Dependence Logic with Generalized Quantifiers: Axiomatizations

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Abstract. We prove two completeness results, one for the extension of dependence logic by a monotone generalized quantifier Q with weak interpretation, weak in the sense that the interpretation of Q varies with the structures. The second result considers the extension of dependence logic where Q is interpreted as "there exist uncountably many." Both of the axiomatizations are shown to be sound and complete for $\mathrm{FO}(Q)$ consequences.

1 Introduction

Generalized quantifiers constitute a well-studied method of extending the expressive power of first order logic. A more recent extension of first order logic is obtained by adding dependence atoms, permitting the expression of partially ordered quantification. In this paper we study the combination of the two methods, adding to first order logic both generalized quantifiers Q and dependence atoms as defined in [1]. It was shown in [2] that the resulting logic, $\mathrm{D}(Q)$, properly extends both $\mathrm{FO}(Q)$ and dependence logic. We analyse further the expressive power and give natural axioms for the new logic. There are theoretical limits to the extent that the axioms can be complete but we give partial completeness results in the sense that completeness is shown with respect to $\mathrm{FO}(Q)$ consequences.

Generalized quantifiers were introduced by Mostowski [3]. The most important of them is perhaps the quantifier

$$\mathbb{M} \vDash Q_1 x \phi(x, \bar{b}) \iff \mathbb{M} \vDash \phi(a, \bar{b}) \text{ for uncountably many } a \in M$$

owing to the beautiful axiomatization of it by Keisler [4]. On the other hand, generalized quantifiers have made an entrance to both linguistics [5] and computer science [6]. In natural language we can use generalized quantifiers to analyse constructs such as

<div align="center">Most boys run,</div>

where we think of "most" as a generalized quantifier. Other natural language quantifiers are "two thirds", "quite a few", "many", etc. There are various so-called polyadic lifts of such quantifiers, for example,

Ramsey lift:

<div align="center">At least two thirds of the boys in your class like each other. [7]</div>

Branching lift:

   A few boys in my class and a few girls in your class have dated each other. [8]

Resumption lift:

                          Most neighbours like each other. [7]

   In computer science, or more exactly finite model theory, we have the *counting quantifiers*

$$\mathbb{M} \vDash \exists_{\geq k} x \phi(x, \bar{b}) \iff \mathbb{M} \vDash \phi(a, \bar{b}) \text{ for at least } k \text{ elements } a \in M$$

which, although first order definable, have turn out relevant, but also the non-first order

$$\mathbb{M} \vDash \exists_{\text{even}} x \phi(x, \bar{b}) \iff \mathbb{M} \vDash \phi(a, \bar{b}) \text{ for an even number of } a \in M$$

and other similar ones. The lifts, also called vectorizations, are important in finite model theory, too. For example, the resumption lift sequence of the transitive closure quantifiers characterises in ordered models NLOGSPACE [9], and the resumption lift sequence of the so-called alternating transitive closure quantifier characterises, even in unordered models, least fixpoint logic [10].

   Dependence logic was introduced in [11]. It gives compositional semantics to the partially ordered quantifiers of [12]:

$$\begin{pmatrix} \forall x \ \exists y \\ \forall u \ \exists v \end{pmatrix} \phi(x, y, u, v, \bar{z}) \iff \exists f \exists g \forall x \forall u \phi(x, f(x), u, g(u), \bar{z}).$$

The compositional analysis is

$$\begin{pmatrix} \forall x \ \exists y \\ \forall u \ \exists v \end{pmatrix} \phi(x, y, u, v, \bar{z}) \iff \forall x \exists y \forall u \exists v (=(\bar{z}, u, v) \wedge \phi(x, y, u, v, \bar{z})),$$

where $=(\bar{z}, u, v)$ is a so-called *dependence atom*, see section 2.1 for its semantics. Dependence logic has the same expressive power as existential second order logic [13]. Thus dependence logic alone cannot express, for example, uncountability, in fact not even finiteness.

   The idea of combining partially ordered quantifiers and generalized quantifiers was first suggested by Barwise [8]. He used this combination to analyse lifts such as the Ramsey lift and the branching lift. It was proved in [7] that the polyadic lifts of monotone (unbounded) generalized quantifiers lead to a strong hierarchy, giving immediately the result that there is no finite number of generalized quantifiers, including partially ordered quantifiers, which would be able to express all Ramsey lifts of a given monotone (unbounded) quantifier. The same is true of branching lifts, and to a lesser extent of resumption lifts [7].

   The situation is quite different with the extension of *dependence* logic (rather than *first order* logic) by a monotone generalized quantifier. All the mentioned polyadic lifts (and vectorizations) can be readily defined (in all arities). Let us see how this is done for the Ramsey lift of a monotone quantifier Q.

$$\exists A \in Q \ \forall x \in A \ \forall y \in A \ \phi(x, y, \bar{z})$$

$$\Longleftrightarrow$$

$$\exists w (=(\bar{z}, w) \wedge Qx \exists y (\ y = w \wedge$$
$$\forall u \exists v (=(\bar{z}, u, v) \wedge (x = u \to v = w) \wedge$$
$$\forall u' \exists v' (=(\bar{z}, u', v') \wedge (u = u' \to v' = w) \wedge$$
$$((v = w \wedge v' = w) \to \phi(u, u', \bar{z}))))))$$

Resumption and branching can be handled similarly.

Thus putting generalized quantifiers and dependence atoms together results in a powerful combination extending far beyond either generalized quantifiers alone or dependence atoms alone.

This paper is organised as follows. In Section 2 we review the basics on dependence logic and generalized quantifiers in the dependence logic context. In Section 3 we present a system of natural deduction for the extension $D(Q, \check{Q})$ of dependence logic by a monotone generalized quantifier Q and its dual $\check{Q}$, and show that these rules are sound. Finally in Section 4 two completeness results for $FO(Q)$ consequences are shown for $D(Q, \check{Q})$. In the first result Q has the so-called weak interpretation, and in the second Q is interpreted as Q_1, that is, "there exist uncountably many."

