of $\mathrm{LST}(L(I,Q^{ec}))$ being actually equal to the first Mahlo cardinal. As Corollary 16 shows, we have to start from a cardinal substantially larger than a Mahlo, even a strong cardinal is not enough. So we start from a supercompact cardinal.

Theorem 20 *It is consistent, relative to the consistency of a supercompact cardinal, that* $\mathrm{LST}(L(I,Q^{ec}))$ *is the first Mahlo cardinal.*

Proof. Suppose κ is supercompact. We then make every $\rho < \kappa$ non-Mahlo. Suppose ρ is Mahlo. Let $\mathcal{P}_\rho$ be the set of closed bounded sets of singular cardinals $< \rho$ inversely ordered by end-extension, i.e. a weaker condition is an initial segment of a stronger condition. For every regular $\lambda < \rho$ the forcing notions $\mathcal{P}_\rho$ contains a λ-closed dense set $\{C : \max(C) > \lambda\}$. Therefore $\mathcal{P}_\rho$ cannot collapse cardinals $< \rho$ or change their cofinality. Moreover, $\mathcal{P}_\rho$ does not add new bounded subsets to ρ. On the other hand, $|\mathcal{P}_\rho| = \rho$, so $\mathcal{P}_\rho$ preserves all cardinals and cofinalities. In particular $\mathcal{P}_\rho$ kills the Mahloness of ρ but preserves inaccessibility of ρ. Now we iterate this forcing. Suppose μ_α, $\alpha < \delta$, is an increasing sequence of Mahlo cardinals. Let $R_0 = \mathcal{P}_{\mu_0}$. Suppose $\mathcal{P}_\alpha$ has been defined. Let $\mathcal{P}_\alpha \Vdash \tilde{\mathrm{Q}}_\alpha = \mathcal{P}_{\mu_\alpha}$ and $R_{\alpha+1} = R_\alpha \star \tilde{\mathrm{Q}}_\alpha$. For limit α let R_α be the direct limit of the previous stages, if α is inaccessible, and inverse limit otherwise. This will ensure that each R_α, α inaccessible, will have the α-c.c. and will therefore preserve the Mahloness of each μ_β, $\beta \geq \alpha$. Let $R = R_\kappa$. Now $V^R \models \kappa = \mathrm{LST}(L(I, Q^{ec})$. To see why, suppose $\mathcal{A}$ is a structure with universe λ, where $\lambda \geq \kappa$. Since κ is supercompact, there is $j : V \to M$, M transitive, such that ${}^\lambda M \subseteq M$ and $j(\kappa) > \lambda$. Note that $j(R) = R \star \mathcal{P}_\kappa \star R_{>\kappa}$, where $R_{>\kappa}$ contains a κ-closed dense set. Since R has the κ-c.c. and $R_{>\kappa}$ is sufficiently closed, j can be extended to $j^* : V^R \to M^{j(R)}$. Now we can continue as in the proof of Theorem 8. $\Box$

4 The First Inaccessible Cardinal

In this section we prove the main result of this paper:

Theorem 21 *If ZFC+ "There is a supercompact cardinal" is consistent, so is ZFC+ "There is an inaccessible cardinal"+ "*$\mathrm{LST}(L(I))$ *is the first inaccessible cardinal".*

The assumption of the consistency of a supercompact cardinal seems, on the basis of present technology, almost unavoidable. By Theorem 15 above, we know that the existence of $\mathrm{LST}(L(I))$ implies the negation of $\Box_{\lambda,\lambda}$ for every large enough λ. The only known way to get a model in which this holds is to start from a strongly compact cardinal. But the definition of strong compactness is not sufficient for getting reflection principles which seem to be necessary for getting the existence of $\mathrm{LST}(L(I))$, so the assumption of a supercompact cardinal seems natural enough.

4.1 Outline of the Proof

We start with a supercompact cardinal κ. In our final model κ will be the first inaccessible cardinal, while preserving enough of the reflection properties of a supercompact cardinal, so that in the model κ will be $\mathrm{LST}(L(I))$.

In the process of achieving this we force a closed unbounded set C of singular cardinals below κ. This will make κ non-Mahlo. We then collapse cardinals between consecutive elements of C so that none of them can be inaccessible. Thus κ has become the first inaccessible. But we have to be careful about the way in which we collapse cardinals in order to maintain enough reflection properties of κ, so that κ will be $\mathrm{LST}(L(I))$. The argument that $\mathrm{LST}(L(I)) = \kappa$ in the final model is similar to the argument of Theorem

4. Namely, suppose $\mathcal{P}$ is the forcing used to get our final model and $\mathcal{A}$ is a name for a finitary structure in $V^{\mathcal{P}}$ with domain λ. Let $i : V \to M$ be an elementary embedding of the universe such that $i(\kappa) > \lambda$ and ${}^{\lambda}M \subseteq M$, where κ is the critical point of i (V is our ground model). $\mathcal{P}$ will be a forcing such that $\mathcal{P}$ as a forcing notion is a regular subforcing of $i(\mathcal{P})$. In $V^{i(\mathcal{P})}$ we can define an embedding $i^* : V^{\mathcal{P}} \to M^{i(\mathcal{P})}$ extending i. By our assumption $i^* \restriction \mathcal{A} \in V^{\mathcal{P}}$, and $i^* \restriction \mathcal{A}$ is an embedding of $\mathcal{A}$ into $i^*(\mathcal{A})$. We would like i^* to preserve formulas of $L(I)$. Given that we are done because

$$M^{i(\mathcal{P})} \models \text{“}i^*(\mathcal{A}) \text{ has an } L(I)\text{-elementary substructure of cardinality } \lambda < i^*(\kappa)\text{”}.$$

By elementarity,

$$V^{\mathcal{P}} \models \text{“}\mathcal{A} \text{ has an } L(I)\text{-elementary substructure of cardinality} < \kappa\text{”}.$$

To get i^* to preserve formulas of $L(I)$ we need that $i(\mathcal{P})/\mathcal{P}$ collapses no cardinals $\leq \lambda$.

Suppose that when we collapsed cardinals between consecutive members of C we had some function $f : \kappa \to \kappa$ such that for a member δ of C no cardinal was collapsed between δ and $f(\delta)$. Let us also assume that $\lambda < i(f)(\kappa)$. (Note that κ is a limit point of $i^*(C)$.) So no cardinal between κ and $i(f)(\kappa)$ will be collapsed. In particular, all cardinals between κ and λ are preserved by $i(\mathcal{P})/\mathcal{P}$.

Another issue is that κ is supposed to be a limit point of $i^*(C)$ hence in $M^{i(\mathcal{P})}$ it is supposed to be singular. In $V^{\mathcal{P}}$ it is supposed to be regular, indeed inaccessible. So we need $i(\mathcal{P})/\mathcal{P}$ to make some regular cardinals singular. Since $i(\mathcal{P})$ “looks like $\mathcal{P}$”, we need $\mathcal{P}$ to make enough regular cardinals singular, so that $M^{i(\mathcal{P})} \models$ “κ is singular”.

The standard way of making a regular cardinal singular is by forcing with Prikry forcing on a measurable cardinal. Since we shall have to do it for many cardinals below κ, we shall have to iterate Prikry type forcings for somewhat supercompact cardinals below κ.

So the forcing notion we shall use will be an iteration of several steps:

(a) Iterated Prikry type forcing for every λ^+-supercompact $\lambda < \kappa$, where besides changing the cofinality of λ to ω we do some preparatory forcing for the additional steps, which will be relevant only to κ. We denote this forcing by Q_{λ} and the iteration up to κ by $\mathcal{P}_{\kappa}$.

(b) On κ we force a closed unbounded set C such that every limit point of C is singular. We denote this forcing by $\mathrm{NM}(\kappa)$ (From "Non-Mahlo").

