Reflection principles for the continuum[*]

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APRIL 27, 2001

Abstract

Let HC' denote the set of sets of hereditary cardinality less than 2^ω. We consider reflection principles for HC' in analogy with the Levy reflection principle for HC. Let $\mathcal{B}$ be a class of complete Boolean algebras. The principle $\mathrm{Max}(\mathcal{B})$ says: If $R(x_1, \ldots, x_n)$ is a property which is provably persistent in extensions by elements of $\mathcal{B}$, then $R(a_1, \ldots, a_n)$ holds whenever $a_1, \ldots, a_n \in HC'$ and $R(a_1, \ldots, a_n)$ has a positive $I\!B$-value for some $I\!B \in \mathcal{B}$. Suppose $\mathcal{C}$ is the class of Cohen algebras. We prove that $Con(ZF)$ implies $Con(ZFC + \mathrm{Max}(\mathcal{C}))$. For a different principle, let $\mathcal{CCC}$ be the class of all CCC algebras. We prove that $ZF +$ Levy schema, and $ZFC + \mathrm{Max}(\mathcal{CCC})$ are equiconsistent. $\mathrm{Max}(\mathcal{CCC})$ implies MA, while $\mathrm{Max}(\mathcal{C})$ implies $\neg MA$. We give applications of these reflection principles to Löwenheim-Skolem theorems of extensions of first order logic. For example, $\mathrm{Max}(\mathcal{C})$ implies that the Löwenheim number of the extension of first order logic by the Härtig quantifier is less than 2^ω.

1 INTRODUCTION

In this paper we study new set-theoretic axioms which give the continuum a very large cardinality. As applications we get Löwenheim-Skolem type results for some powerful extensions of first order logic.

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[*] MSC2000 classification numbers 03E50, 03E35 and 03E40. Keywords: Reflections principle, Martin's axiom, Boolean valued models.

[†] The second author has for many years been unable to reach the first author to get his approval of the final manuscript. Therefore the second author takes all resposibility for any errors or shortcomings of the paper.

One of the fundamental properties of the universe of sets is the fact that

$$HC = \{x \mid x \text{ is hereditarily countable}\}$$

reflects all Σ_1-properties, that is, if $a \in HC$ and $P(a)$ is a true Σ_1-property of a then $HC \models P(a)$. If $2^\omega > \omega_1$, there is an interesting variant of HC:

$$HC' = \{x \mid x \text{ is hereditarily of power} < 2^\omega\}.$$

The basic observation underlying this paper is that while HC' trivially reflects all Σ_1-properties, it may, in a suitable model of set theory, reflect much more. Typically, it may reflect all properties which are Σ_1 with respect to the class of all cardinals.

The strongest and perhaps the most interesting reflection principle to be considered is the following: Let $\mathcal{B}$ be a class of complete Boolean algebras. The principle

$$\text{Max}(\mathcal{B})$$

says: If $R(x_1, \ldots, x_n)$ is a property which is provably persistent in extensions by elements of $\mathcal{B}$, then $R(a_1, \ldots, a_n)$ holds whenever $a_1, \ldots, a_n \in HC'$ and $R(a_1, \ldots, a_n)$ has a positive $\mathbb{B}$-value for some $\mathbb{B} \in \mathcal{B}$.

The relevant classes $\mathcal{B}$ to be considered here are the class $\mathcal{C}$ of all Cohen algebras (for exploding 2^ω) and the class $\mathcal{CCC}$ of all CCC algebras. The main results are:

1. $Con(ZF) \leftrightarrow Con(ZFC + \text{Max}(\mathcal{C}))$.
2. $Con(ZF + \text{Levy schema}) \leftrightarrow Con(ZFC + \text{Max}(\mathcal{CCC}))$.
3. $\text{Max}(\mathcal{CCC}) \rightarrow MA$.

The principle $\text{Max}(\mathcal{C})$ is inconsistent with MA and hence inconsistent with $\text{Max}(\mathcal{CCC})$. Thus we have two mutually inconsistent reflection principles which both make HC' reflect all properties Σ_1 with respect to the class of all cardinals.

Abstract logics relevant in the applications of these reflection principles are logics the satisfaction relation of which is preserved by $\mathcal{C}$ or $\mathcal{CCC}$-extensions. An example of such a logic is the logic $L(I)$ with the Härtig-quantifier

$$IxyA(x)B(y) \leftrightarrow |\{\ a \mid A(a)\ \}| = |\{\ b \mid B(b)\ \}|$$

and its extensions by infinitary operations or new quantifiers like the Magidor-Malitz quantifier

$$Q^2xyA(x,y) \leftrightarrow \exists \text{ uncountable } X \text{ s.t.} \forall \{a,b\} \in [X]^2 A(a,b).$$

The principles $\text{Max}(\mathcal{CCC})$ and $\text{Max}(\mathcal{C})$ imply that the Löwenheim number of the infinitary logic

$$L_{2^\omega\omega}(I)$$

(for example) is 2^ω.

As an example of the use of MA in this connection, we have the following definability result (Theorem 53):

If MA holds and the Löwenheim number of $L(I)$ is $< 2^\omega$, then the decision problem of $L(I)$ is Σ_1^2-definable.

If $V = L$, then the decision problem of $V(I)$ is not Σ_n^m for any $m, n < \omega$ [12].

We shall also consider generalisations to $H(2^{\omega_1})$ (in subsection 2.5) and a strong form of the downward Löwenheim-Skolem theorem (in subsection 3.4).

The paper divides into two parts. The first part is purely set-theoretic. The reflection principles are introduced and studied in detail. The main results are Theorem 18, Theorem 25 and Theorem 27. The second part consists of applications to model theory and to the study of abstract logics in particular. The key observation is that our reflection principles can be used to yield downward Löwenheim-Skolem-theorems to abstract logics satisfying certain persistency criteria (Proposition 43).

The material of this paper was prepared in the early 1980's. The second author eventually lost contact with the first author and the manuscript remained unpublished. Meanwhile the second author has experienced growing interest in the material of the paper, and has eventually decided to make the paper accessible by publishing it. As a result of the history of the paper, the second author has not been able to get final comments from the first author. Therefor the second author is solely responsible for any errors or shortcomings of the paper.

2 SET THEORY

The important notion of *persistency* of a predicate under a class of Boolean algebras is introduced and studied in Section 2.1. In Section 2.2 we introduce the new *reflection* and *maximality* principles. The special case of Cohen-algebras is studied in Section 2.3 and the case of CCC-algebras in Section 2.4. The final section of this part surveys some other possibilities. For example, we extend some results from 2^ω to 2^{ω_1}.

2.1   Basic notions: persistency

One of the key notions of this paper is that of persistency of a predicate of set theory under a family of Boolean extensions. In this section we develop the required theory and examples related to persistency.

We shall work in the conservative extension of ZFC set theory obtained by introducing a predicate symbol for every definable relation.

The following predicates, among others, will be relevant in this paper:

$$
\begin{array}{rcl}
Cd(\alpha) & : & \alpha \text{ is a cardinal,} \\
Rg(\alpha) & : & \alpha \text{ is a regular cardinal,} \\
Sln(T) & : & T \text{ is a Souslin tree,} \\
Cd^L(\alpha) & : & \alpha \text{ is a cardinal of } L.
\end{array}
$$

Definition 1 Let $I\!B$ be a complete Boolean algebra and $P(x_1, \ldots, x_n)$ a predicate (of set theory). We say that $P(x_1, \ldots, x_n)$ is *preserved under extension by* $I\!B$ if for all

3

$a_1, \ldots, a_n$:
$$P(a_1, \ldots, a_n) \text{ implies } [\![P(a_1, \ldots, a_n)]\!]^{\mathbb{B}} = \mathbf{1}.$$

The predicate $P(x_1, \ldots, x_n)$ is *absolute under extension by* $\mathbb{B}$ if both $P(x_1, \ldots, x_n)$ and $\neg P(x_1, \ldots, x_n)$ are persistent under extension by $\mathbb{B}$.

Example 2 Let $P(x_1, \ldots, x_n)$ be an arbitrary predicate. Let $P^L(x_1, \ldots, x_n)$ be the predicate
$$x_1 \in L \wedge \ldots \wedge x_n \in L \wedge L \models P(x_1, \ldots, x_n).$$

Then the predicate $P^L(x_1, \ldots, x_n)$ is absolute under extension by any complete Boolean algebra. This is because $P^L(x_1, \ldots, x_n)$ is provably equivalent to a formula consisting of a string of quantifiers over ordinals followed by a Σ_0-formula.

Our prime interest will be in predicates which are persistent with respect to certain classes of complete Boolean algebras.

Definition 3 Let $\mathcal{B}$ be a class of complete Boolean algebras. We say that a predicate $P(x_1, \ldots, x_n)$ is $\mathcal{B}$-*persistent* if it is persistent under extension by any $\mathbb{B} \in \mathcal{B}$. The $\mathcal{B}$-*absoluteness* of $P(x_1, \ldots, x_n)$ is defined similarly. We say that $P(x_1, \ldots, x_n)$ is provably $\mathcal{B}$-persistent if it is provable in ZFC that $\mathcal{B}$ is a class of complete Boolean algebras and $P(x_1, \ldots, x_n)$ is $\mathcal{B}$-persistent. The provable $\mathcal{B}$-absoluteness of $P(x_1, \ldots, x_n)$ is defined similarly.

Note, that the predicates $P^L(x_1, \ldots, x_n)$ of Example 2 are provably $\mathcal{B}$-absolute for the class $\mathcal{B}$ of all complete Boolean algebras.

Example 4 Let $\mathcal{CCC}$ denote the class of all complete Boolean algebras with the countable chain condition (CCC). The predicates $Cd(x)$ and $Rg(x)$ are provably $\mathcal{CCC}$-absolute, as is well-known.

Example 5 If there is a Souslin tree T, then T gives rise to a $\mathbb{B} \in \mathcal{CCC}$ such that

$$[\![Sln(\check{T})]\!]^{\mathbb{B}} = \mathbf{0}.$$

Indeed, if $\mathbb{B}$ is the regular-open algebra of the reverse ordering of T, and G is the canonical generic ultrafilter on $\mathbb{B}$, then $\cup G$ is an uncountable branch in T "killing" the Souslinity of T. Thus, if ZF is consistent, then $Sln(x)$ is not provably $\mathcal{CCC}$-absolute.

Example 6 For any cardinal κ of uncountable cofinality let C_κ denote the notion of forcing
$$\{p \mid p \text{ is a finite partial mapping } \omega \times \kappa \to 2\}$$

with the partial ordering $\supseteq$. This is the notion of forcing which explodes 2^ω to κ. Let $\mathbb{C}_\kappa$ be the regular-open algebra of C_κ and $\mathcal{C}$ the class of all $\mathbb{C}_\kappa$, $cf(\kappa) > \omega$. We refer to $\mathcal{C}$ as the class of *Cohen-algebras*.

4

Note that $\mathcal{C} \subseteq \mathcal{CCC}$. The predicate $Sln(x)$ is provably $\mathcal{C}$-absolute. To see this, we observe at first that $\neg Sln(x)$ is trivially provably $\mathcal{C}$-persistent, for once a Souslin tree is killed there is no way it can come to life again in a future Boolean extension. On the other hand, it is well-known that adding Cohen-reals does not destroy a Souslin tree (see Proposition 40).

There is a weaker notion of preservation that we shall also exploit. It is related to embeddings between Boolean algebras. The motivation for considering particular embeddings rather than arbitrary ones is that some predicates may be only persistent in certain canonical and nice embeddings and fail to persist in some non-standard ones.