2 Preliminaries

2.1 Dependence Logic

In this section we give a brief introduction to dependence logic. For a detailed account see [11].

The syntax of dependence logic extends the syntax of first order logic with new atomic formulas, the dependence atoms. There is one dependence atom for each arity. We write the atom expressing that the term t_n is uniquely determined by the values of the terms $t_1, \ldots, t_{n-1}$ as $=(t_1, \ldots, t_n)$. We consider formulas where negation can only appear in front of formulas without dependence atoms. For a vocabulary τ, $D[\tau]$ denotes the set of τ-formulas of dependence logic. The set $FV(\phi)$ of free variables of a formula ϕ is defined as in first order logic except that

$$FV(=(t_1, \ldots, t_n)) = FV(t_1) \cup \ldots \cup FV(t_n).$$

To define a compositional semantics for dependence logic we use *sets of assignments*, called *teams*, instead of single assignments as in first order logic. An assignment is a function $s : V \to M$ where V is a finite set of variables and M is the universe under consideration. A team on M is a set of assignments for some fixed finite set of variables V. If $V = \emptyset$ there is only one assignment, the empty function $\emptyset$. Observe that the team of the empty assignment $\{\emptyset\}$ is different from the empty team $\emptyset$.

– Given an assignment $s : V \to M$ and $a \in M$ let $s[a/x] : V \cup \{x\} \to M$ be the assignment:

$$s[a/x] : y \mapsto \begin{cases} s(y) & \text{if } y \in V \setminus \{x\}, \text{ and} \\ a & \text{if } x = y. \end{cases}$$

– Let $X[M/y]$ be the team

$$\{\, s[a/y] \mid s \in X, a \in M \,\},$$

– and whenever $f : X \to M$, let $X[f/y]$ denote the team

$$\{\, s[f(s)/y] \mid s \in X \,\}.$$

The domain of a non-empty team X, denoted $\mathrm{dom}(X)$, is the set of variables V. The interpretation of the term t in the model $\mathbb{M}$ under the assignment s is denoted by $t^{\mathbb{M},s}$. We write $s(\bar{x})$ for the tuple obtained by pointwise application of s to the finite sequence $\bar{x}$ of variables.

The satisfaction relation for dependence logic $\mathbb{M}, X \vDash \phi$ is defined as follows. Below, the notation $\mathbb{M}, s \vDash \phi$ refers to the ordinary satisfaction relation of first order logic. We also assume that $\mathrm{FV}(\phi) \subseteq \mathrm{dom}(X)$.

1. For formulas ψ without dependence atoms: $\mathbb{M}, X \vDash \psi$ iff $\forall s \in X : \mathbb{M}, s \vDash \psi$.
2. $\mathbb{M}, X \vDash \,=\!(t_1, \ldots, t_{n+1})$ iff $\forall s, s' \in X : \bigwedge_{1 \leq i \leq n} t_i^{\mathbb{M},s} = t_i^{\mathbb{M},s'} \to t_{n+1}^{\mathbb{M},s} = t_{n+1}^{\mathbb{M},s'}$
3. $\mathbb{M}, X \vDash \phi \wedge \psi$ iff $\mathbb{M}, X \vDash \phi$ and $\mathbb{M}, X \vDash \psi$
4. $\mathbb{M}, X \vDash \phi \vee \psi$ iff there are Y and Z such that $X = Y \cup Z$, and $\mathbb{M}, Y \vDash \phi$ and $\mathbb{M}, Z \vDash \psi$
5. $\mathbb{M}, X \vDash \exists y \phi$ iff there is $f : X \to M$, such that $\mathbb{M}, X[f/y] \vDash \phi$
6. $\mathbb{M}, X \vDash \forall y \phi$ iff $\mathbb{M}, X[M/y] \vDash \phi$.

We define $\mathbb{M} \vDash \sigma$ for a sentence σ to hold if $\mathbb{M}, \{\, \emptyset \,\} \vDash \sigma$. Let us make some easy remarks.

– Every formula is satisfied by the empty team.
– The satisfaction relation is downwards closed: If $\mathbb{M}, X \vDash \phi$ and $Y \subseteq X$ then $\mathbb{M}, Y \vDash \phi$.
– The satisfaction relation is local: $\mathbb{M}, X \vDash \phi$ iff $\mathbb{M}, Y \vDash \phi$ where

$$Y = \{\, s \upharpoonright \mathrm{FV}(\phi) \mid s \in X \,\}.$$

The expressive power for sentences of dependence logic is the same as that of existential second order logic.

2.2 D(Q)

The notion of a generalized quantifier goes back to Mostowski [3] and Lindström [14]. In [1] semantics for generalized quantifiers in the framework of dependence logic was introduced. We will review the definitions below.

Let Q be a quantifier of type $\langle k \rangle$, meaning that Q is a class of τ-structures, where the signature τ has a single k-ary relation symbol. Also, assume that Q is monotone increasing, i.e., for every M and every $A \subseteq B \subseteq M^k$, if $A \in Q_M$ then also $B \in Q_M$, where $Q_M = \{\, R \subseteq M^k \mid (M, R) \in Q \,\}$.

The formulas of dependence logic extended with a quantifier Q, $\mathrm{D}(Q)$, is built up from $\mathrm{FO}(Q)$-formulas and dependence atoms using the connectives $\wedge$ and $\vee$, and the

quantifier expressions $\exists x$, $\forall x$ and Qx in the usual way. We write $\phi \rightarrow \psi$ as a shorthand for $\neg\phi \vee \psi$, where ϕ is a formula without dependence atoms.