(c) We collapse cardinals between consecutive members of C making sure that if $\beta \in C$, then no cardinals are collapsed between β and $f(\beta)$ for an appropriate function $f : \kappa \to \kappa$. (We denote this forcing $\mathrm{Col}(C)$).

The challenge will be to make sure that $\mathrm{NM}(\kappa) * \mathrm{Col}(C)$ embeds nicely into $\mathrm{Q}_\kappa * \mathrm{Col}(C)$ so that if $R = \mathcal{P}_\kappa * \mathrm{NM}(\kappa) * \mathrm{Col}(C)$, then R embeds nicely into $i^*(R)$. This will be achieved by embedding $\mathrm{NM}(\kappa) * \mathrm{Col}(C)$ into $\mathrm{Q}_\kappa * \mathrm{Col}(C)$. Note that Q_κ is the κ-th stage in the iteration of $i^*(\mathcal{P}_\kappa) = \mathcal{P}_{i^*(\kappa)}$. We hope that these remarks make the following definition of the forcing notion somewhat less frightening.

4.2 The Forcing Construction

Our first step is to define the function $f : \kappa \to \kappa$ that will determine intervals where all the cardinals will be preserved. We assume that our ground model satisfies G.C.H. and that there is no inaccessible above κ. A classical lemma of Laver [6] proves the following:

Lemma 22 *Let κ be supercompact. Then there exists a function $h : \kappa \to V_\kappa$ such that for every x and every $\mu \geq \kappa$ there is a μ-supercompact embedding $j : V \to M$ (i.e. $M^\mu \subseteq M$, $j(\kappa) > \mu$, $j(\alpha) = \alpha$ for $\alpha < \kappa$), such that $j(h)(\kappa) = x$.*

An easy corollary of Laver's lemma is the following:

Lemma 23 *Let κ be supercompact such that there is no inaccessible cardinal above κ. Then there is a function $f : \kappa \to \kappa$ such that for all $\alpha < \kappa$, $\alpha < f(\alpha)$, $f(\alpha)$ is regular, there is no inaccessible cardinal λ with $\alpha < \lambda \leq f(\alpha)$, and for all $\mu \geq \kappa$ there is a μ-supercompact embedding $j : V \to M$ with $\mu < j(f)(\kappa)$.*

Proof Let h be the Laver-function from Lemma 22. Let $f(\alpha) = (h(\alpha))^+$ if $h(\alpha)$ is an ordinal $> \alpha$ such that there is no inaccessible cardinal λ with $\alpha < \lambda \leq h(\alpha)$. Define $f(\alpha) = \alpha^+$ otherwise. Apply Lemma 22 for μ and $x = \mu$. Note that in M there is no inaccessible cardinal λ with $\kappa < \lambda \leq$

$j(h)(\kappa) = \mu^+$ so for $j(f)(\kappa)$ the first possibility of the definition of f holds and hence $j(f)(\kappa) = \mu^+ > \mu$. $\square$

So from now on we fix such a function f. Note that the first inaccessible cardinal is closed under f. The cardinals that we shall be interested in will be cardinals $\lambda \leq \kappa$ such that λ is λ^+-supercompact and λ is closed under the function f. For such λ we define the forcing notion $\mathrm{NM}(\lambda)$, which is intended to make λ non-Mahlo by forcing a closed unbounded set C of cardinals such that every limit point of C is singular. For technical simplicity it will be convenient to assume that if $\beta \in C$ and β' is the minimal member of C above β, then β' is inaccessible and $f(\beta) < \beta'$.

Definition 24 *Suppose λ is a λ^+-supercompact cardinal. Then $\mathrm{NM}(\lambda)$ is the set of all closed bounded subsets C of λ such that*

(a) *Every member of C is a cardinal.*

(b) *If β is a limit point of C, then β is singular.*

(c) *If $\beta \in C$, and β' is the first point of C above β, then β' is inaccessible.*

The partial order $\leq$ on $\mathrm{NM}(\lambda)$ is defined by $D \leq C$ iff $D, C \in \mathrm{NM}(\lambda)$ and D is an end-extension of C.

So the successor members of C are all regular, and limit points of C are closed under f. It is easy to see that if $C \in \mathrm{NM}(\lambda)$, and C contains a point above $\mu < \lambda$, then $\{D : D \leq C\}$ is $\mu-$*closed.* Hence it follows that forcing with $\mathrm{NM}(\lambda)$ introduces no new μ-sequences of ordinals when $\mu < \lambda$. So λ remains regular, and since no new bounded subsets of λ are introduced, λ remains strongly inaccessible. Also, it is easy to see that if $G \subseteq \mathrm{NM}(\lambda)$ is a generic filter, then $\bigcup G$ is a closed unbounded subset of λ. Every limit point of $\bigcup G$ is singular, so in the generic extension λ is a non-Mahlo inaccessible cardinal.

Since we are going to define many partial orders, we shall denote each of the relevant partial orders by $\leq$. Only in case of a possible confusion we shall add the subscript indicating the forcing notion ($\leq_P$ for the partial order of $\mathcal{P}$). Also in some cases it will be convenient to define a preorder on the forcing notion (so we may write $p \leq q$ and $q \leq p$), so that we really mean the forcing notion is the equivalence classes of the relation "$p \leq q$ and $q \leq p$".

Given two regular cardinals $\mu < \rho$, $\mathrm{Col}(\mu, < \rho)$ is the usual Levy collapse of all the cardinals δ with $\mu < \delta < \rho$ to μ. It is a μ-closed forcing notion and if ρ is inaccessible or the successor of a cardinal ν such that $\nu^{<\nu} = \nu$, the forcing notion $\mathrm{Col}(\mu, < \rho)$ satisfies the ρ-c.c.

Let C be a closed set of cardinals. For $\beta \in C \cap \sup(C)$ let β' be the next point of C after β. We assume that if $\beta \in C$, then β' is inaccessible and $f(\beta) < \beta'$. The forcing notion

$$\mathrm{Col}(C)$$

is defined to be the Easton product of $\mathrm{Col}(f(\beta), < \beta')$ for $\beta \in C \cap \sup(C)$. (Easton product means that it is the collection of all functions g defined on $C \cap \sup(C)$ such that $g(\beta) \in \mathrm{Col}(f(\beta), < \beta')$ and for regular δ the set $\{\beta < \delta : g(\beta) \neq \emptyset\}$ is bounded in δ.) Note that for our case, if $C \subseteq \lambda$ is the closed unbounded set introduced by $\mathrm{NM}(\lambda)$, then the Easton condition simply means that if $g \in \mathrm{Col}(C)$ then the cardinality of $\{\beta : g(\beta) \neq \emptyset\}$ is less than λ.

It is easy to see that if C is $\mathrm{NM}(\lambda)$-generic and the first member of C is below the first inaccessible, then if we force with $\mathrm{Col}(C)$, then λ will be the first inaccessible.

4.2.1 Atomic Step

Let λ be a λ^+-supercompact cardinal $\leq \kappa$. We shall describe a variation Q_λ of Prikry forcing for making λ singular of cofinality ω, while at the same time introducing a generic object over V to $\mathrm{NM}(\lambda)$. Like Prikry forcing, Q_λ will introduce no new unbounded subsets of λ. Q_λ has an additional role which we shall explain below. Note that we have

$$V \subseteq V^{\mathrm{NM}(\lambda)} \subseteq V^{Q_\lambda}.$$

(Note also that in $V^{\mathrm{NM}(\lambda)}$ the cardinal λ is still regular.)