Definition 7 A *complete Boolean embedding* is a triple

$$E = (\mathbb{B}_1, j, \mathbb{B}_2)$$

where $\mathbb{B}_1$ and $\mathbb{B}_2$ are complete Boolean algebras and j a complete embedding of $\mathbb{B}_1$ into $\mathbb{B}_2$. A predicate $P(x_1, \ldots, x_n)$ is *persistent under E*, if for all $a_1, \ldots, a_n \in V^{\mathbb{B}_1}$:

$$j[\![P(a_1, \ldots, a_n)]\!]^{\mathbb{B}_1} \leq [\![P(ja_1, \ldots, ja_n)]\!]^{\mathbb{B}_2}.$$

If $\mathcal{E}$ is a class of complete Boolean embeddings, and $P(x_1, \ldots, x_n)$ is persistent under every $E \in \mathcal{E}$, we say that $P(x_1, \ldots, x_n)$ is $\mathcal{E}$-*persistent*. If $\mathcal{E}$ is provably a class of complete Boolean embeddings, and $P(x_1, \ldots, x_n)$ is $\mathcal{E}$-persistent, provably in ZFC, we say that $P(x_1, \ldots, x_n)$ is *provably $\mathcal{E}$-persistent*. $\mathcal{E}$-absoluteness and provable $\mathcal{E}$-absoluteness are defined similarly.

Note, that persistency under extension by $\mathbb{B}$ is equivalent to persistency under the trivial embedding $(\mathbf{2}, j, \mathbb{B})$. In other words, if $\mathcal{E}$ is a class of complete Boolean embeddings, and

$$\mathcal{E}' = \{\mathbb{B} \mid (\mathbf{2}, j, \mathbb{B}) \in \mathcal{E}\}$$

then $\mathcal{E}$-persistency implies $\mathcal{E}'$-persistency. The converse is not true in general ($\mathcal{E}'$ may be empty), but it can be proved for sufficiently regular $\mathcal{E}$.

Definition 8 A class $\mathcal{E}$ of complete Boolean embeddings is *divisible*, if for any triple $(\mathbb{B}_1, j, \mathbb{B}_2) \in \mathcal{E}$ there are sets $h, \mathbb{D} \in V^{\mathbb{B}_1}$ and an isomorphism k such that

(i) $V^{\mathbb{B}_1} \models (\mathbf{2}, h, \mathbb{D}) \in \mathcal{E}$,

(ii) The diagram of Figure 1 commutes, where i is the canonical embedding.

Let

$$\mathcal{CCC}_e$$

be the class of complete Boolean embeddings between elements of $\mathcal{CCC}$. It was proved in [11, pp. 214-215] that $\mathcal{CCC}_e$ is divisible. Let
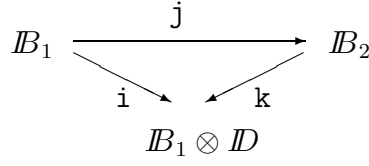
$$\mathcal{C}_e$$

Figure 1: Divisibility.

be the class of complete Boolean embeddings

$$(\mathbb{C}_\kappa, j, \mathbb{C}_\lambda)$$

where $\kappa \leq \lambda$, $cf(\kappa) > \omega, cf(\lambda) > \omega$ and j is canonically generated by the identity $id : C_\kappa \to C_\lambda$ via the embeddings $[\ ]_\kappa : C_\kappa \to \mathbb{C}_\kappa$,

$$[p]_\kappa = \{f \in {}^{\omega \times \kappa}2 \mid p \subseteq f\}.$$

The class $\mathcal{C}_e$ is divisible, as is easily seen.

Proposition 9 *Suppose $\mathcal{E}$ is a divisible class of complete embeddings and*

$$\mathcal{E}' = \{\mathbb{B} \mid (\mathbf{2}, j, \mathbb{B}) \in \mathcal{E}\}.$$

If a predicate is provably $\mathcal{E}'$-persistent, then it is $\mathcal{E}$-persistent.

Proof. Suppose $(\mathbb{B}_1, j, \mathbb{B}_2) \in \mathcal{E}$ and $a_1, \ldots, a_n \in V^{\mathbb{B}_1}$. Find $\mathbb{D} \in V^{\mathbb{B}_1}$ such that (i) and (ii) of Definition 8 hold. Let $R(x_1, \ldots, x_n)$ be a provably $\mathcal{E}'$-persistent perdicate. We show:

$$j[\![R(a_1, \ldots, a_n)]\!]^{\mathbb{B}_1} \leq [\![R(ja_1, \ldots, ja_n)]\!]^{\mathbb{B}_2}. \tag{1}$$

As $R(x_1, \ldots, x_n)$ is $\mathcal{E}'$-persistent in $V^{\mathbb{B}_1}$, we have

$$
\begin{aligned}
kj[\![R(a_1, \ldots, a_n)]\!]^{\mathbb{B}_1} &= i[\![R(a_1, \ldots, a_n)]\!]^{\mathbb{B}_1} \\
&\leq i[\![[\![R(a_1, \ldots, a_n)]\!]^{\mathbb{D}} = \mathbf{1}]\!]^{\mathbb{B}_1}.
\end{aligned}
$$

By the very definition of i,

$$V^{\mathbb{B}_1} \models i[\![[\![R(a_1, \ldots, a_n)]\!]^{\mathbb{D}} = \mathbf{1}]\!]^{\mathbb{B}_1} = [\![R(a_1, \ldots, a_n)]\!]^{\mathbb{D}}.$$

By the basic properties of $\otimes$, this means

$$i[\![[\![R(a_1, \ldots, a_n)]\!]^{\mathbb{D}} = \mathbf{1}]\!]^{\mathbb{B}_1} = [\![R(ia_1, \ldots, ia_n)]\!]^{\mathbb{B}_1 \otimes \mathbb{D}}.$$

Using condition (ii) of Definition 8 yields

$$[\![R(ia_1, \ldots, ia_n)]\!]^{\mathbb{B}_1 \otimes \mathbb{D}} = k[\![R(ja_1, \ldots, ja_n)]\!]^{\mathbb{B}_2}.$$

Thus we have proved

$$kj[\![R(a_1,\ldots,a_n)]\!]^{I\!B_1} \leq k[\![R(ja_1,\ldots,ja_n)]\!]^{I\!B_2},$$

which implies (1). $\square$

Corollary 10 *The predicates $Cd(x)$ and $Rg(x)$ are provably $\mathcal{CCC}_e$-absolute and the predicate $Sln(x)$ is provably $\mathcal{C}_e$-absolute.*

2.2 Basic notions: reflection

Reflection is one of the basic properties of the universe of sets. The idea is that any property of the whole universe is already permitted by a subuniverse which is a set. Moreover there are special sets which are particularly useful in reflection, like the set HC of hereditarily countable sets. The fact that HC reflects all Σ_1-properties can be viewed as an indication of the inaccessibility of ω_1 with respect to Σ_1-operations. We propose reflection principles which have a similar effect on 2^ω with respect to certain generalized Σ_1-operations.

Definition 11 The hereditary cardinality $HC(x)$ of a set x is the cardinality of the transitive closure $TC(x)$ of x. For any cardinal κ denote

$$H(\kappa) = \{x \mid HC(x) < \kappa\}.$$

HC denotes $H(\omega_1)$. We use HC' to denote $H(2^\omega)$.

Let us now recall the usual reflection principle of set theory. By a c.u.b. class of cardinals we understand a proper class of cardinals which is closed under sups of its subsets.

Theorem 12 (*Reflection principle*) *Let $R(y, x_1, \ldots, x_n)$ be a predicate of set theory. There is a c.u.b. class C of cardinals κ such that $H(\kappa)$ reflects the predicate $R(y, x_1, \ldots, x_n)$ that is*

$$\forall x_1, \ldots, x_n \in H(\kappa)(\exists y R(y, x_1, \ldots, x_n) \rightarrow \exists y \in H(\kappa)R(y, x_1, \ldots, x_n)).$$

Moreover, if $R(y, x_1, \ldots, x_n)$ is Σ_1, we may choose C to be the class of all uncountable cardinals.

Definition 13 Let $R_1, \ldots, R_n$ be predicates of set theory. The Σ_1-predicates of the extended language $\{\epsilon, R_1, \ldots, R_n\}$ are called $\Sigma_1(R_1, \ldots, R_n)$-predicates.

If R is a $\mathcal{B}$-absolute predicate, it is trivial that all $\Sigma_1(R)$-predicates are $\mathcal{B}$-persistent.

Definition 14 Let $R_1, \ldots, R_n$ be predicates. The schema

$$\forall x_1 \ldots x_m \in HC'(\exists y R(y, x_1, \ldots, x_m) \to \exists y \in HC' R(y, x_1, \ldots, x_m))$$

for all $\Sigma_1(R_1, \ldots, R_n)$-predicates $R(y, x_1, \ldots, x_n)$, is denoted by

$$\mathrm{Refl}(R_1, \ldots, R_n). \tag{2}$$

If $\mathcal{B}$ is a class of complete Boolean algebras, the union of all schemata (2), where $R_1, \ldots, R_n$ range over all provably $\mathcal{B}$-persistent predicates, is denoted by

$$\mathrm{Refl}(\mathcal{B}).$$

Note, that if R is a Σ_0-predicate, then $\mathrm{Refl}(R)$ is provable. It is not provable in general, of course. For example, the schema $\mathrm{Refl}(Cd)$ says that 2^ω is so large that it is closed under any $\Sigma_1(Cd)$-function on ordinals. Examples of such functions are

$$
\begin{aligned}
f(\alpha) &= \aleph_\alpha \\
f(\alpha, \beta) &= \aleph_{\alpha+\beta} \\
f(\alpha, 0) &= \aleph_\alpha, f(\alpha, \beta+1) = \aleph_{f(\alpha,\beta)}, \\
&\quad f(\alpha, \nu) = \cup_{\beta<\nu} f(\alpha, \beta), \text{ for limit } \nu \\
f(\alpha) &= \quad \text{the } \alpha\text{'th } WC \text{ cardinal in } L.
\end{aligned}
$$

Similarly, $\mathrm{Refl}(Rg)$ says that 2^ω is closed under all $\Sigma_1(Rg)$-functions, e.g. under

$$f(\alpha) = \text{ the } \alpha\text{'th weakly inaccessible cardinal.}$$

Thus $\mathrm{Refl}(Cd)$ is a strong axiom of infinity for the continuum. Even more so is the schema $\mathrm{Refl}(\mathcal{CCC})$ as the following simple lemma shows:

Lemma 15 *$\mathrm{Refl}(\mathcal{CCC})$ implies the schema $L_{2^\omega} \prec L$.*

Proof. We use the so called Tarski-criterion for elementary equivalence. So let us assume $R(y, x_1, \ldots, x_n)$ is a predicate, $a_1, \ldots, a_n \in L_{2^\omega}$, and

$$\exists y(y \in L \wedge R^L(y, a_1, \ldots, a_n)).$$

The predicate

$$\exists y(y \in L_\alpha \wedge R^L(y, a_1, \ldots, a_n))$$

is a provably $\mathcal{CCC}$-persistent predicate of $\alpha, a_1, \ldots, a_n$, whence by $Refl(\mathcal{CCC})$,

$$\exists y(y \in L_{2^\omega} \wedge R^L(y, a_1, \ldots, a_n)).$$

□

Note, that

$$\mathrm{Refl}(Cd) \subseteq \mathrm{Refl}(Rg) \subseteq \mathrm{Refl}(\mathcal{CCC}) \subseteq \mathrm{Refl}(\mathcal{C}).$$

In the following definition we introduce another approach to reflection, one that leads to even more powerful principles.

Definition 16 Let $\mathcal{B}$ be a class of complete Boolean algebras. The schema

$$\forall x_1, \ldots, x_n \in HC'(R(x_1, \ldots, x_n) \leftrightarrow (\exists \mathbb{B} \in \mathcal{B})[\![R(x_1, \ldots, x_n)]\!]^{\mathbb{B}} > \mathbf{0}),$$

where $R(x_1, \ldots, x_n)$ ranges over provably $\mathcal{B}$-persistent predicates, is denoted by

$$\mathrm{Max}(\mathcal{B}).$$

The intuition behind $\mathrm{Max}(\mathcal{B})$ is that it demands elements of HC' to have all properties that they could have in some forcing extension of type $\mathcal{B}$ and that they would have in any further forcing extension of type $\mathcal{B}$. The relation to reflection is revealed by the following result:

Proposition 17 *Suppose $\mathbb{B}$ is a class of complete Boolean algebras such that*

(i) $\mathcal{C} \subseteq \mathcal{B}$.