An assignment s satisfies a formula $Q\bar{x}\,\phi$ in a structure $\mathbb{M}$,

$$\mathbb{M}, s \vDash Q\bar{x}\,\phi, \text{ if the set } \{\,\bar{a} \in M^k \mid \mathbb{M}, s[\bar{a}/\bar{x}] \vDash \phi\,\} \text{ is in } Q_M.$$

In the context of teams we say that a team X satisfies a formula $Q\bar{x}\,\phi$,

$$\mathbb{M}, X \vDash Q\bar{x}\,\phi, \text{ if there exists } F : X \rightarrow Q_M \text{ such that } \mathbb{M}, X[F/\bar{x}] \vDash \phi, \qquad (1)$$

where $X[F/\bar{x}] = \{\,s[\bar{a}/\bar{x}] \mid \bar{a} \in F(s), s \in X\,\}$. This definition works well only with monotone (increasing) quantifiers, see [1] for details.

The following easy proposition suggests that we indeed have the right truth condition for monotone quantifiers:

Proposition 1 ([1]).

(i) $\mathrm{D}(Q)$ *is downwards closed.*

(ii) $\mathrm{D}(Q)$ *is local, in the sense that* $\mathbb{M}, X \vDash \phi$ *iff* $\mathbb{M}, (X \restriction FV(\phi)) \vDash \phi$.

(iii) *Viewing* $\exists$ *and* $\forall$ *as generalized quantifiers of type* $\langle 1 \rangle$, *the truth conditions in* (1) *are equivalent to the truth conditions of dependence logic.*

(iv) *For every* $\mathrm{D}(Q)$ *formula* ϕ *we have* $\mathbb{M}, \emptyset \vDash \phi$.

As proved in [2], the expressive power of $\mathrm{D}(Q)$ sentences corresponds to that of a certain natural extension of existential second order logic by Q.

In order to get a prenex normal form for all formulas we will focus on the logics $\mathrm{D}(Q, \check{Q})$, where $\check{Q}$ is the dual of Q, i.e,

$$\check{Q} = \{\,(M, M^k \setminus R) \mid R \subseteq M^k, (M, R) \notin Q\,\},$$

instead of $\mathrm{D}(Q)$. Note that, according to our definition of $\mathrm{D}(Q)$, a formula $Qx\phi$ may be negated only if ϕ is a $\mathrm{FO}(Q)$ formula.

We will consider monotone increasing quantifiers Q satisfying two non-triviality assumptions: $(M, \emptyset) \notin Q$ and $(M, M^k) \in Q$ for all M. In [2] the following normal form for sentences of $\mathrm{D}(Q)$ was shown for such non-trivial quantifiers. In the following we will use the notations $\bar{x}^i$ and $\bar{x}_i$ interchangeably.

Theorem 1. *Every* $\mathrm{D}(Q)$ *sentence in negation normal form, where* Q *is non-trivial, can be written as*

$$\mathcal{H}^1\bar{x}_1 \ldots \mathcal{H}^m\bar{x}_m \exists y_1 \ldots \exists y_n \big(\bigwedge_{1 \leq i \leq n} =(\bar{z}^i, y_i) \wedge \theta \big),$$

where $\mathcal{H}^i$ *is either* Q *or* $\forall$ *and* θ *is a quantifer-free* FO*-formula.*

In the present paper a similar normal form for all $\mathrm{D}(Q, \check{Q})$ formulas is obtained in Proposition 3.

A *weak semantics* can be given for $\mathrm{D}(Q)$ (and $\mathrm{FO}(Q)$, etc) by regarding Q as an interpreted symbol rather than a logical constant in the following way (see [4] and [15]

for more on this). A weak model is a structure together with an interpretation of Q, often denoted by q. We define $T \vDash_w \sigma$ to hold if every weak model (M, q) of T satisfies σ. In this paper we require the interpretation q of Q to be monotone increasing and non-trivial (In essence this is the *monotone logic* of [16]). In the weak semantics for $\mathrm{D}(Q, \check{Q})$ we require that the interpretation $\check{q}$ of $\check{Q}$ is the dual of the interpretation q of Q. Thus, if $T \cup \{\sigma\}$ consists of $\mathrm{D}(Q, \check{Q})$ sentences, then $T \vDash_w \sigma$ if every model $(M, q, \check{q})$ of T satisfies σ.

3 Natural deduction for $\mathrm{D}(Q, \check{Q})$

In this section we present a set of natural deduction rules for the logic $\mathrm{D}(Q, \check{Q})$, where Q is monotone and satisfies the non-triviality conditions: $(M, \emptyset) \notin Q$ and $(M, M^k) \in Q$ for all M. Observe that then also the dual quantifier $\check{Q}$ satisfies these conditions. To simplify notation, we will restrict attention to type $\langle 1 \rangle$ quantifiers.

We use an abbreviation $\bar{x} = \bar{y}$ for the formula $\bigwedge_{1 \leq i \leq \mathrm{len}(\bar{x})} x_i = y_i$, assuming of course that $\bar{x}$ and $\bar{y}$ are tuples of the same length $\mathrm{len}(\bar{x})$. The substitution of a term t to the free occurrences of x in ψ is denoted by $\psi[t/x]$. Analogously to first order logic, no variable of t may become bound in such a substitution. For tuples $\bar{t} = (t_1, \ldots, t_n)$ and $\bar{x} = (x_1, \ldots, x_n)$ we write $\psi[\bar{t}/\bar{x}]$ to denote the simultaneous substitution of x_i by t_i for $1 \leq i \leq n$.