Like Prikry-forcing, Q_λ will introduce no new bounded subsets of λ. The final stage of our iteration will be forcing with $\mathrm{NM}(\kappa)$ followed by $\mathrm{Col}(D)$, where D is the closed unbounded set introduced by $\mathrm{NM}(\kappa)$. We shall need to embed the forcing $\mathrm{NM}(\kappa) * \mathrm{Col}^{V^{\mathrm{NM}(\kappa)}}(D)$ into $\mathrm{Q}_\kappa * \mathrm{Col}^{V^{\mathrm{Q}_\kappa}}(D)$. Unfortunately, $\mathrm{Col}^{V^{\mathrm{Q}_\kappa}}(D)$ in V^{Q_κ} is different from $\mathrm{Col}^{V^{\mathrm{NM}(\kappa)}}$ (because, for instance, in V of $\mathrm{NM}(\kappa)$ the cardinal κ is regular, but in V^{Q_κ} it is singular of cofinality ω, so the meaning of the Easton Product used in the definition of $\mathrm{Col}(D)$ is very different). But note that because $V^{\mathrm{NM}(\kappa)}$ and V^{Q_κ} have the same bounded subsets of κ, $\mathrm{Col}^{V^{\mathrm{NM}(\kappa)}}(D)$ is a sub-partial order of $\mathrm{Col}^{V^{\mathrm{Q}_\kappa}}(D)$. Also, if $G \subseteq \mathrm{Col}^{V^{\mathrm{Q}_\kappa}}(D)$ is a filter, $G \cap V$ is a filter in $\mathrm{Col}^{V^{\mathrm{NM}(\kappa)}}(D)$. The issue is the genericity of $G \cap V$ over $V^{\mathrm{NM}(\kappa)}$. In general, $G \cap V$ does not have to be generic over $V^{\mathrm{NM}(\kappa)}$. The additional role of Q_λ (for all $\lambda \leq \kappa$) will be to introduce a condition in $h \in \mathrm{Col}^{V^{\mathrm{Q}_\kappa}}(D)$ such that if $h \in G \subseteq \mathrm{Col}^{V^{\mathrm{Q}_\kappa}}(D)$ is a generic filter on V^{Q_κ}, then $G \cap V$ is generic over $V^{\mathrm{NM}(\lambda)}$.

Let R^β_λ be the set of triples (c, h, γ), where $c \in \mathrm{NM}(\lambda)$, $\min(c) > \beta$, $h \in \mathrm{Col}(c)$, and $\sup(c) < \gamma < \lambda$. We can treat R^β_λ as a forcing notion by ordering it as follows $(c', h', \gamma') \leq (c, h, \gamma)$ iff c' is an end-extension of c, h' is an end-extension of h, namely $h = h' \restriction c$, and $\gamma < \gamma'$.

For the next definition we fix a λ-complete ultrafilter U_β on R^β_λ extending the λ-complete filter of dense open subsets of R^β_λ. The ultrafilter U_β exists because the intersection of less than λ dense open subsets of R^β_λ contains a dense open subset and $|R^\beta_\lambda| = \lambda$. Note that since λ is λ^+-supercompact every λ-complete filter on a set of size λ can be extended to λ-complete ultrafilter.

Definition 25 *Let X_λ be the set of all sequences*

$$\langle (c_0, h_0, \alpha_0), ..., (c_{n-1}, h_{n-1}, \alpha_{n-1}) \rangle, \tag{3}$$

where

(i) $(c_i, h_i, \alpha_i), 0 \leq i < n$, *is strictly increasing in* R^0_λ.

(ii) *Each* α_i *is closed under* f.

(iii) $(c_{i+1}, h_{i+1}, \alpha_{i+1}) \in R^{\alpha_i}_\lambda$ *for* $0 \leq i < n-1$.

(iv) c_0 *contains some point below the first inaccessible.*

Let Q_λ be the set of all sets p of the form

$$\langle (c_0, h_0, \alpha_0), ..., (c_{n-1}, h_{n-1}, \alpha_{n-1}), T \rangle, \tag{4}$$

where

(i) $(c_0, h_0, \alpha_0), ..., (c_{n-1}, h_{n-1}, \alpha_{n-1}) \in X_\lambda$.

(iii) *T is a tree consisting of sequences*

$$\langle(c_0, h_0, \alpha_0), ..., (c_{n-1}, h_{n-1}, \alpha_{n-1}), ..., ((c_k, h_k), \alpha_k)\rangle \in X_\lambda$$

where $n \leq k < \omega$ such that

$$\begin{aligned}\{(c, h, \gamma) : \ &\langle(c_0, h_0, \alpha_0), ..., (c_{n-1}, h_{n-1}, \alpha_{n-1}), ...\\ &..., (c_{k-1}, h_{k-1}, \alpha_{k-1}), (c, h, \gamma)\rangle \in T\} \in U_{\alpha_{k-1}}.\end{aligned}$$

The intuitive meaning of the forcing is rather clear. The finite sequence $\alpha_0, ..., \alpha_{n-1}$ in the "stem" (3) of a condition (4) is an initial segment of the ω-sequence that will be cofinal in λ. The tree T is the tree of possible candidates for extending the stem (3). As usual for Prikry type forcings, we require to have a large set of possible candidates to be members of the ω-sequence leading up to λ. The sets c_i are initial segments of the generic object in NM(λ) that will be introduced by Q_λ, h_i is a partial information about an object that will eventually be a condition in $\mathrm{Col}^{V^{Q_\lambda}}(D)$, where D will be the club introduced by the NM(λ) generic filter. These remarks motivate the definition of the partial order in Q_λ:

Definition 26 *(The partial order of Q_λ). Suppose*

$$p = \langle(c_0, h_0, \alpha_0), ..., (c_{n-1}, h_{n-1}, \alpha_{n-1}), T\rangle,$$

and

$$q = \langle(c_0^*, h_0^*, \alpha_0^*), ..., (c_{k-1}^*, h_{k-1}^*, \alpha_{k-1}^*), T^*\rangle,$$

are in Q_λ. We say that q extends p, in symbols $q \leq p$, if

(i) *$n \leq k$ and $\langle(c_0^*, h_0^*, \alpha_0^*), ..., (c_{k-1}^*, h_{k-1}^*, \alpha_{k-1}^*)\rangle \in T$.*

(ii) *$T^* \subseteq T$.*

If $n = k$ above, we say that q is a direct extension *of p, in symbols $q \leq^* p$.*

Notation: If $p = \langle(c_0, h_0, \alpha_0), ..., (c_{n-1}, h_{n-1}, \alpha_{n-1}), T\rangle$ is in Q_λ we call n the *length* of p and denote it by $n(p)$. Similarly

$$\begin{aligned}\alpha(p) &= \langle\alpha_0, ..., \alpha_{n-1}\rangle && \text{the } \alpha\text{-part of } p\\ c(p) &= c_0 \cup ... \cup c_{n-1} && \text{the } c\text{-part of } p\\ h(p) &= h_0 \cup ... \cup h_{n-1} && \text{the } h\text{-part of } p.\end{aligned}$$

Note that if $\mu < \lambda$ and $\min(A) > \mu$, then every decreasing sequence of direct extension of length $\leq \mu$ has a lower bound which is a direct extension of all the members of the sequence. To see this one uses the fact that the ultrafilters U_γ that are used in the definition of Q_λ are all λ-complete.

The following lemma is a typical one for Prikry type forcing. Its proof is a straightforward generalization of similar lemmas proved before (e.g. Magidor [10], [4]):

Lemma 27 *Let Φ be a statement in the forcing language for Q_λ and $p \in Q_\lambda$. Then there exists a direct extension q of p such that q decides Φ.*

As usual, it follows from the lemma that Q_λ introduces no new bounded subsets of λ.