(ii) *Cd is provably $\mathcal{B}$-persistent.*

Then $\mathrm{Max}(\mathcal{B})$ implies $\mathrm{Refl}(\mathcal{B})$.

Proof. To prove $\mathrm{Refl}(\mathcal{B})$ we consider $R(y, x_1, \ldots, x_n)$, a provably $\mathcal{B}$-persistent predicate, and sets $a_1, \ldots, a_n \in HC'$ such that $\exists y R(y, a_1, \ldots, a_n)$. Let b be a set such that $R(b, a_1, \ldots, a_n)$. The main point is that (i) implies the existence of $\mathbb{B} \in \mathcal{B}$ such that $[\![b \in HC']\!]^{\mathbb{B}} = \mathbf{1}$. Consider the predicate

$$S(x_1, \ldots, x_n) \leftrightarrow \exists \alpha \exists y(\quad HC(y) < |\alpha| \wedge |\alpha| \leq 2^\omega \wedge$$
$$R(y, x_1, \ldots, x_n)).$$

This predicate is provably $\mathcal{B}$-persistent (here one uses (ii)). As $[\![S(a_1, \ldots, a_n)]\!]^{\mathbb{B}} = \mathbf{1}$, we have $S(a_1, \ldots, a_n)$ by $\mathrm{Max}(\mathcal{B})$. Thus $\exists y \in HC' R(y, a_1, \ldots, a_n)$. $\square$

Our later results show that $\mathrm{Refl}(\mathcal{B})$ does not in general imply $\mathrm{Max}(\mathcal{B})$. An overview of the principles introduced is given in Figure 2. The schema $\mathrm{Max}(\mathcal{CCC})$ does not only say that the continuum is large. We shall prove later that it also implies Martin's axiom (MA). Curiously enough, $\mathrm{Max}(\mathcal{C})$ implies $\neg MA$. Thus we have a whole spectrum of axioms of infinity, or reflection principles for 2^ω, starting from the weakest $\mathrm{Refl}(Cd)$ and ending with the strong maximality principles, emerging to completely different directions.

2.3 Cohen-extensions

In this section we shall consider various models for $\mathrm{Max}(\mathcal{C})$. After proving the mere consistency of $\mathrm{Max}(\mathcal{C})$ we consider $\mathrm{Max}(\mathcal{C})$ together with the regularity of 2^ω. This turns out to be equiconsistent with the so called Levy schema.

Theorem 18 $Con(ZF)$ *implies* $Con(ZFC + Max(\mathcal{C}))$.

$$\text{Max}(\mathcal{CCC}) \qquad\qquad \text{Max}(\mathcal{C})$$
$$\text{Refl}(\mathcal{C})$$
$$\text{Refl}(\mathcal{CCC})$$
$$\downarrow$$
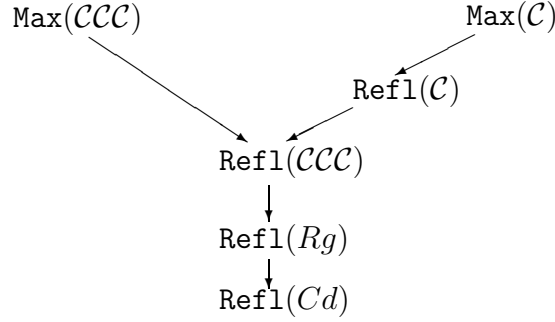$$\text{Refl}(Rg)$$
$$\downarrow$$
$$\text{Refl}(Cd)$$

Figure 2: Reflection and maximality principles.

Proof. Let $R_1(x_1, \ldots, x_{n_1}), \ldots, R_m(x_1, \ldots, x_{n_m})$ be provably $\mathcal{C}$-persistent predicates. We shall find a $I\!\!D \in \mathcal{C}$ such that $V^{I\!\!D}$ satisfies

$$\forall x_1 \ldots x_{n_i} \in HC'(R_i(x_1, \ldots, x_{n_i}) \leftrightarrow (\exists I\!\!B \in \mathcal{C}) [\![R_i(x_1, \ldots, x_{n_i})]\!]^{I\!\!B} > \mathbf{0}) \qquad (3)$$

for $i = 1, \ldots, m$. This together with the Compactness Theorem will give the theorem. Let $\Phi_i(\kappa, x_1, \ldots, x_{n_i})$ be the predicate

$$Cd(\kappa) \wedge x_1, \ldots, x_{n_i} \in V^{\mathbb{C}_\kappa} \wedge V^{\mathbb{C}_\kappa} \models (x_1, \ldots, x_{n_i}) \in HC' \wedge R_i(x_1, \ldots, x_{n_i}),$$

where $\mathbb{C}_\kappa$ is as in Example 6. By the Reflection Principle, there is a cardinal κ of uncountable cofinality such that $H(\kappa)$ reflects every formula $\Phi_i(y, x_1, \ldots, x_{n_i})$, $i = 1, \ldots, m$. We claim that $V^{\mathbb{C}_\kappa}$ satisfies (3) for $i = 1, \ldots, m$. For this end, let $1 \le i \le m$. The implication from the left to the right in (3) is trivial. Suppose then λ is a cardinal of uncountable cofinality and $a_1, \ldots, a_{n_i} \in V^{\mathbb{C}_\kappa}$ so that

$$V^{\mathbb{C}_\kappa} \models a_1, \ldots, a_{n_i} \in HC' \ \wedge \ [\![R_i(a_1, \ldots, a_{n_i})]\!]^{\mathbb{C}_\lambda} > \mathbf{0}.$$

Let $\mu = \text{Max}(\kappa, \lambda)$. It follows from $\mathbb{C}_\kappa \otimes \mathbb{C}_{\check\lambda} \cong \mathbb{C}_\mu$ and from the homogeneity of $\mathbb{C}_\lambda$ that

$$V^{\mathbb{C}_\mu} \models R_i(a_1, \ldots, a_{n_i}).$$

Let f be a permutation of $\mathbb{C}_\kappa$ so that if f is canonically extended to $\mathbb{C}_\mu$ and to $V^{\mathbb{C}_\mu}$, then it means no loss of generality to assume that $fa_1, \ldots, fa_{n_i} \in H(\kappa)$. Thus we have $\Phi_i(\mu, fa_1, \ldots, fa_{n_i})$. By the choice of κ, we have $\Phi_i(\nu, fa_1, \ldots, fa_{n_i})$ for some $\nu < \kappa$. By the $\mathcal{C}$-persistency of $R_i(x_1, \ldots, x_{n_i})$,

$$V^{\mathbb{C}_\kappa} \models R_i(fa_1, \ldots, fa_{n_i}).$$

But f defines an isomorphism of $V^{\mathbb{C}_\kappa}$. Thus

$$V^{\mathbb{C}_\kappa} \models R_i(a_1, \ldots, a_{n_i}).$$

as desired. $\square$

Looking at the above proof one sees that the same inference yields the following result:

10

Theorem 19 *Suppose $P_1, \ldots, P_n$ are provably $\mathcal{C}$-persistent predicates. There is a $\mathbb{B} \in \mathcal{C}$ such that*

$$V^{\mathbb{B}} \models \mathit{Refl}(P_1, \ldots, P_n).$$

Moreover, if κ is any regular cardinal $> \omega$, we can also ensure $cf(2^\omega) = \kappa$ in $V^{\mathbb{B}}$. $\square$

We can even make 2^ω regular, but for this we need large cardinals in the ground model, since $\mathit{Refl}(Cd)$ together with the regularity of 2^ω imply weak inaccessibility of 2^ω.

Recall the *Levy schema*:

Every *c.u.b.* class of ordinals contains a regular cardinal.

This schema implies the existence of a c.u.b. class of hyperinaccessible cardinals (and more). On the other hand, $H(\kappa)$ satisfies the schema whenever κ is Mahlo.

Lemma 20 *If 2^ω is regular and $\mathit{Refl}(\mathcal{CCC})$, then Levy schema holds in L_{2^ω}.*

Proof. Suppose D is a c.u.b. class of ordinals in L_{2^ω}. Since Lemma 15 implies $L_{2^\omega} \prec L$, the definition of D gives rise to a c.u.b. class C of ordinals in L such that $D = C \cap 2^\omega$. Thus 2^ω is a regular element of C. Using $L_{2^\omega} \prec L$ again, we get an L-regular element for D. $\square$

Corollary 21 *The following theories are equiconsistent:*

1. *ZF + Levy schema,*
2. *ZFC + Max($\mathcal{C}$) + 2^ω is regular,*
3. *ZFC + Refl($\mathcal{CCC}$) + 2^ω is regular.*

Proof. To prove $(1) \rightarrow (2)$ it suffices to follow the proof of Theorem 18 and choose κ regular. The implication $(2) \rightarrow (3)$ follows from Proposition 17. Finally $(3) \rightarrow (1)$ by Lemma 20. $\square$

Let $\mathcal{M}$ be the class of product measure algebras with the canonical embeddings. It is proved in [3, p. 48] that $\mathcal{M}$ is divisible. Imitating the proof of Theorem 18 we can, assuming the consistency of the schema:

Every c.u.b. class of ordinals contains a measurable cardinal.

prove the consistency of Max($\mathcal{M}$) + 2^ω is real-valued measurable. Clearly, Max($\mathcal{M}$) implies Refl($\mathcal{CCC}$).

We shall now show that Max($\mathcal{C}$) is inconsistent with MA. We are indebted to U. Avraham for pointing out how this is proved and for letting us include his proof in this paper. Consider the predicate

$$P(x) \leftrightarrow \quad x = \langle x_\alpha \mid \alpha < \beta \rangle \text{ is a sequence of infinite subsets of } \omega \text{ such that}$$
$$\alpha < \gamma < \beta \rightarrow x_\gamma - x_\alpha \text{ is finite and there is no infinite}$$
$$y \subseteq \omega \text{ such that } y - x_\alpha \text{ is finite for all } \alpha < \beta.$$

Proposition 22   *1.* $ZFC \vdash \exists x P(x)$,

  2. $ZFC \vdash \forall x (P(x) \rightarrow \exists I\!B \in \mathcal{CCC}(V^{I\!B} \models \neg P(x)))$,

  3. $ZFC + MA \vdash \neg(\exists x \in HC')P(x)$,

  4. $P(x)$ *is provably* $\mathcal{C}$*-persistent.*

Proof.   Claim (1) follows from Zorn's Lemma. To prove (2), suppose $P(x)$ holds for $x = \langle x_\alpha | \alpha < \beta \rangle$. Let

$$\mathcal{P} = \{\langle t, u \rangle | t \subset \omega \text{ and } u \subset \beta \text{ are finite }\}$$

and

$$\langle t, u \rangle \leq_{\mathcal{P}} \langle t', u' \rangle \text{ if and only if } t \supseteq t', u \supseteq u' \text{ and } t - x_\alpha = t' - x_\alpha \text{ for } \alpha \in u'.$$

Obviously, $\mathcal{P}$ has CCC and forces $\neg P(x)$. To prove (3) it suffices to observe that if $x \in HC'$ satisfies $P(x)$, then the above $\mathcal{P}$ contradicts MA. $\square$

Corollary 23 *The predicate* $P(x)$ *is provably* $\mathcal{C}$*-persistent but provably not* $\mathcal{CCC}$*-persistent.*

Corollary 24 *Refl*$(\mathcal{C})$ *is inconsistent with* MA.

Remark: Max$(\mathcal{C})$ is consistent with $\square$, the existence of a Souslin tree and the existence of a Kurepa tree, for we may assume $V = L$ in the ground model in the proof of Theorem 18. On the other hand, Jensen and Schlechta [4] prove that it is consistent, relative to the consistency of a Mahlo cardinal, that there are no Kurepa trees in CCC-extensions of the universe. Thus we get the consistency of Max$(\mathcal{C})$ with the non-existence of Kurepa trees, relative to the consistency of a Mahlo cardinal.