We use the standard first order introduction and elimination rules for conjunction, existential quantifier, and universal quantifier. We also adopt rules for commutativity and associativity of disjunction, and the usual identity axioms for $\mathrm{FO}(Q, \check{Q})$ formulas. The rest of the rules are listed below:

1. Disjunction:

$$\frac{\phi}{\phi \vee \psi} \; \vee\mathrm{I} \qquad \frac{\psi}{\phi \vee \psi} \; \vee\mathrm{I} \qquad \frac{\phi \vee \psi \quad \overset{[\phi]}{\underset{\gamma}{\vdots}} \quad \overset{[\psi]}{\underset{\gamma}{\vdots}}}{\gamma} \; \vee\mathrm{E}$$

   where γ is a $\mathrm{FO}(Q, \check{Q})$ formula.
2. Negation and duality:

$$\frac{\overset{[\phi]}{\underset{\bot}{\vdots}}}{\neg\phi} \; \neg\mathrm{I} \qquad \frac{\overset{[\neg\phi]}{\underset{\bot}{\vdots}}}{\phi} \; \mathrm{RAA} \qquad \frac{\phi \quad \neg\phi}{\bot} \; \bot\mathrm{I} \qquad \frac{\check{Q}x\phi}{\neg Qx\neg\phi}$$

   where ϕ is a $\mathrm{FO}(Q, \check{Q})$ formula.
3. Disjunction substitution:

$$\frac{\phi \vee \psi \quad \overset{[\psi]}{\underset{\gamma}{\vdots}}}{\phi \vee \gamma}$$

4. Extending scope:

$$\frac{\mathcal{H}x\phi \vee \psi}{\mathcal{H}x(\phi \vee \psi)} \qquad \frac{Qx\phi \wedge \psi}{Qx(\phi \wedge \psi)}$$

where $\mathcal{H} \in \{Q, \check{Q}, \exists, \forall\}$, and the prerequisite for applying these rules is that x does not appear free in ψ. The rule on the right is also assumed for $\check{Q}$.

5. Unnesting:

$$\frac{=(t_1, ..., t_n)}{\exists z(=(t_1, ..., z, ..., t_n) \wedge z = t_i)}$$

where z is a new variable.

6. Dependence distribution: let

$$\phi = \exists y_1 \ldots \exists y_n (\bigwedge_{1 \leq j \leq n} =(\bar{z}^j, y_j) \wedge \phi_0),$$

$$\psi = \exists y_{n+1} \ldots \exists y_m (\bigwedge_{n+1 \leq j \leq m} =(\bar{z}^j, y_j) \wedge \psi_0).$$

where ϕ_0 and ψ_0 are quantifier-free formulas without dependence atoms, and y_i, for $1 \leq i \leq n$, does not appear in ψ and y_i, for $n + 1 \leq i \leq m$, does not appear in ϕ. Then,

$$\frac{\phi \vee \psi}{\exists y_1 \ldots \exists y_m (\bigwedge_{1 \leq j \leq m} =(\bar{z}^j, y_j) \wedge (\phi_0 \vee \psi_0))}$$

7. Dependence introduction:

$$\frac{\exists x \forall y \phi}{\forall y \exists x(=(\bar{z}, x) \wedge \phi)} \qquad \frac{\exists x Q y \phi}{Qy \exists x(=(\bar{z}, x) \wedge \phi)}$$

where $\bar{z}$ lists the variables in $\mathrm{FV}(\phi) - \{x, y\}$. Similar for $\check{Q}$.

8. Monotonicity of Q and $\check{Q}$:

$$\frac{Qx\phi \quad \begin{matrix} [\phi] \\ \vdots \\ \psi \end{matrix}}{Qx\psi}$$

where the prerequisite for applying this rule is that the variable x cannot appear free in any non-discharged assumption used in the derivation of ψ, except for ϕ. Similar for $\check{Q}$.

9. Bound variables:

$$\frac{Qx\phi}{Qy\phi[y/x]},$$

where y does not appear in ϕ. Similar for $\check{Q}$.

Observe that $\mathrm{FO}(Q, \check{Q}) \equiv \mathrm{FO}(Q)$, but syntactically $\mathrm{FO}(Q, \check{Q})$ includes more formulas.

3.1   Soundness of the rules

In this section we show the soundness of the rules introduced in the previous section under any monotone and non-trivial interpretation of Q. Clearly this is the same as soundness in the weak semantics for Q.

The following lemmas will be needed in the proof.

Lemma 1. *Let $\phi(x)$ be a $\mathrm{D}(Q, \check{Q})$ formula, and t a term such that in the substitution $\phi(t/x)$ no variable of t becomes bound. Then for all $\mathbb{M}$ and teams X, where $(FV(\phi) - \{x\}) \cup \mathrm{Var}(t) \subseteq \mathrm{dom}(X)$*

$$\mathbb{M}, X \vDash \phi(t/x) \Leftrightarrow \mathbb{M}, X(F/x) \vDash \phi(x),$$

where $F \colon X \to A$ is defined by $F(s) = t^{\mathbb{M}}\langle s \rangle$.

Proof. Analogous to Lemma 8 in [17].

It is easy to verify that Lemma 1 gives the following familiar property concerning changing free variables.

Lemma 2 (Change of free variables). *Let the free variables of $\phi \in \mathrm{D}(Q, \check{Q})$ be $x_1, \ldots, x_n$ and let $y_1, \ldots, y_n$ be distinct variables. Then for all structures $\mathbb{M}$ and teams X with domain $\{x_1, \ldots, x_n\}$ it holds that*

$$\mathbb{M}, X \vDash \phi \Leftrightarrow \mathbb{M}, X' \vDash \phi(\bar{y}/\bar{x}),$$

where X' is the team with domain $\{y_1, \ldots, y_n\}$ containing the assignments $s' \colon y_i \mapsto s(x_i)$ for $s \in X$.