Lemma 28 *Let $G \subseteq Q_\lambda$ be generic over V. Let $D_G = \{c(p) : p \in G\}$. Then $D_G \subseteq \mathrm{NM}(\lambda)$ generates a generic filter of $\mathrm{NM}(\lambda)$ over V.*

Proof: It is immediate that for $p, p' \in G$ either $c(p) \leq c(p')$ or $c(p') \leq c(p)$, so D_G generates a filter. We just have to prove its genericity. Let $E \subseteq \mathrm{NM}(\lambda)$, $E \in V$, be dense open in $\mathrm{NM}(\lambda)$. We have to show that $E \cap D_G \neq \emptyset$. Let

$$p = \langle (c_0, h_0, \alpha_0), ..., (c_{n-1}, h_{n-1}, \alpha_{n-1}), T \rangle \in Q_\lambda.$$

We show that an extension of p forces that $E \cap D_G \neq \emptyset$. Let us look at the set $\bar{D}$ of all conditions $(c, h, \gamma) \in R_\lambda^{\alpha_{n-1}}$ such that $\alpha_{n-1} < \min(c)$ and $c(p) \cup c \in E$. Clearly $\bar{D}$ is dense open in $R_\lambda^{\alpha_{n-1}}$ and hence belongs to $U_{\alpha_{n-1}}$. Hence p has an extension q which forces $E \cap D_G \neq \emptyset$.

□ (Lemma)

We abuse notation by denoting also $\cup D_G$ by D_G. It is a club in λ in which limit points are all singulars. Its minimal point is below the first inaccessible cardinal. (Recall clause (iii) of Definition 25.) As usual, $\bigcup\{\alpha(p) : p \in G\}$ is an ω-sequence cofinal in λ, so λ has cofinality ω in $V[G]$.

Now let us consider the h-parts of conditions in G. Let $H_G = \bigcup\{h(p) : p \in G\}$.

Lemma 29 $V[G] \models H_G \in \mathrm{Col}(D_G)$.

Proof: H_G is clearly a partial function defined on $\beta \in D_G$. Note that if p, q are in G, each of them with stem length $> n$, then if α_n is the n-th coordinate (in both of them) of their α-part, then $h(p) \restriction \alpha_n = h(q) \restriction \alpha_n$. It means that

$$H_G \restriction \alpha_n = h(p) \text{ for any } p \text{ of stem length } > n \text{ belonging to } G. \quad (5)$$

Now $h(p) \in \mathrm{Col}(c(p))$, so for every $\beta' \in \mathrm{dom}(H_G)$ we have

$$H_G(\beta) \in \mathrm{Col}(f(\beta), < \beta'),$$

where β' is the first member of D_G above β. By (5) also $H_G \restriction \alpha_n$ has Easton support, but since $\lambda = \sup_{n<\omega} \alpha_n$ is singular, it follows that the support constraint in $\mathrm{Col}^{V[G]}(D)$ is satisfied by H_G and hence $H_G \in \mathrm{Col}(D_G)$. $\square$ (Lemma 29)

The next lemma explains the role of the h part of the conditions in Q_λ.

Lemma 30 *Let $G \subseteq \mathrm{Q}_\lambda$ be generic over V and D_G, H_G as above. Let $G^* \subseteq \mathrm{Col}^{V[G]}(D_G)$ be generic over $V[G]$ such that $H_G \in G^*$. Let $G^{**} = \mathrm{Col}^{V[D_G]}(D_G) \cap G^*$. Then G^{**} is a generic filter in $\mathrm{Col}^{V[D_G]}(D_G)$ over $V[D_G]$. (Note that $\mathrm{Col}^{V[D_G]}(D_G) \subseteq \mathrm{Col}^{V[G]}(D_G)$.)*

Proof: G^{**} is obviously a filter. We have to verify genericity. So let $\mathring{E}$ be an $\mathrm{NM}(\lambda)$-term, which is forced to be a dense open subset of $\mathrm{Col}^{V[D_G]}(D_G)$ and let E_G be its realization in $V[D_G]$. Let $p \in Q_\lambda$. We shall extend p to a condition q such that $q \Vdash G^* \cap \mathring{E} \neq \emptyset$. Then, since $E_G \in V[D_G]$, $q \in G$ implies

$$G^{**} \cap E_G = G^* \cap V[D_G] \cap E_G \neq \emptyset.$$

Assume

$$p = \langle (c_0, h_0, \alpha_0), ..., (c_{n-1}, h_{n-1}, \alpha_{n-1}), T \rangle.$$

Consider a condition $c \cup \bar{c}$ in $\mathrm{NM}(\lambda)$ with $\beta = \min(\bar{c}) > \alpha_{n-1}$, and $h \cup \bar{h} \in \mathrm{Col}(c \cup \bar{c})$. Assume for a while that $c \cup \bar{c} \in D_G$. Note that in $V[D_G]$, $\mathrm{Col}(D_G)$ can be represented as the cartesian product $\mathrm{Col}(c \cup \{\beta\}) \times \mathrm{Col}(D_G - \beta)$. Since w.l.o.g. β is inaccessible, $\mathrm{Col}(c \cup \{\beta\})$ has cardinality β. $\mathrm{Col}(D_G - \beta)$ is β^+-closed. So by standard arguments the following set is dense open subset of $\mathrm{Col}(D_G - \beta)$ in $V[D_G]$:

$$\begin{aligned} E^* = \{ g : \; & g \in \mathrm{Col}(D_G - \beta), \\ & \{g' : g' \in \mathrm{Col}(c \cup \{\beta\}), g \cup g' \in E\} \\ & \text{is dense open in } \mathrm{Col}(c \cup \{\beta\}) \}. \end{aligned}$$

So there is an extension c^*_β of $c \cup \bar{c}$ in $\mathrm{NM}(\lambda)$ such that some $\mathring{g}_\beta$ satisfies

$$c^*_\beta \Vdash \mathring{g}_\beta \in \mathrm{Col}(D_G - \beta) \wedge \mathring{g}_\beta \in E^* \wedge \mathring{g}_\beta \leq \bar{h}.$$

(Note that since $c^*_\beta \leq c \cup \bar{c}$ and $\bar{h} \in \mathrm{Col}(\bar{c})$, $c^*_\beta \Vdash \bar{h} \in \mathrm{Col}(D_G - \beta)$.) Without loss of generality we can assume $\mathring{g}_\beta$ is essentially a bounded subset of λ and since forcing with $\mathrm{NM}(\lambda)$ does not add bounded subsets of λ, we can assume that we have a particular g_β which we can assume to be in $\mathrm{Col}(c^*_\beta)$ such that

$$c^*_\beta \Vdash \check{g}_\beta \in \mathrm{Col}(D_G - \beta) \wedge \check{g}_\beta \in E^* \wedge \check{g}_\beta \leq \bar{h}.$$

The condition c^*_β must be of the form $c \cup c^{**}_\beta$ where c^{**}_β is an end extension of $\bar{c}$.

We have shown that the set D of $(c_n, h_n, \alpha_n) \in R^\beta_\lambda$ such that

$$c_n \Vdash h_n \in \mathrm{Col}(D_G - \beta), h \in E^*, h_n \leq \bar{h}$$

is dense open in R^β_λ

Let $((c_n, h_n), \alpha_n) \in D$ such that

$$\langle (c_0, h_0, \alpha_0), ..., (c_{n-1}, h_{n-1}, \alpha_{n-1}), (c_n, h_n, \alpha_n)) \rangle \tag{6}$$

extends $\langle (c_0, h_0, \alpha_0), ..., (c_{n-1}, h_{n-1}, \alpha_{n-1}) \rangle$ in T, and let T' be the set of sequences extending (6) in T. Finally, let q be the condition consisting of (6) followed by T'. Now q extends p in Q_λ.