2.4   CCC-extensions

The principle Max$(\mathcal{CCC})$ differs drastically from Max$(\mathcal{C})$. This follows from the first results of this section which show that MA is a consequence of Max$(\mathcal{CCC})$. Thus Max$(\mathcal{CCC})$ is proof-theoretically at least as strong as Levy schema. In fact we prove the equiconsistency of Max$(\mathcal{CCC})$ and Levy schema. The section ends with a study of some weaker principles.

The following characterization of MA is due to J. Stavi and independently to J. Bagaria [1]:

Theorem 25 *Let* $\mathfrak{M}$ *be a countable model of* ZFC. *Then* $\mathfrak{M} \models MA$ *if and only if*

$$HC'^{\mathfrak{M}} \prec_{\Sigma_1} \mathfrak{N}$$

for every CCC *extension* $\mathfrak{N}$ *of* $\mathfrak{M}$.

Proof. We prove the 'if' part first. Let $\mathcal{P}$ be a CCC partially ordered set and D a set of power $< 2^\omega$ of dense subsets of $\mathcal{P}$. We may assume that $\mathcal{P}, D \in HC'$. Let $\mathfrak{M}$ be the ground model, and $\mathfrak{N}$ a generic extension obtained by forcing with $\mathcal{P}$. Then $\mathfrak{N} \models$ "$\mathcal{P}$ has a D-generic filter". As this is a Σ_1-property of $\mathcal{P}$ and D, we can use the assumed reflection property of $HC'^{\mathfrak{M}}$ to obtain the desired result:

$$\mathfrak{M} \models \text{"$\mathcal{P}$ has a D-generic filter"}.$$

To prove the converse, suppose $\mathfrak{N}$ is a generic extension of $\mathfrak{M}$ obtained by forcing with a CCC notion of forcing $\mathcal{P}$. To prove

$$HC'^{\mathfrak{M}} \prec_{\Sigma_1} \mathfrak{N},$$

suppose $R(x, y)$ is a Σ_0-predicate, $a \in HC'^{\mathfrak{M}}$ and $b \in \mathfrak{N}$ such that $\mathfrak{N} \models R(a, b)$. We want to find a set $y \in HC'^{\mathfrak{M}}$ such that $HC'^{\mathfrak{M}} \models R(a, y)$.

Let κ be the hereditary cardinality of $a \cup b$. If κ is not in $HC'^{\mathfrak{M}}$, one needs but an application of the Reflection Principle to find another such b for which $\kappa \in HC'^{\mathfrak{M}}$. We shall now define an infinitary sentence $\phi \in L_{2^\omega \omega}$ in $\mathfrak{M}$. A consideration of Boolean valued models of ϕ will lead us to the desired conclusion. The sentence ϕ will have two non-logical symbols only: the binary predicate-symbol xEy (for "membership") and a constant-symbol $\mathbf{a}$. If c is any set, let

$$\phi_c(x) = \forall y(yEx \leftrightarrow \bigvee_{d \in c} \phi_d(y)).$$

Let $h(x, y)$ be the Σ_1-formula of the language $\{E\}$ which intuitively says: "the hereditary cardinality of x is $\leq y \in On$". Finally, let ϕ be the conjunction of

1. $\forall xy(\forall z(zEx \leftrightarrow zEy) \rightarrow x = y)$
2. $\phi_a(\mathbf{a})$
3. $\bigwedge_{c \in TC(\{a\})} \exists x \phi_c(x)$
4. $\bigwedge_{\alpha \leq \kappa} \exists x \phi_\alpha(x)$
5. $\forall x \exists \alpha(\phi_\kappa(\alpha) \wedge h(x, \alpha))$
6. $\exists y \exists \alpha(R(\mathbf{a}, y) \wedge h(y, \alpha) \wedge \phi_\kappa(\alpha))$.

Clearly, the structure $(H(\kappa^+), \in, a)$ is a model of ϕ in $\mathfrak{N}$. To end the proof, we only need to show that ϕ has a model in $\mathfrak{M}$, for then by the Σ_1-nature of (6), we obtain a set $y \in HC'^{\mathfrak{M}}$ such that $R(a, y)$.

So let $\mathfrak{A}$ be a model of ϕ in $\mathfrak{N}$. By the Löwenheim-Skolem theorem of $L_{2^\omega \omega}$ we may assume $\mathfrak{A}$ has power $< (2^\omega)^{\mathfrak{M}}$ in $\mathfrak{N}$ (note, that $(2^\omega)^{\mathfrak{M}}$ is regular in $\mathfrak{N}$). We may assume that the domain A of $\mathfrak{A}$ is in fact an element of $HC'^{\mathfrak{M}}$.

Suppose $\mathbb{B}$ is the regular-open algebra of $\mathcal{P}$ and G is a generic filter on $\mathbb{B}$ such that $\mathfrak{N} = \mathfrak{M}[G]$. We can easily turn $\mathfrak{A}$ into a $\mathbb{B}$-valued first order structure $\mathfrak{A}'$ in $\mathfrak{M}$ such that the $\mathbb{B}$-value b_0 of ϕ in $\mathfrak{A}'$ is in G.

If ψ is a subformula of ϕ with the free variables $x_1, \ldots, x_n$ and $a_1, \ldots, a_n \in A$, let $D(\psi, a_1, \ldots, a_n)$ be the set

$$\{[\![\psi_\alpha(a_1, \ldots, a_n)]\!]^{\mathfrak{A}'} \mid \psi_\alpha(x_1, \ldots, x_n) \in \Psi\}$$

if ψ is $\bigwedge \Psi$ or $\bigvee \Psi$, and

$$\{[\![\theta(a_1, \ldots, a_i, a, a_{i+1}, \ldots, a_n)]\!]^{\mathfrak{A}'} \mid a \in A\}$$

if ψ is $\forall x_i \theta$ or $\exists x_i \theta$.

Let $\mathcal{D}$ be the set of all such sets $D(\psi, a_1, \ldots, a_n)$. Note, that $\mathcal{D}$ has power $< 2^\omega$ in $\mathfrak{M}$. By MA there is a homomorphism $h : \mathbb{B} \to \mathbf{2}$ such that $h(b_0) = \mathbf{1}$ and h preserves all sups and infs over sets in $\mathcal{D}$. It is routine to check that the model obtained from $\mathfrak{A}'$ by reducing with h satisfies ϕ. As observed above, this suffices for the proof of the claim. $\square$

Corollary 26 (i) *MA follows from Max($\mathcal{CCC}$).*

(ii) *Max($\mathcal{CCC}$) is inconsistent with Refl($\mathcal{C}$) and hence with Max($\mathcal{C}$).*

As MA implies the regularity of 2^ω, Max($\mathcal{CCC}$) implies that L_{2^ω} is a model of $ZFC+$ Levy schema (see Corollary 21). On the other hand, starting with Levy schema we can construct a model for Max($\mathcal{CCC}$).

Theorem 27 *The following two theories are equiconsistent:*

1. *$ZF+$ Levy schema,*
2. *$ZFC + $ Max($\mathcal{CCC}$).*

Proof. We assume (1) and prove (2). Without loss of generality, assume $V = L$ (in fact we only need a definable wellordering of the universe). We shall construct a sequence $(\mathbb{B}_\alpha)_{\alpha \in On}$ of complete Boolean algebras and then choose a κ such that $V^{\mathbb{B}_\kappa} \models$ Max($\mathcal{CCC}$).

Let $F : On \to V$ be the Gödel function. Let $(\cdot, \cdot, \cdot)$ be a bijection $On^2 \times \omega \to On$ such that $\alpha \leq (\alpha, \beta, \gamma)$ for all α, β, γ, and let $(\cdot)_0, (\cdot)_1, (\cdot)_2$ be mappings such that $((\alpha)_0, (\alpha)_1, (\alpha)_2) = \alpha$ for all α.

Let $\mathbb{B}_0 = \mathbf{2}$. If $\mathbb{B}_\alpha$ is defined, we let $\mathbb{B}_{\alpha+1}$ be determined as follows: Suppose $F((\alpha)_1) = a$ and $F((\alpha)_2) = N$.

Case 1 It is true in $V^{\mathbb{B}_\alpha}$ that $a = \langle a_1, \ldots, a_n \rangle$ and N is the Gödel number of a provably $\mathcal{CCC}$-persistent predicate $R(x_1, \ldots, x_n)$ such that $[\![R(a_1, \ldots, a_n)]\!]^{\mathbb{B}} = \mathbf{1}$ holds for some $\mathbb{B} \in \mathcal{CCC}$. In this case choose such a $\mathbb{B}$ and let (in a canonical way) $\mathbb{B}_{\alpha+1} \cong \mathbb{B}_\alpha \otimes \mathbb{B}$ so that $\mathbb{B}_\alpha$ is a complete subalgebra of $\mathbb{B}_{\alpha+1}$.

Case 2 Otherwise. In this case let $\mathbb{B}_{\alpha+1} = \mathbb{B}_\alpha$.

For limit ν we let $\mathbb{B}_\nu$ be the direct limit of $\mathbb{B}_\alpha, \alpha < \nu$. The construction of $(\mathcal{B}_\alpha)_{\alpha \in On}$ is finished.

By Levy schema we may choose a regular κ such that $H(\kappa)$ reflects the formula

$$(\alpha, \beta, N) = \gamma \wedge F(\beta) = x$$

and

$$\alpha < \kappa \to \mathbb{B}_\alpha \in H(\kappa).$$

We claim that

$$V^{\mathbb{B}_\kappa} \models \mathrm{Max}(\mathcal{CCC}).$$

For this end suppose it is true in $V^{\mathbb{B}_\kappa}$ that

1. $a = \langle a_1, \ldots, a_n \rangle \in H(\kappa)$,

2. $R(x_1, \ldots, x_n)$ is a provably $\mathcal{CCC}$-persistent predicate with Gödel number N,

3. $[\![R(a_1, \ldots, a_n)]\!]^{\mathbb{D}} > \mathbf{0}$ for some $\mathbb{D} \in \mathcal{CCC}$.

Define (in a canonical way)

$$\mathbb{B} \cong \mathbb{B}_\kappa \otimes \mathbb{D}$$

so that $\mathbb{B}_\kappa$ is a complete subalgebra of $\mathcal{B}$. By originally cutting down $\mathbb{D}$ (if needed) we can make sure

$$V^{\mathbb{B}} \models R(a_1, \ldots, a_n).$$

It follows from the regularity of κ that we can find an $\alpha < \kappa$ such that $a_1, \ldots, a_n \in V_\alpha^{\mathbb{B}}$. Let $\alpha \le \beta < \kappa$ such that $\beta = (\alpha, \gamma, N)$ where $F(\gamma) = a$. As $\mathbb{B}_\beta$ is a complete subalgebra of $\mathbb{B}$ and the class $\mathcal{CCC}_e$ is divisible, there is a $\mathbb{C} \in V^{\mathbb{B}_\beta}$ such that $V^{\mathbb{B}_\beta} \models \mathbb{C} \in \mathcal{CCC}$ and $\mathbb{B}_\beta \otimes \mathbb{C} \cong \mathbb{B}$. Thus we have

$$V^{\mathbb{B}_\beta} \models [\![R(a_1, \ldots, a_n)]\!]^{\mathbb{C}} = 1.$$

By construction, $\mathbb{B}_{\beta+1} \cong \mathbb{B}_\beta \otimes \mathbb{C}$ for some such $\mathbb{C}$ whence

$$V^{\mathbb{B}_{\beta+1}} \models R(a_1, \ldots, a_n).$$

Now

$$V^{\mathbb{B}_\kappa} \models R(a_1, \ldots, a_n).$$

by the $\mathcal{CCC}$-persistency of $R(x_1, \ldots, x_n)$. Thus, to end the proof, we have to show

$$V^{\mathbb{B}_\kappa} \models \kappa = 2^\omega.$$

From the above considerations it follows that if $\lambda < \kappa$ has uncountable cofinality, then $V^{\mathbb{B}_\kappa} \models \lambda \le 2^\omega$ since

$$V^{\mathbb{B}_\kappa} \models [\![\lambda \text{ has power } \le 2^\omega]\!]^{\mathbb{C}_\lambda} = \mathbf{1}.$$

15

This gives $V^{\mathbb{B}_\kappa} \models \kappa = 2^\omega$. On the other hand, $\mathbb{B}_\kappa = \bigcup_{\alpha<\kappa} \mathbb{B}_\alpha$, where $|\mathbb{B}_\alpha| < \kappa$ for $\alpha < \kappa$. Thus $|\mathbb{B}_\kappa| \le \kappa$ and therefore $V^{\mathbb{B}_\kappa} \models 2^\omega \le \kappa$. $\square$

Remark: Max($\mathcal{CCC}$) is consistent with $\square$, for we have $V = L$ in the ground model in the proof of Theorem 27. By a result of Jensen, $\square$ gives a CCC notion of forcing which adds a Kurepa tree. Hence Max($\mathcal{CCC}$) $+ \square$ implies the existence of a Kurepa tree. On the other hand, it is consistent relative to the consistency of a Mahlo cardinal, that there are no Kurepa trees in CCC-extensions [4]. Thus Max($\mathcal{CCC}$) + "there are no Kurepa trees" is consistent relative to the consistency of a Mahlo cardinal.