Proposition 2. *Assume that Q is monotone and non-trivial. Let $T \cup \{\phi\}$ be a set of sentences of $\mathrm{D}(Q, \check{Q})$. If $T \vdash \phi$, then $T \vDash \phi$.*

Proof. We prove the statement that if $T \vdash \phi$, where $T \cup \{\phi\}$ is a set of *formulas*, then for any $\mathbb{M}$ and X where $\mathrm{dom}(X) \supseteq \mathrm{FV}(T) \cup \mathrm{FV}(\phi)$, if $\mathbb{M}, X \vDash T$ then $\mathbb{M}, X \vDash \phi$. This is done by using induction on the length of derivation.

It suffices to consider the rules 2 (duality), 4, 7, 8, and 9 since the soundness of the other rules can be proved analogously to [17] using the fact that $\mathrm{D}(Q, \check{Q})$ is local and has downwards closure (see (ii) and (i) of Proposition 1). In particular, Lemma 1 is used in the soundness proofs of the rules $\exists\, \mathrm{I}$ and $\forall\, \mathrm{E}$.

   2 Assume $\mathbb{M}, X \vDash \check{Q}x\phi$ then, since $\check{Q}x\phi$ is a $\mathrm{FO}(Q, \check{Q})$ formula we have $\mathbb{M}, s \vDash \check{Q}x\phi$ for all $s \in X$. This clearly implies that $\mathbb{M}, s \vDash \neg Qx\neg\phi$ for all $s \in X$, which is equivalent to $\mathbb{M}, X \vDash \neg Qx\neg\phi$.
   4 These two rules preserve logical equivalence analogously to Lemma 3.2 in [2].
   7 The soundness of this rule follows from the logical equivalence

$$Qy\exists x(=(\bar{z}, x) \wedge \phi) \equiv \exists x Qy\phi$$

   the proof of which is analogous to the case where Q is replaced by $\forall$ (see [17]).

8 Assume that we have a natural deduction proof of $Qx\psi$ from the assumptions

$$\{\gamma_1, \ldots, \gamma_k\}$$

with the last rule 13. Let $\mathbb{M}$ and X be such that $\mathbb{M}, X \vDash \phi_i$, for $1 \leq i \leq k$. By the assumption, we have a shorter deduction of $Qx\phi$ from the assumptions $\{\gamma_{n_1}, \ldots, \gamma_{n_l}\}$ and a deduction of ψ from the assumptions $\{\phi, \gamma_{n_{l+1}}, \ldots, \gamma_{n_m}\}$. Hence by the induction assumption it holds that $\mathbb{M}, X \vDash Qx\phi$. Therefore, there is $F \colon X \to Q_M$ such that $\mathbb{M}, X[F/x] \vDash \phi$. Since the variable x cannot appear free in the formulas $\gamma_{n_{l+1}}, \ldots, \gamma_{n_m}$ it follows that $\mathbb{M}, X[F/x] \vDash \gamma_i$, for $i \in \{n_{l+1}, \ldots, n_m\}$. Now by the induction assumption we get that $\mathbb{M}, X[F/x] \vDash \psi$ and $\mathbb{M}, X \vDash Qx\psi$.

9 This rule preserves logical equivalence by Lemma 2. $\square$

Note that since Proposition 2 holds for every monotone non-trivial quantifier Q we get also soundness for weak semantics: If $T \vdash \phi$ then $T \vDash_w \phi$.

4   Completeness results for $\mathrm{FO}(Q, \check{Q})$ consequences

4.1   Deriving a normal form for $\mathrm{D}(Q, \check{Q})$

In this section we show that from each formula $\phi \in \mathrm{D}(Q, \check{Q})$ we can derive a logically equivalent formula in the following normal form:

$$\mathcal{H}^1 x_1 \ldots \mathcal{H}^m x_m \exists y_1 \ldots \exists y_n \big(\bigwedge_{1 \leq i \leq n} =(\bar{x}^i, y_i) \wedge \theta \big), \tag{2}$$

where $\mathcal{H}^i$ is either Q, $\check{Q}$ or $\forall$, and θ is a quantifier-free FO-formula.

Proposition 3. *Let ϕ be a formula of $\mathrm{D}(Q, \check{Q})$. Then $\phi \vdash \phi'$, where ϕ' is of the form* (2)*, and ϕ' is logically equivalent to ϕ.*

Proof. The proof of this Proposition is analogous to the proof of the corresponding result for dependence logic formulas in [17]. See the full version of this article [18].

$\square$

4.2   Completeness for $\mathrm{D}(Q, \check{Q})$

In this section we prove a completeness result for $\mathrm{D}(Q, \check{Q})$ with respect to $\mathrm{FO}(Q, \check{Q})$ consequences of $\mathrm{D}(Q, \check{Q})$-sentences, with weak semantics. Analogously to [17], we approximate $\mathrm{D}(Q, \check{Q})$-sentences in the normal form (2) by an infinite set of $\mathrm{FO}(Q, \check{Q})$ sentences. We use an extra predicate R to encode a team witnessing the satisfiability of the quantifier prefix $\mathcal{H}^1 x_1 \ldots \mathcal{H}^m x_m$.

Let σ be

$$\mathcal{H}^1 x_1 \ldots \mathcal{H}^m x_m \exists y_1 \ldots \exists y_n \big(\bigwedge_{1 \leq i \leq n} =(\bar{x}^i, y_i) \wedge \theta(x_1, \ldots, x_m, y_1, \ldots, y_n) \big),$$

where each $\mathcal{H}^i$ is either Q, $\check{Q}$ or $\forall$.