Assume $q \in G$. The c part of q extends $c \cup c^{**}_{\alpha_n}$. The h part extends $h \cup g_{\alpha_n}$. Now D_G is an end extension of $c \cup c^{**}_{\alpha_n}$. In $V[G]$ we can also represent $\mathrm{Col}(D_G)$ as $\mathrm{Col}(c \cup \{\alpha_n\}) \times \mathrm{Col}(D_G - \alpha_n)$. Note that H_G extends $h \cup g_{\alpha_n}$ and $H_G \in G^*$, so $g_{\alpha_n} \in G^* \cap V[D_G] = G^{**}$. Hence in $V[D_G]$ the set

$$E' = \{g' : g' \in \mathrm{Col}(c \cup \{\alpha\}), (g', g_{\alpha_n}) \in E_G\}$$

is dense open.

But $\mathrm{Col}(c \cup \{\alpha_n\})$ is the same in $V[D_G]$ and in $V[G]$. (Because it is a set of bounded subsets of λ and all bounded subsets of λ in $V[D_G]$ and in $V[G]$ are also in V.) The filter $G^* \cap \mathrm{Col}(c \cup \{\alpha_n\})$ is generic in $\mathrm{Col}(c \cup \{\alpha_n\})$ over $V[G]$, so $G^* \cap E' \neq \emptyset$. Let $g' \in G^* \cap E'$, $(g', g_{\alpha_n}) \in E_G$, $(g', g_{\alpha_n}) \in G^*$, $(g', g_{\alpha_n}) \in V[D_G]$. So $(g', g_\beta) \in E_G \cap G^* \cap V[D_G] = E_G \cap G^{**} \neq \emptyset$. $\square$ (Lemma 30).

We also want to show that Q_λ collapses no cardinals. By Lemma 27 forcing with Q_λ introduces no new bounded subsets of λ, so no cardinals $< \lambda$ are collapsed.

We assume G.C.H., Q_λ has cardinality $2^\lambda = \lambda^+$ so no cardinals above λ^+ are collapsed. So the only cardinal we have to consider is λ^+.

Lemma 31 *In V^{Q_λ} the cardinal λ^+ is still a cardinal.*

Proof: If two conditions in Q_λ are incompatible, they must have a different stem. There are only λ stems. So Q_λ satisfies the λ^+-chain condition. Hence λ^+ is preserved. □ (Lemma 31)

The forcing $\mathrm{Col}(D_G)$ of course collapses cardinals but the following lemma will be useful:

Lemma 32 *Let G, G^*, G^{**} be as in Lemma 30. Then:*

(a) The only V-cardinals collapsed in $V[G][G^*]$ are the cardinals in the interval $(f(\beta), \beta')$, where $\beta \in D_G$ and β' is the next member of D_G above β.

(b) The only V-cardinals collapsed in $V[D_G][G^{**}]$ are the cardinals in the interval $(f(\beta), \beta')$, where $\beta \in D_G$ and β' is the next member of D_G above β.

(c) $V[G][G^*]$ and $V[D_G][G^{**}]$ have the same cardinals.

(d) $V[G][G^*]$ and $V[D_G][G^{**}]$ have the same bounded subsets of λ.

Proof: (a) is standard, after we have Lemma 31. The only possible problem is again λ^+, but if it collapsed, it becomes singular of cofinality $< \lambda$. The forcing $\mathrm{Col}(D_G)$ is such that every μ-sequence of ordinals is introduced by $\mathrm{Col}(D_G \restriction \rho)$ for some $\rho < \lambda$, so it is of cardinality $< \lambda$. So λ^+ is not collapsed.

(b) follows from (a) for cardinals $< \lambda$. G^* is generic over $V[D_G]$ with respect to a forcing notion of size λ, so no cardinal above λ is collapsed.

(c) follows immediately from (a) and (b).

(d) follows from the fact that a bounded subset of λ introduced by $\mathrm{Col}(D_G)$ is introduced by $\mathrm{Col}(D_G \cap \beta)$ for some $\beta < \lambda$. This is true for both $V[D_G]$ and $V[G]$. $\mathrm{Col}(D_G \cap \beta)$ is the same in $V[D_G]$ and $V[G]$, and $G^{**} \cap \mathrm{Col}(D_G \cap \beta) = G^* \cap \mathrm{Col}(D_G \cap \beta)$. So (d) follows. □ (Lemma 32)

4.2.2 Iteration

We would like to iterate the forcing Q_λ for all λ^+-supercompact $\lambda < \kappa$. The scheme of iteration we shall use is the scheme introduced by Gitik [3]. Our terminology follows (with minor changes) the terminology of [3].

Definition 33 *Suppose λ is a regular cardinal. A forcing notion $\langle P, \leq\rangle$ is said to be* λ-Prikry *if there is a partial order $\leq^* \subseteq \leq$ on P such that*

(a) *Every $\leq^*$-decreasing sequence of length less than λ has a $\leq^*$-lower bound.*

(b) *For every statement Φ in the forcing language for for P and for every $p \in P$ there is $q \in P$ such that $q \leq^* p$ and q decides Φ.*

We call $\leq^*$ the direct *extension relation.*

Note that we do *not* assume that any two strict extensions of p are necessarily compatible. The remarks above show that if λ is λ^+-supercompact and U is a normal ultrafilter on $\mathcal{P}_\lambda(\lambda^+)$, then Q_λ is a λ-Prikry forcing notion.

When we refer to Prikry forcing notions in the sequel we assume that they are given with $\leq^*$, that is, they are of the form $\langle P, \leq, \leq^*\rangle$. We shall also assume that each forcing notion P is given with its maximal element $\mathbf{1}_P$.

Definition 34 *An iteration*

$$\langle\langle P_\alpha : \alpha \leq \mu\rangle, \langle Q_\alpha : \alpha < \mu\rangle\rangle$$

is called a Gitik iteration of Prikry forcings *if the following holds: Each P_α is a forcing notion of sequences of length α such that*

(i) *If $p = \langle \tau_\beta : \beta < \alpha\rangle \in P_\alpha$ and $\gamma < \alpha$, then $p \restriction \gamma = \langle \tau_\beta : \beta < \gamma\rangle \in P_\gamma$.*

(ii) *If $p = \langle \tau_\beta : \beta < \alpha\rangle \in P_\alpha$ and $\gamma < \alpha$ then $p \restriction \gamma \Vdash_{P_\gamma} \tau_\gamma \in Q_\gamma$, where Q_γ is a P_γ-name forced over P_γ to denote a γ-Prikry forcing with the partial orders $\leq_\gamma, \leq^*_\gamma$.*

(iii) *The sequence has Easton support, namely for every regular $\gamma \leq \alpha$ the set $\{\beta < \gamma : \tau_\beta \neq \mathbf{1}_{\mathrm{Q}_\beta}\}$ has cardinality $< \gamma$.*

(iv) Q_γ *is the trivial forcing notion unless both γ is Mahlo and $\Vdash_{P_\beta} |Q_\beta| < \gamma$ for every $\beta < \gamma$.*

The partial order on P_α is defined as follows: Suppose $q = \langle \tau^*_\beta : \beta < \alpha \rangle$ and $p = \langle \tau_\beta : \beta < \alpha \rangle$. Then $q \leq p$ if there is a finite set $B \subseteq \{\beta < \alpha : \tau_\beta \neq \mathbf{1}_{Q_\beta}\}$ such that:

(a) *If $\beta \notin B$ such that $\tau_\beta \neq \mathbf{1}_{Q_\beta}$, then $q \restriction \beta \Vdash \tau^*_\beta \leq^*_\beta \tau_\beta$.*

(b) *If $\beta \in B$ or $\tau_\beta = \mathbf{1}_{Q_\beta}$, then $q \restriction \beta \Vdash \tau^*_\beta \leq_\beta \tau_\beta$.*