We end this section with some remarks on weaker reflection and maximality principles which are independent of Levy schema.

Definition 28 For any set-variable A and predicates $P_1, \ldots, P_n$ we use

$$\text{Refl}_A(P_1, \ldots, P_n)$$

to denote the principle

$$\forall x_1 \ldots x_m \in A(\exists y \phi(x_1, \ldots, x_m, y) \to \exists y \in HC' \phi(x_1, \ldots, x_m, y))$$

where $\phi(x_1, \ldots, x_m, y)$ ranges over $\Sigma_1(P_1, \ldots, P_n)$-predicates. The schema

$$\text{Refl}_A(\mathcal{B})$$

for a class $\mathcal{B}$ of complete Boolean algebras is defined similarly. We use

$$Max_A(\mathcal{B})$$

to denote the schema

$$\forall x_1 \ldots x_m \in A(\phi(x_1, \ldots, x_m) \leftrightarrow (\exists \mathbb{B} \in \mathcal{B})(\llbracket \phi(x_1, \ldots, x_m) \rrbracket^{\mathbb{B}} > \mathbf{0}))$$

where $\phi(x_1, \ldots, x_m)$ ranges over provably $\mathcal{B}$-persistent predicates.

The proof of Proposition 17 gives:

Proposition 29 *Suppose $\mathcal{B}$ is a class of complete Boolean algebras such that $\mathcal{C} \subseteq \mathcal{B}$, and Cd is provably $\mathcal{B}$-persistent. Then $Max_A(\mathcal{B})$ implies $Refl_A(\mathcal{B})$.*

The proof of Theorem 27 gives (mutatis mutandis):

Theorem 30 *Suppose A is any set. There is a complete Boolean algebra $\mathbb{B} \in \mathcal{CCC}$ such that*
$$V^{\mathbb{B}} \models Max_A(\mathcal{CCC}) + MA.$$

A set A of particular interest in this connection is the set

$$HCF = H(cf(2^\omega)).$$

Another consequence of the proof of Theorem 27 is:

Theorem 31 $Con(ZF)$ *implies* $Con(ZFC + Max_{HCF}(\mathcal{CCC}))$.

Proof. We proceed as in the proof of Theorem 27 up to the choice of κ. We can pick a κ which reflects the formula indicated but cannot make sure κ is regular. Let $\lambda = cf(\kappa)$. Note that the choice of κ imposes no upper bound on λ, and we may require $\lambda = \aleph_{17}$, for example. The imitation of Theorem 27 proceeds now with $H(\kappa)$ replaced by $H(\lambda)$. In particular we get

$$V^{I\!B_\kappa} \models \lambda = 2^\omega \ \wedge \ Max_{H(\lambda)}(\mathcal{CCC}).$$

$\square$

Note however, that $Max_{HCF}(\mathcal{CCC})$ together with MA imply $\mathrm{Max}(\mathcal{CCC})$.

2.5 Other extensions

In addition to $\mathcal{C}$ and $\mathcal{CCC}$, there are many other classes $\mathcal{B}$ that one can consider in connection with $\mathrm{Max}(\mathcal{B})$. Of these classes we shall touch upon a few in this section. The proofs do not feature any new aspects, so they are mostly omitted or just sketched.

As the first example we shall consider the most obvious of all classes of Boolean algebras: the class $\mathcal{BA}$ of all complete Boolean algebras. Note that the class $\mathcal{BA}_e$ of all complete Boolean embeddings is divisible.

Using the fact that the predicate "x is countable" is $\mathcal{BA}$-persistent, one can observe that

$$\mathrm{Refl}(\mathcal{BA}) \rightarrow 2^\omega = \omega_1.$$

From the $\mathcal{BA}$-persistency of the predicate Cd^L it follows that

$$\mathrm{Refl}(\mathcal{BA}) \rightarrow \omega_1 \text{ is hyperinaccessible in } L.$$

Moreover, the proof of Lemma 15 gives

$$\mathrm{Refl}(\mathcal{BA}) \rightarrow L_{\omega_1} \prec L.$$

Thus $\mathrm{Refl}(\mathcal{BA})$ implies that L_{ω_1} satisfies Levy schema. On the other hand, the proof of Theorem 27 easily carries over to $\mathcal{BA}$, giving:

Theorem 32 *The following theories are equiconsistent:*

1. *ZF+ Levy schema,*
2. *$ZFC + Max(\mathcal{BA})$,*
3. *$ZFC + Refl(\mathcal{BA})$.*

An interesting variant of $\mathcal{CCC}$ is the class

$$\mathcal{CCC}_n$$

of all $I\!B \in \mathcal{CCC}$ with dense subset of power $\leq \aleph_n$ (n a fixed natural number). The schema $\mathrm{Refl}(\mathcal{CCC}_n)$ is inconsistent, as can be seen by considering the predicate

$$R(\alpha) \leftrightarrow \alpha = 2^{\aleph_n}.$$

However, we have the following result:

17

Theorem 33 *If $Con(ZF)$, then $Con(ZFC + Max(\mathcal{CCC}_n))$ for all $n < \omega$.*

Proof. We follow the proof of Theorem 27. Note at first that $\mathcal{CCC}_n$ is divisible and $\mathbb{B} \otimes \mathbb{D} \in \mathcal{CCC}_n$ whenever $\mathbb{B} \in \mathcal{CCC}_n$ and $V^{\mathbb{B}} \models \mathbb{D} \in \mathcal{CCC}_n$. ([12] contains proofs). We construct the sequence $(\mathbb{B}_\alpha)_{\alpha \in On}$ as in the proof of Theorem 27 adding $\mathcal{CCC}_n$, rather than $\mathcal{CCC}$-algebras, at successor stages. Then $\mathbb{B}_\alpha \in \mathcal{CCC}_n$ for all $\alpha < \omega_{n+1}$. If $n > 0$, let $\kappa = \omega_n$. Now one can easily prove

$$V^{\mathbb{B}_\kappa} \models 2^\omega = \kappa$$

and

$$V^{\mathbb{B}_\kappa} \models \forall a_1, \ldots, a_n \in H(\kappa) \forall \mathbb{D} \in \mathcal{CCC}_n(\quad [\![R(a_1, \ldots, a_n)]\!]^{\mathbb{D}} > \mathbf{0}$$
$$\to R(a_1, \ldots, a_n))$$

where $R(x_1, \ldots, x_n)$ is an arbitrary provably $\mathcal{CCC}_n$-persistent predicate. If $n = 0$, let $\kappa = \omega_1$. The above two claims can be proved using the fact that if $V^{\mathbb{B}_\kappa} \models \mathbb{D} \in \mathcal{CCC}_0$, then without loss of generality assume $\mathbb{D} \in V^{\mathbb{B}_\alpha}$ for some $\alpha < \kappa$. $\square$

We also have for $n > 0$:

$$\mathrm{Max}(\mathcal{CCC}_n) \to 2^\omega \geq \aleph_n + MA(\aleph_n)$$

but $\mathrm{Max}(\mathcal{CCC}_0)$ is consistent with the existence of a Souslin tree, since the algebra(s) of $\mathcal{CCC}_0$ are strongly $\mathcal{CCC}$.

Let us now turn to reflection principles for 2^{ω_1}. Let $\mathrm{Refl}_1(P), \mathrm{Refl}_1(\mathcal{B})$ and $Max_1(\mathcal{B})$ denote the principles obtained from $\mathrm{Refl}(P), \mathrm{Refl}(\mathcal{B})$ and $\mathrm{Max}(\mathcal{B})$ by replacing HC' by $H(2^{\omega_1})$. The relevant analogue of $\mathcal{C}$ in this connection is the class $\mathcal{C}_1$ of all $\mathbb{C}_{1,\kappa} = RO(\mathcal{P}_{1,\kappa})$, where

$$\mathcal{P}_{1,\kappa} = \{f| \text{ is a countable partial mapping } \kappa \times \omega_1 \to 2\},$$

$$f \leq_{\mathcal{P}_{1,\kappa}} g \text{ if and only if } f \supseteq g.$$

The appropriate analogue of $\mathcal{CCC}$ is more problematic. An interesting candidate is the class $\mathcal{S}$ of all $RO(\mathcal{P})$, where $\mathcal{P}$ has the properties

(P1) Every descending countable sequence in $\mathcal{P}$ has a greatest lower bound.

(P2) If $p_\alpha \in \mathcal{P}$ for $\alpha < \omega_2$, then there is a *c.u.b.* set $C \subseteq \omega_2$ and a regressive function $f : \omega_2 \to \omega_2$ such that if $\alpha \in C$ and $cf(\alpha) > \omega$, then the set

$$A = \{p_\beta | cf(\beta) > \omega, f(\alpha) = f(\beta)\}$$

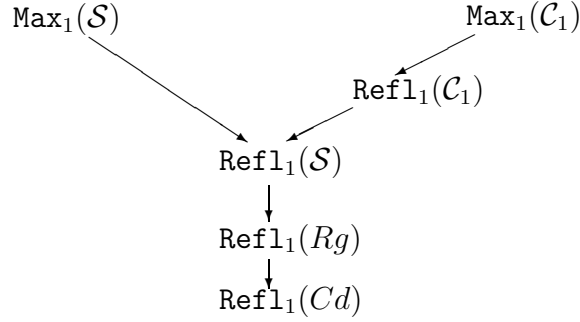is well-met $(p, q \in A \to p \wedge q \in A)$.

18

Figure 3: Reflection principles for 2^{ω_1}.

Note, that (P1) is a strengthening of the countable closure condition, and (P2) is a stregthening of the $\aleph_1$-linkedness condition ($\exists f : \mathcal{P} \to \omega_1$ such that $\forall \alpha : f^{-1}(\alpha)$ is pairwise compatible) which generalizes the $\aleph_2$-chain condition. The class $\mathcal{S}$ has arisen in connection with attempts (by Baumgartner, Laver and Shelah, idependently) to find useful generalizations of MA for higher cardinalities. Conditions (P1) and (P2) are due to Shelah [10].

It is essentially proved in [10, Lemma 1.2] that the class $\mathcal{S}_e$ of complete embeddings

$$(RO(\mathcal{P}), i, RO(Q))$$

where $\mathcal{P}$ and $Q \supseteq \mathcal{P}$ satisfy (P1) and (P2), and i is generated by inclusion $\mathcal{P} \to \mathcal{Q}$, is divisible. Evidently, $\mathcal{C}_1 \subseteq \mathcal{S}$. Thus we have the relations of Figure 3.

The proof of Theorem 18 yields:

Theorem 34 $Con(ZF)$ *implies* $Con(ZFC + 2^\omega = \omega_1 + Max_1(\mathcal{C}_1))$.