We define finite approximations $A^i\sigma$ of σ as follows. The first approximation, $A^1\sigma$, is

$$\forall\bar{x}\exists\bar{y}\big(R(\bar{x}) \rightarrow \theta(\bar{x},\bar{y})\big).$$

The second approximation $A^2\sigma$ is

$$\forall\bar{x}_1\exists\bar{y}_1\forall\bar{x}_2\exists y_2\big(R(\bar{x}_1)\wedge R(\bar{x}_2) \rightarrow \theta(\bar{x}_1,\bar{y}_1)\wedge\theta(\bar{x}_2,\bar{y}_2)\wedge$$
$$\bigwedge_{1<i<n}(\bar{x}_1^i = \bar{x}_2^i \rightarrow y_{i,1} = y_{i,2}))$$

With the notational convention that $(x_{i_1},\ldots,x_{i_k})_j$ is the sequence $(x_{i_1,j},\ldots,x_{i_k,j})$. By generalizing this construction we get the k-th approximation:

$$\forall\bar{x}_1\exists\bar{y}_1\ldots\forall\bar{x}_k\exists\bar{y}_k\big(\bigwedge_{1\leq j\leq k}R(\bar{x}_j) \rightarrow \bigwedge_{1\leq j\leq k}\theta(\bar{x}_j,\bar{y}_j)\wedge$$
$$\bigwedge_{\substack{1\leq i\leq n\\1\leq j,j'\leq k}}(\bar{x}_j^i = \bar{x}_{j'}^i \rightarrow y_{i,j} = y_{i,j'}))$$

Also we need a sentence saying that R is of the right kind, witnessing the quantifier prefix: Let $B\sigma$ be

$$\mathcal{H}^1 x_1\ldots\mathcal{H}^m x_m R(x_1,\ldots,x_m).$$

We will adopt the following rule in our deduction system:

(10) Approximation rule:

$$
\begin{array}{cc}
[B\sigma] & [A^n\sigma] \\
\ddots & \ddots \\
\end{array}
$$
$$\dfrac{\sigma \qquad\qquad \psi}{\psi}\text{ (Approx)}$$

where σ is a sentence in normal form, and R does not appear in ψ nor in any uncancelled assumptions in the derivation of ψ, except for $B\sigma$ and $A^n\sigma$.

Proposition 4. *Adding the approximation rule to the inference system results in a sound system for* $\mathrm{D}(Q,\check{Q})$ *with regard to weak semantics.*

Proof. We plug in the following induction step to the proof of Proposition 2:

Assume that there is a derivation of ψ from Γ ending with the approximation rule. Then there are shorter derivations from Γ of σ and from $\Gamma', B\sigma, A^n\sigma$ of ψ, where $\Gamma' \subseteq \Gamma$ is such that R does not occur in Γ'. By the induction hypothesis we get $\Gamma \vDash \sigma$ and $\Gamma', B\sigma, A^n\sigma \vDash \psi$. We will prove that $\Gamma \vDash \psi$, by assuming $\mathbb{M}, X \vDash \Gamma$ for some non-empty X and proving that $\mathbb{M}, X \vDash \psi$.

Assume σ is of the form

$$\mathcal{H}^1 x_1\ldots\mathcal{H}^m x_m\exists y_1\ldots\exists y_n\big(\bigwedge_{1\leq i\leq n} =(\bar{x}^i, y_i)\wedge\theta\big).$$

where θ is a quantifier free first order formula.

From the fact that $\mathbb{M}, X \vDash \sigma$ we get $\mathbb{M} \vDash \sigma$ and thus there is a (non-empty) team Y such that

$$\mathbb{M}, Y \vDash \exists y_1 \ldots \exists y_n \big(\bigwedge_{1 \leq i \leq n} =(\bar{x}^i, y_i) \wedge \theta \big).$$

Let $r \subseteq M^m$ be the relation $Y(\bar{x}) = \{ s(\bar{x}) \mid s \in X \}$ corresponding to Y. Then $(\mathbb{M}, r) \vDash B\sigma$ and, it should also be clear that $(\mathbb{M}, r) \vDash A^n \sigma$. Since R does not occur in Γ' we have $(\mathbb{M}, r), X \vDash \Gamma', B\sigma, A^n \sigma$. By the induction hypothesis $(\mathbb{M}, r), X \vDash \psi$, and since R does not occur in ψ we have $\mathbb{M}, X \vDash \psi$.   □

Lemma 3. *If T is a set of* $\mathrm{FO}(Q, \check{Q})$*-sentences consistent in the deduction system described above then there are a countable recursively saturated model $\mathbb{M}$ and an interpretation q of Q such that $(\mathbb{M}, q, \check{q}) \vDash T$.*

Proof. First translate T to $T^\neg$ in which each $\check{Q}x\phi$ is replaced by $\neg Qx\neg\phi$. By using the same argument as in [4,15] we may reduce $\mathrm{FO}(Q)$ to FO by replacing subformulas of the form $Qx\phi$ with new relation symbols $R_\phi(\bar{y})$, $\bar{y}$ being the free variables of $Qx\phi$. This will reduce the set $T^\neg$ to a set T^*. Let T' be T^* together with the translations of the universal closures of

– $(\phi \to \psi) \to (Qx\phi \to Qx\psi)$, for all ϕ and ψ; and
– $Qx\phi \to Qy(\phi[y/x])$, for all ϕ such that the substitution is legal.

Now T' is consistent by the same argument as in [15]. Let $\mathbb{M}^*$ be a countable recursively saturated model of T', and M its reduct to the original signature. Now we may define q to be

$$\{ A \subseteq M \mid A \supseteq \{ a \in M \mid \mathbb{M}^*, s[a/x] \vDash \phi^* \} \text{ for some } \phi \text{ s.t. } M^*, s \vDash Qx\phi^* \} .$$

Proposition 2.3.4 in [15] shows that $(\mathbb{M}, q) \vDash T^\neg$, and thus $(\mathbb{M}, q, \check{q}) \vDash T$.   □

The main result of this section can now be stated as follows.