(Namely, we can take a non-direct extension of any point β in which $\tau_\beta = \mathbf{1}_{Q_\beta}$ and only at finitely many points β in which $\tau_\beta \neq \mathbf{1}_{Q_\beta}$.) The direct extension *for P_α is defined as $q \leq^* p$ if in the above definition we can take $B = \emptyset$.*

Let us fix now a Gitik iteration $\langle \langle P_\alpha : \alpha \leq \mu \rangle, \langle Q_\alpha : \alpha < \mu \rangle \rangle$ of Prikry forcings. Now Lemma 1.3 of Gitik [3] is essentially:

Lemma 35 *Let α be Mahlo such that $\Vdash_{P_\gamma} |Q_\gamma| < \alpha$ for all $\gamma < \alpha$. Then P_α has cardinality $\leq \alpha$ and it satisfies the $\alpha-$c.c..*

Lemma 1.4 of Gitik [3] is essentially:

Lemma 36 *Let Φ be a statement for the forcing language for P_μ and $p \in P_\mu$. Then there is $q \leq^* p$ such that $q \Vdash \Phi$ or $q \Vdash \neg\Phi$.*

It follows that if α is the first such that Q_α is not the trivial forcing notion, then P_μ is α-Prikry. Also, if α is a Mahlo cardinal such that $\Vdash_{P_\gamma} |Q_\gamma| < \alpha$ for all $\gamma < \alpha$, then in V^{P_α} we can consider

$$\langle \langle P_\beta / P_\alpha : \alpha \leq \beta \leq \mu \rangle, \langle Q_\beta : \alpha \leq \beta < \mu \rangle \rangle$$

to be a Gitik iteration of Prikry forcing notions, so in particular we get:

Lemma 37 *If $\alpha < \mu$ is a Mahlo cardinal such that $\Vdash_{P_\gamma} |Q_\gamma| < \alpha$ for all $\gamma < \alpha$, then*

(i) *P_μ / P_α is an α-Prikry forcing notion.*

(ii) *Every bounded subset of α in V^{P_μ} belongs already to V^{P_α}. (So, in particular, no α satisfying the above requirement is collapsed.)*

The next lemma is a variation of Lemma 1.5 in Gitik [3] and it deals with the preservation of λ^+-supercompact cardinals λ by the Gitik iterations:

Lemma 38 *Assume GCH. Let $\alpha \leq \mu$ be α^+-supercompact such that $\Vdash_{P_\beta} |Q_\beta| < \alpha$ for all $\beta < \alpha$. Let*

$$A = \{\beta < \alpha : \Vdash_{P_\beta} \text{“}Q_\beta \text{ is the trivial forcing notion”}\}.$$

Let U be a normal ultrafilter on $\mathcal{P}_\alpha(\alpha^+)$ such that $\{p : p \cap \alpha \in A\} \in U$. Then in V^{P_α} the filter U can be extended to a normal ultrafilter on $\mathcal{P}_\alpha(\alpha^+)$. In particular, α remains α^+-supercompact.

Proof: Let j be the ultrapower embedding $j : V \to M \cong V^{\mathcal{P}_\alpha(\alpha^+)}/U$. Note that $j(\langle\langle P_\beta : \beta \leq \alpha\rangle, \langle Q_\beta : \beta < \alpha\rangle\rangle)$ is in M a Gitik iteration of Prikry forcings of length $j(\alpha)$. Let us denote the new iteration $\langle\langle P^*_\beta : \beta \leq j(\alpha)\rangle, \langle Q^*_\beta : \beta < j(\alpha)\rangle\rangle$. Since $\Vdash_{P_\beta} |Q_\beta| < \alpha$ for all $\beta < \alpha$, we get that for all $\beta < \alpha$ $|P_\beta| < \alpha$, $Q^*_\beta = Q_\beta$, $P^*_\beta = P_\beta$ and also $P^*_\alpha = P_\alpha$. Our assumption that $\{p : p \cap \alpha \in A\} \in U$ translates into

$$\Vdash_{P_\alpha} \text{“}Q_\alpha \text{ is the trivial forcing notion.”}$$

So

$$M^{P_\alpha} \models \text{“}P_{j(\alpha)}/P_\alpha \text{ is an } \alpha^{++}\text{-Prikry forcing notion.”}$$

The forcing P_α satisfies the α-c.c. and has cardinality α and we assume GCH, so we can enumerate in V in a sequence of length α^{++} all P_α-terms forced to denote subsets of $\mathcal{P}_\alpha(\alpha^+)$. Let this list be $\langle \mathring{A}_\delta : \delta < \alpha^{++}\rangle$. Note that since M is closed under α^+-sequences, initial segments of the sequence $\langle j(\mathring{A}_\delta) : \delta < \alpha^{++}\rangle$ are in M. Now we argue in V^{P_α}. By induction define a $\leq^*$-decreasing sequence $\langle p_\delta : \delta < \alpha^{++}\rangle$ in $P_{j(\alpha)}/P_\alpha$ such that for each $\delta < \alpha^{++}$ the condition $p_{\delta+1}$ decides the statement ‘$j''\alpha^+ \in j(A_\delta)$’. ($j(\mathring{A}_\delta)$ is a $P^*_{j(\alpha)} = j(P_\alpha)$ -term, but in M^{P_α} we can consider it to be a $P^*_{j(\alpha)}/P_\alpha$ -term. By $P^*_{j(\alpha)}/P_\alpha$ being a Prikry type forcing we can find such a condition $p_{\delta+1}$ $\leq^*$-extending p_δ. Every initial segment of the sequence $\langle p_\delta : \delta < \alpha^{++}\rangle$ is in M, so at limit stages δ we can take p_δ to be a $\leq^*$-lower bound for $\langle p_\eta : \eta < \delta\rangle$.) Now in V^{P_α} define the ultrafilter U^* extending U by

$$A_\delta \in U^* \iff p_{\delta+1} \Vdash j''\alpha^+ \in j(\mathring{A}_\delta).$$

It is easy to check that U^* is well-defined and that it is a normal ultrafilter in V^{P_α} extending U. $\square$ (Lemma 38)

4.3 The Final Model

We are now ready to define our final model in which the first inaccessible will be a LST($L(I)$). Let us fix a supercompact cardinal κ. Recall that we assume G.C.H. to hold in V. We start the construction by defining a Gitik iteration $\langle\langle P_\alpha : \alpha \leq \kappa\rangle, \langle Q_\alpha : \alpha < \kappa\rangle\rangle$ of Prikry forcings. of length κ. The iteration will be defined if we inductively define Q_α. By induction it will be clear that for $\alpha < \gamma$, γ Mahlo, we have $\Vdash |Q_\alpha| < \gamma$. So we define:

(i) If α is not α^+-supercompact in V then Q_α is the trivial forcing notion.

(ii) If α is α^+-supercompact in V, then we pick a normal ultrafilter U on $\mathcal{P}_\alpha(\alpha^+)$ such that $A = \{p : p \cap \alpha \text{ non-}(p \cap \alpha)^+\text{-supercompact}\} \in U$.

Then α and U satisfy the assumptions of Lemma 38. So in V^{P_α} the cardinal α is still α^+-supercompact with a normal ultrafilter extending U. Define Q_α to be the Q_α as defined in Section 4.2.1. It is an α-Prikry forcing and its cardinality is 2^α which is less that the next Mahlo cardinal. So the iteration is defined. Since P_κ is of cardinality κ and satisfies the κ-c.c., the cardinal κ is still Mahlo in V^{P_κ}. (In fact, by Lemma 38 it is still κ^+-supercompact.) So now we force over V^{P_κ} with NM(κ) to get a closed unbounded subset D of κ such that each of the limit points of D is singular, and then we force with Col(D). So our final forcing notion is

$$P_\kappa \star \mathrm{NM}(\kappa) \star \mathrm{Col}(D).$$

Forcing with Col(D) makes sure that there are no inaccessible cardinals $< \kappa$. Forcing with NM(κ) keeps κ inaccessible, similarly for Col(D), so in $V^{P_\kappa \star \mathrm{NM}(\kappa) \star \mathrm{Col}(D)}$ the cardinal κ is the first inaccessible. Our goal will be achieved when we show:

Theorem 39 *In $V^{P_\kappa \star \mathrm{NM}(\kappa) \star \mathrm{Col}(D)}$ the cardinal κ is* LST($L(I)$).