A similar relation as $\mathcal{CCC}$ bears to MA can be established between $\mathcal{S}$ and the following Generalized Martin's axiom:

GMA: If $\mathcal{P}$ has the above properties (P1) and (P2) and if D is a set of dense subsets of $\mathcal{P}$ such that $|D| < 2^{\omega_1}$, then there is a D-generic filter on $\mathcal{P}$.

Theorem 35 *Let* $\mathfrak{M}$ *be a countable model of* ZFC. *Then* $\mathfrak{M} \models GMA$ *if and only if*

$$H(2^{\omega_1})^{\mathfrak{M}} \prec_{\Sigma_1} \mathfrak{N}$$

for every $\mathcal{S}$*-extension* $\mathfrak{N}$ *of* $\mathfrak{M}$.

Corollary 36 GMA *follows from* $Max_1(\mathcal{S})$.

We shall now indicate how the proof of the consistency of GMA (due to Shelah [10]) can be modified to yield the consistency of $Max_1(\mathcal{S})$.

Theorem 37 *If $Con(ZF + \ Levy\ schema)$, then $Con(ZFC + Max_1(\mathcal{S}))$.*

Proof. We follow the proof of Theorem 27. Thus a sequence $(\mathbb{B}_\alpha)_{\alpha \in On}$ is constructed keeping an eye on provably $\mathcal{S}$-persistent rather that $\mathcal{CCC}$-persistent predicates. However, inverse limits are taken at limit stages of countable cofinality. It follows from [10, Lemma 1.3] that $\mathcal{B}_\alpha \in \mathcal{S}$ for all $\alpha \in On$. The rest of the proof carries over immediately. $\square$

Remark: $Max_1(\mathcal{C}_1)$ implies $\Diamond$, hence $2^\omega = \omega_1 +$ "there is a Souslin tree", for $\mathbb{C}_{1,\omega_1}$ forces $\Diamond$. $Max_1(\mathcal{C}_1)$ is clearly consistent with the existence of a Kurepa tree. $Max_1(\mathcal{S})$ implies $2^\omega = \omega_1 +$ "there is a Souslin tree". It is not hard to see that GMA, and hence $Max_1(\mathcal{S})$, implies the existence of a Kurepa tree.

3   Abstract logic

There is a fairly simple duality between set theory and abstract logic, especially if sufficently powerful logics are considered. Every abstract logic determines by its very definition a predicate of set theory, and conversely, every predicate of set theory can be associated in a canonical way with an abstract logic. In this part of the paper we combine this duality with the reflection principles of the first part to derive certain results in abstract logic, mainly Löwenheim-Skolem theorems. In Section 3.3 we use MA to obtain a definability result for decision problems of certain logics. In the final section, which is independent of the first part of the paper, we start with a supercompact cardinal and derive a model of set theory with a strong form of the downward Löwemheim-Skolem theorem for $\mathcal{C}$-absolute sublogics of second order logic.

3.1   Persistent logics

In this section we recall quicly some basic notions of abstract logic and point out the crucial relation berween set theory and abstract logic. The fundamental notion of our results in abstract logic is that of persistency of an abstract logic with respect to certain Boolean extensions. This notion is closely related to the notion of persistency of a predicate of set theory.

Definition 38 A *simlarity type* is a set of predicate-, function- and constant-symbols. An abstract logic is a pair

$$L = \langle F_L, S_L \rangle$$

where F_L is a mapping relating similarity types t with classes $F_L(t)$, and S_L is a class of pairs $\langle \mathfrak{A}, \phi \rangle$ such that $\mathfrak{A}$ is a structure of some type t and $\phi \in F_L(t)$.

When no confusion arises, we write

$$\phi \in L \text{ for } \phi \in F_L(t)$$

and

$$\mathfrak{A} \models_L \phi \text{ (or just } \mathfrak{A} \models \phi) \text{ for } \langle \mathfrak{A}, \phi \rangle \in S_L.$$

The definition of the notion of an abstract logic can be given on many different levels of abstraction depending on the particular purpose one has in mind. The above definition is all one needs in this paper. For other definitions the reader is referred to [2] and [6]. Note that the definition takes place most conveniently in class theory.

Definition 39 Let L be an abstract logic and P a predicate of set theory. We say that L is *P-persistent*, if the predicates F_L and S_L are $\Sigma_1(P)$. If, in addition, $\neg S_L$ is $\Sigma_1(P)$, we say that L is *P-absolute*. If $\mathcal{B}$ is a class of complete Boolean algebras, the notions of a *$\mathcal{B}$-persistent* abstract logic and a *B-absolute* abstract logic are defined similarly.

Let W be the generalized quantifier

$$W\,xy\,A(x,y) \leftrightarrow \{(a,b) \mid A(a,b)\} \text{ is a well-founded relation.}$$

If W is added to $L_{\infty\omega}$, an abstract logic

$$L_{\infty\omega}(W)$$

is obtained. This logic is provably P-absolute, whatever P. Let I and R be the generalized quantifiers

$$IxyA(x)B(y) \quad\leftrightarrow\quad |\{a \mid A(a)\}| = |\{b \mid B(b)\}| \text{ (Härtig-quantifier)}$$

$$RxyA(x,y) \quad\leftrightarrow\quad \{(a,b) \mid A(a,b)\} \text{ well-orders its field in the type of } \\ \text{a regular cardinal.}$$

The logics

$$L_{\infty\omega}(I), L_{\infty\omega}(R)$$

which are obtained from $L_{\infty\omega}$ by adding the respective new quantifier, are (respectively) provably Cd- and Rg-absolute. A variety of examples can be found by combining the above or other logics. For example, if Q_α is the quantifier

$$Q_\alpha xA(x) \leftrightarrow |\{a \mid A(a)\}| \geq \aleph_\alpha,$$

then the logic $L_{\infty\omega}(W, I, R, Q_1, \ldots, Q_n, \ldots (n < \omega))$ is provably Rg-absolute. Let C be the quantifier

$$CxyA(x,y) \leftrightarrow \quad \{(a,b) \mid A(a,b)\} \text{ has} \\ \text{the order type of a cardinal of } L.$$

If $V = L$, the rather curious logic $L(C)$ is Δ-equivalent to $L(I)$ and hence also to full second order logic. However, there are Boolean extensions of L, as we shall see, where $L(C)$ is much weaker than $L(I)$. The logic $L_{\infty\omega}(C)$ is provably $\mathcal{BA}$-absolute. Let Q^n be the Magidor-Malitz-quantifier ([8])

$$Q^n x_1, \ldots, x_n A(x_1, \ldots, x_n) \leftrightarrow \quad \exists X(|X| > \omega \wedge \\ A(a_1, \ldots, a_n) \text{ holds for all } (a_1, \ldots, a_n) \in [X]^n).$$

21

Proposition 40 *(1) $L_{\infty\omega}(Q^n)$ is provably $\mathcal{C}$-absolute,*

(2) If there is a Souslin tree, then $L_{\infty\omega}(Q^2)$ is not $\mathcal{CCC}$-absolute.

Proof (1): Suppose $\mathbb{C}_\kappa \in \mathcal{C}$, $A \subseteq \lambda^n$ and $p \in \mathbb{C}_\kappa$ such that

$$p\|\!\!-\ |\mathbf{X}| > \omega\ \wedge\ \mathbf{X} \subseteq \lambda\ \wedge\ \mathbf{X}^n \subseteq A.$$

Let
$$Y = \{\alpha < \lambda | \exists q \leq p(q\|\!\!-\ \alpha \in \mathbf{X})\}.$$

The set Y is uncountable for $p\|\!\!-\ \mathbf{X} \subseteq Y$. If $\alpha \in Y$, let $q_\alpha \leq p$ such that $q_\alpha\|\!\!-\ \alpha \in \mathbf{X}$. If $\alpha_1, \ldots, \alpha_n \in Y$ and $q_{\alpha_1}, \ldots, q_{\alpha_n}$ are pairwise compatible, then $\langle \alpha_1, \ldots, \alpha_n \rangle \in A$. Every $\mathbb{B} \in \mathcal{C}$ is strongly CCC in the sense that every uncountable set of conditions contains an uncountable set of pairwise compatible conditions. Let $Z \subseteq Y$ be uncountable such that $\{q_\alpha | \alpha \in Z\}$ is pairwise compatible. Thus $Z^n \subseteq A$. This shows that the quantifier $\neg Q^n$ is provably $\mathcal{C}$-persistent.

(2): The Souslinity of a tree can be expressed by a sentence of $L_{\infty\omega}(Q^2)$ (see [8]). $\square$

If $P(x_1, \ldots, x_n)$ is any predicate of set theory, we can associate $P(x_1 \ldots x_n)$ with the logic
$$L_{\infty\omega}(Q_P)$$

where Q_P is the quantifier

$Q_P xyx_1, \ldots, x_n(xEy)A_1(x_1), \ldots, A_n(x_n) \leftrightarrow E$ is well-founded extensional and $A_i = \{a_i\}(i = 1, \ldots, n)$ such that $P(a_1, \ldots, a_n)$ holds in the Mostowski-collapse of the universe determined by E.

The definition of $L_{\infty\omega}(Q_P)$ may appear somewhat cumbersome, but the resulting logic has a nice minimality property: $L_{\infty\omega}(Q_P)$ is provably P-absolute, and conversely, if the Δ-closure of any P-absolute logic contains $L_{\omega\omega}(W)$ as a sublogic, it contains also $L_{\omega\omega}(Q_P)$ as a sublogic (see [13]).

3.2   Löwenheim numbers

The Löwenheim number of an abstract logic is, roughly speaking, the least cardinal to which a weak downward Löwenheim-Skolem theorem holds. The main idea of this section, and one of the initial motivations for considering reflection principles for the continuum, is the following: if L is sufficiently persistent and a suitable reflection principle holds, then the Löwenheim number of L is less than the power of the continuum.

Definition 41 Let L be an abstract logic. The Löwenheim number of L,

$$\ell(L)$$

is, if it exists, the least κ such that if $\phi \in L$ has a model then ϕ has a model of power $\leq \kappa$.

As is well-known, $\ell(L_{\kappa^+\omega}) = \kappa$ for regular κ, $\ell(L(W)) = \omega$ and $\ell(L(Q_1)) = \omega_1$. The logic $L_{\infty\omega}$ has no Löwenheim number.

Definition 42 Let L be an abstract logic and A a set. We use

$$L \cap A$$

to denote the abstract logic $\langle F, S \rangle$ where

$$\phi \in F(t) \text{ if and only if } \phi \in F_L(t) \cap A$$

$$\langle \mathfrak{A}, \phi \rangle \in S \text{ if and only if } \langle \mathfrak{A}, \phi \rangle \in S_L \wedge \phi \in A.$$

Proposition 43 *Suppose P is a predicate of set theory and L is a P-persistent abstract logic. If A is a set such that $\mathrm{Refl}_A(P)$ holds, then $\ell(L \cap A) \leq 2^\omega$.*

Proof Let $R(x, y, z)$ be the $\Sigma_1(P)$-predicate

"y is a similarity type, z is a structure of type y, $x \in F_L(y)$ and $\langle x, z \rangle \in S_L$".