Theorem 2. *Let T be a set of sentences of* $\mathrm{D}(Q, \check{Q})$ *and* $\phi \in \mathrm{FO}(Q, \check{Q})$ *a sentence. Then the following are equivalent:*

(I) $T \vDash_w \phi$
(II) $T \vdash \phi$

Lemma 4. *In a countable recursively saturated weak model $(\mathbb{M}, q, \check{q})$ in which $B\sigma$ and $A^n \sigma$ holds for all n, σ holds.*

Proof. Suppose σ is

$$\mathcal{H}^1 x_1 \ldots \mathcal{H}^m x_m \exists y_1 \ldots \exists y_n \big(\bigwedge_{1 \leq i \leq n} =(\bar{x}^i, y_i) \wedge \theta \big).$$

Note that the sentences $A^n \sigma$ can be viewed as the finite approximations as defined in [17] (and see also [19]) of the D sentence σ':

$$\forall \bar{x} \exists \bar{y} \big(\bigwedge_{1 \leq i \leq n} =(\bar{x}^i, y_i) \wedge (R(\bar{x}) \to \theta) \big).$$

Thus by Theorem 2.4 in [19] (see also [17]), we know that $\mathbb{M} \vDash \sigma'$.

Let X be the team $\{\, s : \{\, x_1, \ldots, x_k \,\} \to M \mid (s(x_1), \ldots, s(x_m)) \in R^{\mathbb{M}} \,\}$. To prove that $(\mathbb{M}, q, \check{q}) \vDash \sigma$ we find $F_1, \ldots, F_m$ so that $\{\, \emptyset \,\}\, [F_1/x_1] \ldots [F_m/x_m] = X$.

$$
\begin{aligned}
F_1(\emptyset) &= X \restriction \{\, x_1 \,\}, \\
F_{i+1} &: \{\, \emptyset \,\}\, [F_1/x_1] \ldots [F_i/x_i] \to M \\
F_{i+1}(s) &= \{\, a \in M \mid \exists s' \in X : (s'(x_1), \ldots, s'(x_{i+1})) = (s(x_1), \ldots, s(x_i), a) \,\}.
\end{aligned}
$$

By the assumption $(\mathbb{M}, q, \check{q}) \vDash B\sigma$ it follows that $F_i(s) \in \mathcal{H}_M^i$. Furthermore, since $\mathbb{M} \vDash \sigma'$ we get that

$$
\mathbb{M}, X \vDash \exists y_1 \ldots \exists y_n \big(\bigwedge_{1 \leq i \leq n} =(\bar{x}^i, y_i) \wedge \theta \big).
$$

Therefore

$$
(\mathbb{M}, q, \check{q}) \vDash \mathcal{H}^1 x_1 \ldots \mathcal{H}^m x_m \exists y_1 \ldots \exists y_n \big(\bigwedge_{1 \leq i \leq n} =(\bar{x}^i, y_i) \wedge \theta \big)
$$

as wanted.   □

Proof (Proof of Theorem 2.). (I) $\Rightarrow$ (II): This is just a special case (for sentences) of soundness.

(II) $\Rightarrow$ (I): Suppose $T \nvdash \phi$, where ϕ is a $\mathrm{FO}(Q, \check{Q})$-sentence. We will construct a weak model of $T \cup \{\, \neg\phi \,\}$ showing that $T \nvDash_w \phi$. Replacing T with the set $T' = \{\, B\sigma, A^n\sigma \mid \sigma \in T, n \in \mathbb{N} \,\}$ we can conclude that $T' \cup \{\, \neg\phi \,\} \nvdash \bot$. By applying Lemma 3 we get a weak countable recursively saturated model $(M, q, \check{q})$ of T'. Lemma 4 implies that $(M, q, \check{q}) \vDash T$. Now since $(M, q, \check{q}) \nvDash \phi$, we get $T \nvDash_w \phi$ as wanted.   □

4.3   Completeness for $\mathrm{D}(Q_1, \check{Q}_1)$

We will now prove a completeness result similar to Theorem 2 for the logic $\mathrm{D}(Q, \check{Q})$ where Q is interpreted as Q_1, the quantifier "there exist uncountably many." In this section we consider only structures over uncountable universes.

We add the following two rules from [4] to the system presented in Section 3. Note that the approximation rule of section 4.2 is not included.

$$
\overline{\neg Q x (x = y \vee x = z)}
$$

$$
\frac{Q x \exists y \phi}{\exists y Q x \phi \vee Q y \exists x \phi}
$$

The intuitive meaning of the second rule is that a countable union of countable sets is countable. The first is needed to avoid Q being interpreted as the quantifier "the exists at least two."

For each $D(Q, \check{Q})$ sentence σ

$$\mathcal{H}^1 x_1 \ldots \mathcal{H}^m x_m \exists y_1 \ldots \exists y_n \Big(\bigwedge_{1 \leq i \leq n} =(\bar{x}^i, y_i) \wedge \theta \Big)$$

in normal form we define the Skolem translation $S\sigma$ of σ to be:

$$\mathcal{H}^1 x_1 \ldots \mathcal{H}^m x_m \theta(f_i(\bar{x}^i)/y_i),$$

where the f_i's are new function symbols of the right arity. If σ is a sentence in the signature τ then $S\sigma$ will be in the extended signature $\tau \cup \{ f_1, \ldots, f_n \}$.

The last rule of the deduction system is the following:

$$[S\sigma]$$

$$\vdots$$

$$\frac{\sigma \qquad \psi}{\psi} \text{ (Skolem)}$$

Here σ is a $D(Q, \check{Q})$ sentence in normal form, and the function symbols $f_1, \ldots, f_n$ do not occur in ψ nor in any uncancelled assumption of the derivation of ψ, except for $S\sigma$.