Proof: Denote $V^* = V^{P_\kappa}$. In V^* the cardinal κ is still κ^+-supercompact, so we can define the forcing Q_κ and force with it over V^*. Let G be the generic filter in Q_κ. By Lemma 28 we can define from G an NM(κ) filter D_G which is going to be generic over V^*. We can force further with Col(D_G). Let G^* be the generic filter, where we can assume that $H_G \in G^*$. By Lemma 30 $G^{**} = G^* \cap V^*[D_G]$ is generic over $V^*[D_G]$ with respect to $\mathrm{Col}^{V^*[D_G]}(D_G)$. So $V^*[D_G][G^{**}]$ is a $P_\kappa \star \mathrm{NM}(\kappa) \star \mathrm{Col}(D)$ generic extension of V. By Lemma 32

$V^*[D_G][G^{**}]$ has the same bounded sequences of elements of κ and the same cardinals as $V^*[G][G^*]$.

It is now easily seen that given any generic $D \star H$ over V^* with respect to $\mathrm{NM}(\kappa) \star \mathrm{Col}(D)$, we can (by doing further forcing) assume that $D = D_G$ and $H = G^{**}$, where $G \subseteq Q_\kappa$ is generic over V^*. Assume otherwise. Let $(c, h) \in \mathrm{NM}(\kappa) \star \mathrm{Col}(D)$ force the negation of our claim. Without loss of generality $h \in \mathrm{Col}(c)$. Pick a condition p in Q_λ where the c-part is c and its h-part is h. Assume G is Q_κ-generic (over V^*) such that $p \in G$. Note that $c \in D_G$ and H_G extends h. Pick a generic G^* in $\mathrm{Col}(D_G)$ such that $H_G \in G^*$. Obviously $h \in G^* \cap V^*[D_G] = G^{**}$. So if we consider the generic pair $D_G \star G^{**}$, (c, h) belongs to it, but this is a contradiction. So we proved (Using also Lemma 32):

Lemma 40 $V_1 = V^{P_\kappa \star \mathrm{NM}(\kappa) \star \mathrm{Col}(D)}$ *has a forcing extension which is of the form* $V_2 = V^{P_\kappa \star Q_\kappa \star \mathrm{Col}(D_G)}$, *where* V_1 *and* V_2 *have the same cardinals and the same bounded sequences of elements of* κ.

Of course, the extension we describe in Lemma 40 does not preserve cofinalities because in $V_1 = V^{P_\kappa \star \mathrm{NM}(\kappa) \star \mathrm{Col}(D)}$ the cardinal κ is regular while in $V_2 = V^{P_\kappa \star Q_\kappa \star \mathrm{Col}(D_G)}$ it has cofinality ω.

We resume the proof of Theorem 39. So we are given in $V_1 = V^{P_\kappa \star \mathrm{NM}(\kappa) \star \mathrm{Col}(D)}$ a structure $\mathcal{A} = \langle \lambda, R_1, R_2, ... \rangle$ (without loss of generality we may assume that the universe of the structure is an ordinal λ). We have to get a substructure $\mathcal{A}'$ of $\mathcal{A}$ such that $\mathcal{A}' \prec_{L(I)} \mathcal{A}$ and $|A'| < \kappa$. Without loss of generality we may assume that $\kappa^+ \leq \lambda$.

In V the cardinal κ is supercompact, so we fix in V an embedding $j : V \to M$ such that κ is the critical point of j, $j(\kappa) > \lambda$, $M^\lambda \subseteq M$ and by our definition of the function f we can assume that $j(f)(\kappa) > \lambda$. The structure $\mathcal{A}$ is the realization of a term $\mathring{A}$ in the forcing language for $P_\kappa \star \mathrm{NM}(\kappa) \star \mathrm{Col}(D)$. We can assume $\mathring{A} \in M$ because $|P_\kappa \star \mathrm{NM}(\kappa) \star \mathrm{Col}(D)| = \kappa$.

Consider in M the forcing notion $j(P_\kappa) \star j(\mathrm{NM}(\kappa)) \star j(\mathrm{Col}(D))$. The forcing notion $j(P_\kappa)$ is (in M) an iteration of length $j(\kappa)$. Its first κ steps are exactly like in V (They are defined in V_κ which is the same as in M). The κ-th step of the iteration is Q_κ, so $j(P_\kappa) = P_\kappa \star Q_\kappa \star T$, where T is the iteration in M between κ and $j(\kappa)$.

Lemma 41 *One can force over* V_1 *to get a generic filter in the forcing* $j(P_\kappa) \star j(Q_\kappa) \star j(\mathrm{Col}(D))$ *such that in the resulting model there is an embedding*

$$j^* : V_1 \to M^{j(P_\kappa) \star j(Q_\kappa) \star j(\mathrm{Col}(D))}$$

extending j such that V_1 and $M^{j(P_\kappa)\star j(Q_\kappa)\star j(\mathrm{Col}(D))}$ have the same cardinals $\leq \lambda$.

Proof: By Lemma 40, we can force over V_1, not collapsing any cardinals, to get a model V_2 of the form $V^{P_\kappa \star Q_\kappa \star \mathrm{Col}(D)}$, where the generic filter for Col(D) over $V^{P_\kappa \star \mathrm{NM}(\kappa)}$ is of the form $G^{**} = G^* \cap V^{P_\kappa \star \mathrm{NM}(\kappa)}$, and where furthermore G^* is the generic filter in Col(D) over V^{Q_κ}. The generic filter for $P_\kappa \star Q_\kappa$ provides the generic filter for the first $\kappa + 1$ steps of the iteration of $j(P_\kappa)$ over M. (Note that Q_κ is the same in the sense of M^{P_κ} and V^{P_κ}). Follow this forcing by forcing with T. So we get a generic filter for $P_\kappa \star Q_\kappa \star T = j(P_\kappa)$. Recall that we assumed that κ is the last inaccessible in V so there are no inaccessibles (and hence no α^+-supercompact α) between κ and λ. Since $M^\lambda \subseteq M$, the same is true in M. So the iteration of $j(P_\kappa)$ between κ and λ^+ is the trivial iteration, so T is a Gitik iteration of μ-Prikry forcings for $\lambda^+ \leq \mu$. It means that

$$\mathcal{P}(\lambda) \cap V^{P_\kappa \star Q_\kappa} = \mathcal{P}(\lambda) \cap M^{P_\kappa \star Q_\kappa} = \mathcal{P}(\lambda) \cap V^{j(P_\kappa)}.$$

Hence no cardinals $\leq \lambda$ are collapsed in $M^{j(P_\kappa)}$. We now have to force with $j(\mathrm{NM}(\kappa))$ which is $\mathrm{NM}(j(\kappa))$ in the sense of $M^{j(P_\kappa)}$. The club D introduced by NM(κ) is in $V^{P_\kappa \star Q_\kappa}$ so it belongs to $M^{P_\kappa \star Q_\kappa} \subseteq M^{j(P_\kappa)}$. In $M^{P_\kappa \star Q_\kappa}$ the cardinal κ is singular of cofinality ω. So $D \cup \{\kappa\}$ is a condition in $\mathrm{NM}(j(\kappa))$. (It is a closed subset of $j(\kappa)$, every limit point, including κ, is singular. The other conditions are easily verified.) So we can pick a generic filter D^* in $\mathrm{NM}(j(\kappa))$ such that $D \cup \{\kappa\}$ is an initial segment of it. Forcing with $\mathrm{NM}(j(\kappa))$ over $M^{j(P_\kappa)}$ does not add any bounded subsets of $j(\kappa)$, so again no cardinals $\leq \lambda$ are collapsed.