Suppose $\phi \in L \cap A$ has a model. Then

$$\exists yz R(\phi, y, z).$$

Hence by $\mathrm{Refl}_A(P)$,

$$\exists yz \in HC' R(\phi, y, z),$$

whence ϕ has a model $\mathfrak{A} \in HC'$, that is , ϕ has a model of power $< 2^\omega$. □

Corollary 44 *If L is P-persistent and $\mathrm{Refl}(p)$, then $\ell(L \cap HC') \leq 2^\omega$.*

Corollary 45 (1) *If L is a provably $\mathcal{C}$-persistent abstract logic, then there is a $\mathcal{B} \in \mathcal{C}$ such that*

$$V^\mathcal{B} \models \ell(L \cap HC') \leq 2^\omega.$$

For example, there is a $\mathcal{B} \in \mathcal{C}$ such that

$$V^\mathcal{B} \models \ell(L_{2^\omega\omega}(I)) = 2^\omega.$$

(2) *If L is a provably $\mathcal{CCC}$-persistent abstract logic and A is any set, there is a $\mathcal{B} \in \mathcal{CCC}$ such that*

$$V^\mathcal{B} \models \ell(L \cap A) < 2^\omega + MA.$$

(3) *If $Con(ZF + \text{ Levy schema})$, then $Con(ZFC + MA + \ell(L \cap HC') \leq 2^\omega$ whenever L is a provably $\mathcal{CCC}$-persistent abstract logic).*

We can also bound Löwenheim numbers by 2^{ω_1}:

Proposition 46 *If L is P-persistent, and $Refl_1(P)$, then $\ell(L \cap H(2^{\omega_1})) \leq 2^{\omega_1}$.*

Corollary 47 *$Con(ZF)$ implies $Con(ZFC + CH + \ell(L_{2^{\omega_1}\omega_1}(I)) = 2^{\omega_1})$.*

Finally, we can use our results on $\mathcal{B}A$ to make Löwenheim numbers $= \omega$:

Corollary 48 (1) *If L is $\mathcal{B}A$-persistent and $Refl(\mathcal{B}A)$, then $\ell(L \cap HC) = \omega$.*

(2) *If L is a provably $\mathcal{B}A$-persistent abstract logic, and A is any set, then there is a $\mathcal{B}$ such that*
$$V^{\mathcal{B}} \models \ell(L \cap A) = \omega.$$

(3) *If $Con(ZF + $ Levy schema$)$, then $Con(ZFC + \ell(L \cap HC) = \omega$ for every provably $\mathcal{B}A$-persistent abstract logic $L)$.*

In particular, we get models for
$$\ell(L_{\omega_1\omega}(C)) = \omega.$$

3.3 An application of MA.

In this section we show how
$$\ell(L) < 2^{\omega} + MA$$

can be used to obtain a definability result for the decision problem of L, provided L is a sufficiently regular sublogic of second order logic. Throughout this section, similarity types are assumed to be elements of HF and only abstract logics L such that $F_L(t) \in HF$ for all t are considered. The main result is Theorem 53.

Definition 49 Let L be an abstract logic. The *decision problem* of L is the set
$$V(L) = \{\phi \in L \mid \phi \text{ is valid, i.e. true in all models }\}.$$

Example 50 $V(L(Q_0))$ is the complete Π_1^1-subset of HF. $V(L(W))$ is the complete Π_2^1-subset of HF (see [5]). $V(L(I))$, which we denote by V_I, is not Σ_2^1 or Π_2^1 but can be Δ_3^1 (see [12]). If $V = L$, then V_I is not Σ_n^m for any $m, n < \omega$.

By second order logic L^2 we understand here the extension of $L_{\omega\omega}$ which permits full quantification over (finitary) relations. We may assume without loss of generality that formulae of L^2 are elements of HF.

Definition 51 Let L be an abstract logic. We say that L is an *analytical sublogic* of L^2 if L is a sublogic of L^2 in the usual sense and the relation $\phi \in F_L(t)$ is analytical (i.e. Σ_n^1 for some n) in ϕ and t on HF.

24

Example 52 $L(W)$ is an analytical sublogic of L^2, for we can use the equivalence

$$WxyA(x,y) \leftrightarrow \neg \exists X(\exists x X(x) \wedge \forall y(X(y) \rightarrow \exists z(X(z) \wedge z \neq y \wedge A(z,y))))$$

to embed $L(W)$ into L^2. $L(I)$ is an analytical sublogic of L^2 for

$$IxyA(x)B(y) \leftrightarrow \exists f(f \text{ is a one-one map of } \{a \mid A(a)\} \text{ onto } \{b \mid B(b)\}).$$

$L(R)$ is an analytical sublogic of L^2 for

$$RxyA(x,y) \leftrightarrow \quad \{(a,b) \mid A(a,b)\} \text{ linearly orders its field } \wedge WxyA(x,y) \wedge$$
$$\neg \exists f(f \text{ maps an initial segment of } A$$
$$\text{onto a cofinal subset}).$$

Also $L(C), L(Q_\alpha)$ and $L(Q^n)$ are analytical sublogics of L^2.

Our purpose is to establish:

Theorem 53 *Let L be an analytical sublogic of L^2. If MA holds and $\ell(L) < 2^\omega$, then $V(L)$ is Σ_1^2.*

Before the proof, let us consider the question: Is this the best possible result? It is quite easy to construct for each $n < \omega$ an analytical sublogic L of L^2 so that $\ell(L) = \omega$ and $V(L)$ is not Σ_n^1. So Δ_1^2 would be a natural improvement for the general result. We do not know, whether this improvement is possible. However, if Q is a generalized quantifier definable in L^2 and $\ell(L(Q)) = \omega$, then obviously $V(L(Q))$ is Δ_1^2. On the other hand, some bound on $\ell(L)$ has to be put, since $V(L^2)$ is certainly not Σ_n^m for any $n, m < \omega$. If $V = L$, then $V(L(I))$ is not Σ_n^m for any $m, n < \omega$ [12].

For the proof of the theorem we need some facts from the theory of almost disjoint sets. Two sets $x \subseteq \omega$ and $y \subseteq \omega$ are *almost disjoint* if $x \cap y$ is finite. The standard construction of almost disjoint sets is the following: Let $\{s_n | n < \omega\}$ be an effective enumeration of all finite subsets of ω. If $x \subseteq \omega$ let $d_x = \{n | s_n \text{ is an initial segment of } x\}$. Then $\{d_x | x \subseteq \omega\}$ is a family of almost disjoint sets.

Lemma 54 (Martin-Solovay [9]) *If MA holds and K is a family of almost disjoint sets of integers such that $|K| < 2^\omega$, then for every $A \subseteq K$ there is a set $t_A^K \subseteq \omega$ such that for all $s \in K$:*

$$s \in A \text{ if and only if } s \cap t_A^K \text{ is infinite.}$$

Let A_2 denote second order arithmetic with individual variables $x_1, \ldots, x_n, \ldots$ and set variables $Y_1, Y_2, \ldots$. Let A_2' be the extension of A_2 obtained by adding the type-2-symbols K and the new set variables $X_n^m (m, n < \omega)$ to the language. We shall define a translation of L^2 into A_2'. For this end, suppose the individual variables of L^2 are $v_1, \ldots, v_n, \ldots$ and the relation variables are $R_n^m (m, n < \omega)$ such that R_n^m is m-ary.

For $x, y \subseteq \omega$ let

$$x \otimes y = \{2n | n \in x\} \cup \{2n + 1 | n \in y\}.$$

If K is a family of almost disjoint subsets of ω then so is $\{x \otimes y | x, y \in K\}$. Define

$$x_1 \otimes \ldots \otimes x_n = x_1 \otimes (x_2 \otimes \ldots (x_{n-1} \otimes x_n) \ldots).$$

If $A \subseteq (\mathcal{P}(\omega))^n$, let

$$A^{(n)} = \{x_1 \otimes \ldots \otimes x_n | \langle x_1, \ldots, x_n \rangle \in A\}.$$

We use $I(x)$ to denote the predicate "x is infinite". Define

$$
\begin{aligned}
(v_i = v_j)^* &\quad \text{is} \quad Y_i = Y_j \\
(R_n^m(v_{i_1}, \ldots, v_{i_m}))^* &\quad \text{is} \quad I(X_n^m \cap (Y_{i_1} \otimes \ldots \otimes Y_{i_m})) \\
(\neg \phi)^* &\quad \text{is} \quad \neg \phi^* \\
(\phi \wedge \psi)^* &\quad \text{is} \quad \phi^* \wedge \psi^* \\
(\exists v_i \phi(v_i))^* &\quad \text{is} \quad \exists Y_i(K(Y_i) \wedge \phi(v_i)^*) \\
(\exists R_n^m \phi(R_n^m))^* &\quad \text{is} \quad \exists X_n^m(\phi(R_n^m))^*.
\end{aligned}
$$

Let $AD(K)$ be the formula

> "K is a set of almost disjoint subsets of ω such that the restriction of $\otimes$ to K^2 maps K^2 one-one onto K".

Clearly, AD is a Π_2^1-formula. Moreover for every $\kappa < 2^\omega$ there is a K of power κ such that $AD(K)$.

Lemma 55 *Assume $MA, AD(K)$ and $|K| < 2^\omega$. Suppose*

$$\phi(v_1, \ldots, v_n, R_1^{k_1}, \ldots, R_m^{k_m}) \in L^2,$$

$P_i \subseteq K^{k_1} (i = 1, \ldots, m)$ and $s_1, \ldots, s_n \in K$. Then the following are equivalent:

1. *$\langle K, s_1, \ldots, s_n, P_1, \ldots, P_m \rangle \models \phi(v_1, \ldots, v_n, R_1^{k_1}, \ldots, R_m^{k_m})$*
2. *$\langle \mathcal{P}(\omega), s_1, \ldots, s_n, t_{Q_1}^K, \ldots, t_{Q_m}^K, K \rangle \models \phi^*(Y_1, \ldots, Y_n, X_1, \ldots, X_m, K)$,*

where $Q_i = P_i^{(k_i)} (i = 1, \ldots, m)$.

Proof The proof goes by induction on the length of ϕ. The case ϕ is $v_i = v_j$ is trivial. Let ϕ be $R_i^k(v_{j_1}, \ldots, v_{j_k})$. Then ϕ^* is

$$\phi^*(Y_{j_1}, \ldots, Y_{j_k}, X_i^k, K) = I(X_i^k \cap (Y_{j_1} \otimes \ldots \otimes Y_{j_k})).$$

Now

$$
\begin{aligned}
\langle K, s_{j_1}, \ldots, s_{j_k}, P_i \rangle &\models \phi \quad \text{if and only if} \\
s_{j_1} \otimes \ldots \otimes s_{j_k} &\in P_i^{(k)} \quad \text{if and only if} \\
I(t_{Q_i}^K \cap (s_{j_1} \otimes \ldots \otimes s_{j_k})) &\quad \text{if and only if} \\
\langle \mathcal{P}(\omega), s_{j_1}, \ldots, s_{j_k}, t_{Q_i}^K, K \rangle &\models \phi^*.
\end{aligned}
$$

26

The induction steps for $\neg$, $\wedge$ and v_i are entirely trivial. Suppose then $\phi = \exists R_i^k \psi(\mathbf{v}, \mathbf{R}, R_i^k)$. Suppose ϕ^* is true in $\langle \mathcal{P}(\omega), \mathbf{s}, \mathbf{t}, K \rangle$. Then there is a $t' \in \mathcal{P}(\omega)$ such that ψ^* is true in $\langle \mathcal{P}(\omega), \mathbf{s}, \mathbf{t}, t', K \rangle$. Let $P_i = \{s \in K | I(t' \cap s)\}$. Now for $s \in K$, $I(t' \cap s) \leftrightarrow I(t_{P_i}^K \cap s)$. A simple subinduction on ψ shows that ψ^* holds in $\langle \mathcal{P}(\omega), \mathbf{s}, \mathbf{t}, t_{P_i}^K, K \rangle$. By induction hypothesis $\langle K, \mathbf{s}, \mathbf{P}, P_i \rangle \models \psi$, whence $\langle K, \mathbf{s}, \mathbf{P} \rangle \models \phi$. Conversely, if $\langle K, \mathbf{s}, \mathbf{P} \rangle \models \phi$, then $\langle K, \mathbf{s}, \mathbf{P}, P_i \rangle \models \psi$ for some $P_i \subseteq K$ whence $\langle \mathcal{P}(\omega), \mathbf{s}, \mathbf{t}, t_{P_i}^K, \rangle \models \psi^*$ for some $\mathcal{P}(\omega) \subseteq K$. It follows that $\langle \mathcal{P}(\omega), \mathbf{s}, \mathbf{t} \rangle \models \phi^*$. $\square$

For any $\phi \in L^2$ let ϕ^+ be $(\phi^-)^*$, where ϕ^- is the relativisation of ϕ to a predicate not occuring in ϕ.