Proposition 5. *If $T \vdash \phi$ in the deduction system for $D(Q_1, \check{Q}_1)$ then $T \vDash \phi$.*

Proof. We extend the proof of Proposition 2 to also cover the three new rules:

(1) The soundness of the first rule is easily seen by observing that the formula $Q_1 x(x = y \vee x = z)$ is a $FO(Q_1)$ formula and thus a team satisfies it iff every assignment in the team satisfies the formula.

(2) For the second rule we need to prove that if $\mathbb{M}, X \vDash \Gamma, Q_1 x \exists y \phi$ then $\mathbb{M}, X \vDash \Gamma, \exists y Q_1 x \phi \vee Q_1 y \exists x \phi$. By the assumption we get functions $F : X \to Q_M$ and $f : X[F/x] \to M$ such that $\mathbb{M}, X[F/x][f/y] \vDash \phi$. Thus, for each $s \in X$ there is a binary relation $R_s = \{ (a, f(s[a/y])) \mid a \in F(s) \}$ such that $(M, R_s) \vDash Q_1 x \exists y R(x, y)$. Let

$$Y = \{ s \in X \mid (M, R_s) \vDash \exists y Q_1 x R(x, y) \}$$

and

$$Z = \{ s \in X \mid (M, R_s) \vDash Q_1 y \exists x R(x, y) \}.$$

By the validity of the rule for $FO(Q_1)$ we see that $X = Y \cup Z$.

It should be clear that $Y \vDash \exists y Q_1 x \phi$ since by letting $g(s)$ be such that $(M, R_s) \vDash Q_1 x R(x, g(s))$ and

$$G(s[g(s)/y]) = \{ a \in M \mid (M, R_s) \vDash R(a, g(s)) \},$$

we have that $Y[g/y][G/x] \subseteq X[F/x][f/y]$ and thus by downward closure

$$\mathbb{M}, Y[g/y][G/x] \vDash \phi.$$

Similarly we can prove that $\mathbb{M}, Z \vDash Q_1 y \exists x \phi$, and thus that $\mathbb{M}, X \vDash \Gamma, \exists y Q_1 x \phi \vee Q_1 y \exists x \phi$.

(3) For the Skolem rule assume that there is a derivation of ψ from Γ ending with the Skolem rule. Then there are shorter derivations from Γ of σ and from $\Gamma, S\sigma$ of ψ. By the induction hypothesis we get $\Gamma \vDash \sigma$ and $\Gamma, S\sigma \vDash \psi$. We will prove that $\Gamma \vDash \psi$, by assuming $\mathbb{M}, X \vDash \Gamma$ for some non-empty X and proving that $\mathbb{M}, X \vDash \psi$.

From the proof of Theorem 3.5 in [2] we see that $\mathbb{M} \vDash \sigma$ iff $\mathbb{M} \vDash \exists f_1 \ldots \exists f_k S\sigma$. From $\mathbb{M}, X \vDash \Gamma$ and $\Gamma \vDash \sigma$ we get that $\mathbb{M} \vDash \sigma$ and thus there are $f_1, \ldots, f_k$ such that $(M, f_1, \ldots, f_k) \vDash S\sigma$. Now since the f_i's do not occur in formulas $\Gamma' \subseteq \Gamma$ used in the derivation of ψ, $\mathbb{M}, X \vDash \Gamma$ implies that $(M, f_1, \ldots, f_k), X \vDash \Gamma'$. By locality, we also have $(M, f_1, \ldots, f_k), X \vDash S\sigma$. Therefore, by the induction hypothesis, we get that $(M, f_1, \ldots, f_k), X \vDash \psi$, and, since the f_i's do not occur in ψ, $\mathbb{M}, X \vDash \psi$ follows. $\square$

Theorem 3. *If T is a set of* $\mathrm{D}(Q_1, \check{Q}_1)$ *sentences and ϕ is a* $\mathrm{FO}(Q_1, \check{Q}_1)$ *sentence then* $T \vdash \phi$ *iff* $T \vDash \phi$.

Proof. Assume $T \nvdash \phi$. We build a model of $T' = T \cup \{\neg\phi\} \nvdash \bot$ by translating sentences σ of T into normal form σ_{nf} and considering the $\mathrm{FO}(Q, \check{Q})$ theory $T_S = \{S\sigma_{\mathrm{nf}} \mid \sigma \in T\} \cup \{\neg\phi\}$. This theory is consistent, since otherwise the Skolem rule and Proposition 3 would allow us to derive a contradiction from T'.

Since the deduction system for $\mathrm{D}(Q_1, \check{Q}_1)$ contains Keisler's system [4] we may apply the completeness theorem for $\mathrm{FO}(Q_1)$ and get a model $\mathbb{M}$ of $T_S \cup \{\neg\phi\}$. By the remark made in the proof of Proposition 5 and Proposition 3 $\mathbb{M}$ is also a model of $T \cup \{\neg\phi\}$. Thus $T \nvDash \phi$. $\square$

5 Conclusion

In this article we have presented inference rules and axioms for extensions of dependence logic by monotone generalized quantifiers. We also proved two completeness results for $\mathrm{FO}(Q)$ consequences in the cases where Q either has a weak interpretation or Q is interpreted as "there exist uncountably many." In the first completeness theorem, an important feature of the proof is the approximation of a $\mathrm{D}(Q_1, \check{Q}_1)$ sentence by an infinite set of $\mathrm{FO}(Q)$ sentences. In the second completeness theorem the approximations were replaced by the Skolem rule which however is slightly unsatisfactory due to the extra function symbols f_i used in its formulation. In future work our plan is to further analyze the completeness theorem of $\mathrm{D}(Q_1, \check{Q}_1)$, and replace the Skolem rule with rules that do not rely on the explicit use of the Skolem functions f_i.

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