Now we have to pick a generic filter for Col(D^*). The set $D \cup \{\kappa\}$ is an initial segment of D^* so

$$\mathrm{Col}(D^*) = \mathrm{Col}(D) \star \mathrm{Col}(D^* - \kappa).$$

(The collapses are not in the sense of $M^{j(P_\kappa)}$ but since $M^{j(P_\kappa)}$ agrees with $V^{P_\kappa \star Q_\kappa}$ on P_κ, Col(D) is the same in the sense of $V^{P_\kappa \star Q_\kappa}$ and $M^{j(P_\kappa)}$). The filter G^* is generic in Col(D), so we pick a generic filter for Col(D^*) such that its restriction to $\mathrm{Col}(D \cup \{\kappa\})$ is exactly G^*. Now we reach a crucial point for which we introduced the function f.

We defined Col(D) such that if $\beta \in D$, then Col(D) does not collapse any cardinals between β and $f(\beta)$. Since $\kappa \in D^*$, the forcing Col(D^*) does not collapse any cardinals between κ and $j(f)(\kappa)$. But $j(f)(\kappa) > \lambda$, so $\mathrm{Col}(D^*)$ collapses below λ only cardinals collapsed by $\mathrm{Col}(D^* \cap \kappa) = \mathrm{Col}(D)$. So it means that below λ the models $M^{j(P_\kappa \star \mathrm{NM}(\kappa) \star \mathrm{Col}(D))}$ and $V^{P_\kappa \star \mathrm{NM}(\kappa) \star \mathrm{Col}(D)}$ have the same cardinals.

Denote by H the generic filter over V in $P_\kappa \star \mathrm{NM}(\kappa) \star \mathrm{Col}(D)$. We have defined a generic filter in $j(P_\kappa \star \mathrm{NM}(\kappa) \star \mathrm{Col}(D))$. Let us denote this by H^*. We claim that the particular way we have defined H^* guarantees that H^* satisfies a condition which is known as "the master condition" i.e.

Claim: If $p \in H$, then $j(p) \in H^*$.

Proof: The condition $p \in H$ is of the form (q, s, t) where $q \in P_\kappa$, s is a term denoting an element of $\mathrm{NM}(\kappa)$ in V^{P_κ} and t is a term denoting a member of $\mathrm{Col}(D)$ in $V^{P_\kappa \star \mathrm{NM}(\kappa)}$. The generic filter we picked for $j(P_\kappa)$ extends the generic filter for P_κ, so $q \in H$ implies that $q = j(q) \in H^*$.

The generic filter we picked for $\mathrm{NM}(j(\kappa))$ is an end extension of the generic filter picked for $\mathrm{NM}(\kappa)$, so since s denotes a bounded subset of κ introduced by P_κ, $j(s) = s$ and $s \in H$ implies $j(s) \in H^*$. The generic filter for $\mathrm{Col}(D)$ is $G^{**} = G^* \cap V^{P_\kappa \star \mathrm{NM}(\kappa)}$. The way we picked the generic filter for $\mathrm{Col}(D^*)$ was such that $G^{**} \subseteq H^*$. Note that t denotes a subset of $V_\kappa^{P_\kappa}$ of cardinality $< \kappa$ in V^{P_κ}, so $j(t) = t$. But $t \in G^*$ so $t \in G^{**}$, so $j(t) = t \in H^*$. (We abuse notation by denoting a term and its realization by the same symbol.) So we have actually showed that for $p \in H$ we have $j(p) = (j(q), j(s), j(t)) \in H^*$. □ (Claim)

Once we have the master condition we can as usual define the extension j^* of j by defining j^* in the realization $[t]_H$ of an $P_\kappa \star \mathrm{NM}(\kappa) \star \mathrm{Col}(\kappa)$-term t as

$$j^*([t]_H) = [j(t)]_{H^*}.$$

It is a standard argument that given the assumptions of the claim, j^* is well-defined and it is elementary.

When we described the construction of H^* we argued that the cardinals $\leq \lambda$ in $V[H]$ are the same as in $M[H^*]$. So the lemma is verified. □ (Lemma 41)

So we have two universes $V[H]$ and $M[H^*]$ which agree on cardinals $\leq \lambda$. Moreover, we have $j \subseteq j^*$ which is elementary

$$j^* : V[H] \to M[H^*].$$

The structure $\mathcal{A}$ is in $V[H]$ and because $M^\lambda \subseteq M$ we have $\mathcal{A} \in M[H^*]$ and $j \restriction \mathcal{A} = j \restriction \lambda \in M \subseteq M[H^*]$.

Suppose $\Phi(x_1, ..., x_n)$ is a formula in the logic $L(I)$ and suppose $a_1, ..., a_n \in A$. Now:

$$M[H^*] \models \text{“}\mathcal{A} \models \Phi(a_1, ..., a_n)\text{” iff } V[H] \models \text{“}\mathcal{A} \models \Phi(a_1, ..., a_n)\text{”}. \qquad (7)$$

This is because $M[H^*]$ agrees with $V[H]$ on the cardinals $\leq \lambda$ which are all the cardinals relevant for evaluating the truth of $\Phi(a_1, .., a_n)$. On the other hand, from j^* being elementary we get:

$$V[H] \models \text{“}\mathcal{A} \models \Phi(a_1, ..., a_n)\text{” iff } M[H^*] \models \text{“}j^*(\mathcal{A}) \models \Phi(j^*(a_1), ..., j^*(a_n))\text{”}.$$

Hence

$$M[H^*] \models \text{“}\mathcal{A} \models \Phi(a_1, ..., a_n)\text{” iff } M[H^*] \models \text{“}j^*(\mathcal{A}) \models \Phi(j^*(a_1), ..., j^*(a_n))\text{”}.$$

So

$M[H^*] \models j^* \restriction A = j \restriction A$ is an $L(I)$-elementary embedding of $\mathcal{A}$ into $j^*(\mathcal{A})$.

Since $M[H^*] \models |A| \leq \lambda < j^*(\kappa)$, we get

$M[H^*] \models$ “There is an $L(I)$-elementary substructure of $j^*(\mathcal{A})$ of cardinality $< j^*(\kappa)$.

By j^* being elementary,

$V[H] \models$ “There is an $L(I)$-elementary substructure of $\mathcal{A}$ of cardinality $< \kappa$.

$\square$ (Theorem 39)

This ends the proof of Theorem 21. $\square$

We have shown that, assuming the consistency of a supercompact cardinal, it is consistent to assume that $\text{LST}(L(I))$ exists and moreover, we can consistently assume that it is either the first supercompact cardinal, or something much smaller, namely the first (weakly) inaccessible cardinal. A fortiori, then $\text{LST}(L(I))$ can be consistently equal to $\text{LST}(L^2)$ or also consistently different from $\text{LST}(L^2)$. Moreover, we have shown that even the existence of $\text{LST}(L(I))$ implies the consistency of large cardinals. In many respects the existence of $\text{LST}(L(I))$ seems, in the light of present day knowledge, like Martin's Maximum, and the cardinal $\text{LST}(L(I))$ behaves – be it small or large - as $\aleph_2$ in the presence of Martin's Maximum. But $\text{LST}(L(I))$ makes no claims about the size of the continuum: If it is consistent that there are supercompact cardinals, then it is consistent on the one hand that $\text{LST}(L(I))$ exists and $2^\omega = \aleph_1$ and on the other hand that $\text{LST}(L(I)) = 2^\omega$ ([14]).

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