Corollary 56 *Assume MA and $\kappa < 2^\omega$. The following three conditions are equivalent:*

(1) *ϕ is valid in models of power $\leq \kappa$*

(2) $\exists K \subseteq \mathcal{P}(\omega)(\kappa \leq |K| < 2^\omega \ \wedge \ AD(K) \ \wedge \ \langle \mathcal{P}(\omega), K \rangle \models \phi^+)$

(3) $\forall K \subseteq \mathcal{P}(\omega)((|K| < 2^\omega \ \wedge \ AD(K)) \to \langle \mathcal{P}(\omega), K \rangle \models \phi^+)$.

Proof

(1) $\to$ (3): Suppose $|K| < 2^\omega$ and $AD(K)$. By lemma 56 and (1), ϕ^+ is valid in $\langle \mathcal{P}(\omega), K \rangle$.

(3) $\to$ (2): Let $K \subseteq \mathcal{P}(\omega)$ such that $|K| = \kappa$ and $AD(K)$. By (3) $\langle \mathcal{P}(\omega), K \rangle \models \phi^+$.

(2) $\to$ (1): By Lemma 55 ϕ^- is valid in models of power κ, whence ϕ is valid in models of power $\leq \kappa$. $\square$

Proof of Theorem 53 Suppose L is an analytical sublogic of L^2, $\ell(L) < 2^\omega$ and MA holds. Let $WO(R)$ be the Π_1^1-predicate "R is a well-ordering of $\mathcal{P}(\omega)$". Let $CD(X, Z)$ be the Σ_1^2-predicate "$X \subseteq \mathcal{P}(\omega)$ and $Z \subseteq \mathcal{P}(\omega)$ have the same power". Let $CL(X)$ be the Π_1^2-predicate "$X \subseteq \mathcal{P}(\omega)$ has power $< 2^\omega$". We claim that the following two conditions are equivalent for any $\phi \in L$:

(1) $\phi \in V(L)$,

(2) $\exists R \subseteq \mathcal{P}(\omega)^2 (WO(R) \wedge \forall x \subseteq \omega (CL(R(\cdot, x)) \to \exists X(AD(X) \ \wedge \ CD(X, R(\cdot, x)) \ \wedge \ \langle \mathcal{P}(\omega), X \rangle \models \phi^+)))$.

Suppose at first (1). Let R be a well-ordering of $\mathcal{P}(\omega)$ of type 2^ω. Suppose $x \subseteq \omega$ so that $CL(R(\cdot, x))$. Let X be a subset of $\mathcal{P}(\omega)$ so that $AD(X)$ and $|X| = |R(\cdot, x)|$. By Corollary 56, $\langle \mathcal{P}(\omega), X \rangle \models \phi^+$.

Conversely, suppose (2) holds. Let $x \subseteq \omega$ so that $|R(\cdot, x)| = \ell(L)$. By (2) there is an X such that $AD(X), |X| = \ell(L)$ and $\langle \mathcal{P}(\omega), X \rangle \models \phi^+$. By Corollary 56, ϕ is valid in models of power $\leq \ell(L)$. But then ϕ is valid in all models. The equivalence of (1)

and (2) is proved. To end the proof it suffices to observe that (2) is a Σ_1^2-statement. Thus $V(L)$ has the Σ_1^2-definition

$$\phi \in V(L) \text{ if and only if } \exists t(\phi \in F_L(t) \ \wedge \ \phi \text{ satisfies (2))}.$$

$\square$

3.4 A Löwenheim-Skolem-Tarski theorem

The downward Löwenheim-Skolem theorem related to the Löwenheim numbers is extremely weak. For many logics we have much stronger results. In this section we consider the following property of an abstract logic (with sufficient regularity properties):

$LST(\kappa)$: If $\mathfrak{A}$ is a finitary structure, there is a structure $\mathfrak{B}$ of power $< \kappa$ such that $\mathfrak{B} \prec_L \mathfrak{A}$, that is, for all $b_1, \ldots, b_n \in |\mathfrak{B}|$ and $\phi(x_1, \ldots, x_n) \in L$,

$$\mathfrak{B} \models_L \phi(b_1, \ldots, b_n) \leftrightarrow \mathfrak{A} \models_L \phi(b_1, \ldots, b_n).$$

For example, $L_{\kappa^+\omega}$ has $LST(\kappa^+)$ for all κ. Also, $L(Q_\kappa)$ has $LST(\kappa^+)$. But for many logics even the mere existence of a cardinal κ with $LST(\kappa)$ is unprovable in ZFC. This is the case with $L(I)$, as the following two lemmas show:

Lemma 57 *Suppose $L(I)$ satisfies $LST(\kappa)$. Then there is a weakly inaccessible cardinal $\leq \kappa$.*

Proof Let $\mathfrak{A} = (R(\kappa^+), \epsilon)$. By $LST(\kappa)$ there is a transitive set M of power $< \kappa$ and a monomorphism $i : (M, \epsilon) \to \mathfrak{A}$ which preserves $L(I)$-truth. Moreover, every M-cardinal is a real cardinal. Let λ be the largest cardinal in M. Clearly $i(\lambda) = \kappa > \lambda$. Let γ be the first ordinal moved by i. Trivially, γ is a limit cardinal. Suppose $f \in M$ is a cofinal δ-sequence in γ for some $\delta < \gamma$. Now $i(f)$ is a cofinal δ-sequence in $i(\gamma)$ whence $i(f)(\beta) > \gamma$ for some $\beta < \delta$. But $i(f)(\beta) = i(f(\beta)) = f(\beta) < \gamma$. Thus γ is weakly inaccessible in M, and therefore, $i(\gamma)$ is weakly inaccessible in V. $\square$

Lemma 58 *Suppose $L(I)$ satisfies $LST(\kappa)$. Then $a^\#$ exists for any $a \subseteq \aleph_\omega$.*

Proof Let $\mathfrak{A} = (R(\kappa^+), a, \epsilon)$ where $a \subseteq \aleph_\omega$. Let i, λ and γ be determined as in Lemma 57. Thus $i : (M, a', \epsilon) \to \mathfrak{A}$ for some $a' \in M$. Note that $\aleph_\omega^M = \aleph_\omega$ whence $i(a') = a' = a$. In particular, $a \in M$ and $L_\alpha[a] \subseteq M$ where $\alpha = M \cap On$. Let $j = i|_{L_\alpha[a]}$. Now

$$j : L_\alpha[a] \overset{\equiv}{\to} L_{\kappa^+}[a].$$

It follows that $a^\#$ exists. $\square$

Thus the existence of a κ with $LST(\kappa)$ for $L(I)$ is proof-theoretically a strong assumption. Exactly how strong, remains open. If one goes all the way up to a supercompact cardinal, one can get $LST(\kappa)$ for $L(I)$ and much more:

28

Theorem 59 (M. Magidor [7]) *If κ is a supercompact cardinal, then the infinitary second order logic $L^2_{\kappa\kappa}$ satisfies $LST(\kappa)$. Conversely, if L^2 satisfies $LST(\kappa)$, there is a supercompact cardinal $\leq \kappa$.*

Returning to the topic of this paper, we shall elaborate Magidor's proof of Theorem 59 to get a model of set theory where $L(I)$ satisfies $LST(2^\omega)$. Again, we do not know whether anything less than a supercompact is enough in the proof.

Theorem 60 *Suppose κ is a supercompact cardinal and $\mathcal{P}$ is the notion of forcing C_κ. Let L be a provably $\mathcal{C}$-absolute logic which is provably a sublogic of L^2. Then*

$$\|\!\!-_\mathcal{P} \; L \text{ satisfies } LST(2^\omega).$$

Proof Suppose $\mathfrak{A}$ is a name for a finitary structure with universe λ in the $\mathcal{P}$-forcing language. Let $i : V \to M$ be an elementary embedding of the universe such that $i(\kappa) > \lambda, {}^\lambda M \subseteq M$ and $i``\kappa = \kappa$. We prove at first

$$M \models (\|\!\!-_{i(\mathcal{P})} \; i : \mathfrak{A} \to_L i(\mathfrak{A})). \tag{4}$$

Note that $i(\mathcal{P})$ is C_ν in M for some cardinal ν. Let $p \in i(\mathcal{P}), \phi(x_1, \ldots, x_n) \in L, a_1, \ldots, a_n \in \lambda$ and $q \leq p$ such that in M

$$q\|\!\!-_{i(\mathcal{P})} \mathfrak{A} \models \phi(a_1, \ldots, a_n).$$

The condition q is a finite partial mapping $\omega \times \nu \to 2$. Let I be a set of power κ such that if

$$\mathcal{P}' = \{r \in i(\mathcal{P}) \mid dom(r) \subseteq \omega \times I\},$$

then $\mathcal{P} \subseteq \mathcal{P}'$ and $q \in \mathcal{P}'$. Note that $\mathfrak{A}$ is a term of the $\mathcal{P}'$-forcing language. By the $\mathcal{C}$-absoluteness of L,

$$q\|\!\!-_{\mathcal{P}'} \mathfrak{A} \models \phi(a_1, \ldots, a_n)$$

holds in M. Let $j : \mathcal{P}' \to \mathcal{P}$ be an isomorphism. We have

$$jq\|\!\!-_\mathcal{P} j\mathfrak{A} \models \phi(a_1, \ldots, a_n) \tag{5}$$

in M. But as $L \subseteq L^2$ and ${}^\lambda M \subseteq M$, (5) holds also in V. Hence in M

$$ijq\|\!\!-_{i(\mathcal{P})} ij\mathfrak{A} \models \phi(ia_1, \ldots, ia_n),$$

that is

$$i(j)(iq)\|\!\!-_{i(\mathcal{P})} i(j)(i\mathfrak{A}) \models \phi(ia_1, \ldots, ia_n).$$

Note that $i(j)$ is an isomorphism between $i(\mathcal{P}')$ and $i(\mathcal{P})$, whence in M

$$iq\|\!\!-_{i(\mathcal{P}')} i\mathfrak{A} \models \phi(ia_1, \ldots, ia_n).$$

Let $\mathcal{R}$ denote $C_{i(\nu)}$ in M. We have in M $i(\mathcal{P}') \subseteq \mathcal{R}$ and

$$iq\|\!\!-_\mathcal{R} i\mathfrak{A} \models \phi(ia_1, \ldots, ia_n).$$

29

Now $iq \cup q$ is a joint extension of iq and q in $\mathcal{R}$, because $iq|_{\omega \times \kappa} = q|_{\omega \times \kappa}$ and iq is not defined for values in $\omega \times \nu - \omega \times \kappa$. Thus q does not force $i\mathfrak{A} \models \neg\phi(ia_1, \ldots, ia_n)$ in $\mathcal{R}$. By $\mathcal{C}$-absoluteness of L, q does not force $i\mathfrak{A} \models \neg\phi(ia_1, \ldots, ia_n)$ in $i(\mathcal{P})$ ($\subseteq \mathcal{R}$). Hence there is an $r \leq q$ in $i(\mathcal{P})$ such that

$$r ||{-}_{i(\mathcal{P})} \, i\mathfrak{A} \models \phi(ia_1, \ldots, ia_n).$$

We have proved in M

$$\forall q \leq p \; \exists r \leq q((\quad q||{-}_{i(\mathcal{P})} \, \mathfrak{A} \models \phi(a_1, \ldots, a_n)) \rightarrow$$
$$(r||{-}_{i(\mathcal{P})} \, i\mathfrak{A} \models \phi(ia_1, \ldots, ia_n))). \tag{6}$$

This establishes (4). It follows from (4) that

$$M \models \; ||{-}_{i(\mathcal{P})} \, i(\mathfrak{A}) \text{ has an } L\text{-elementary substructure of power} < i(\kappa).$$

Therefore

$$||{-}_{\mathcal{P}} \, \mathfrak{A} \text{ has an } L\text{-elementary substructure of power } < \kappa.$$

$\square$

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