Three fundamental games in logic

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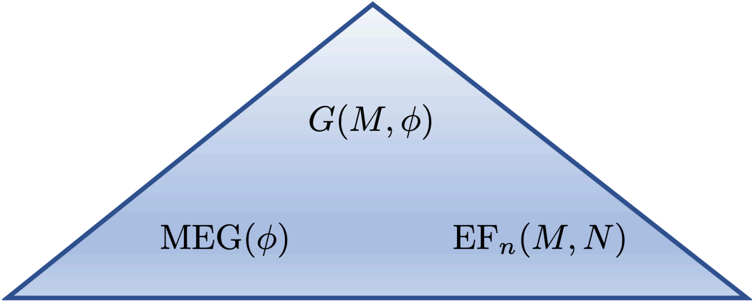
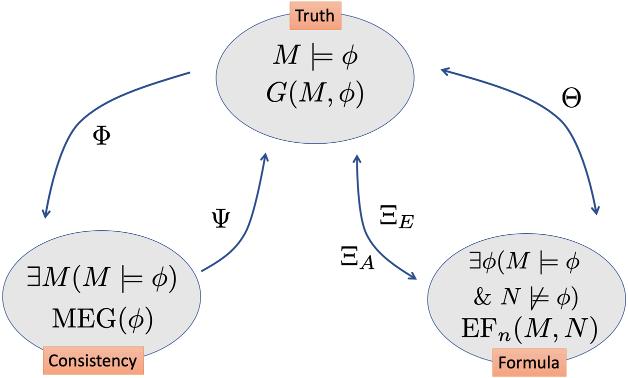
- "The chess-board is the world, the pieces are the phenomena of the universe, the rules of the game are what we call the laws of Nature. The player on the other side is hidden from us." (Thomas Huxley)

1. Evaluation Game: "ϕ is true in M?"
2. Model Existence Game: "ϕ is consistent?"
3. EF (Ehrenfeucht-Fraïssé) game: "some sentence is true in M but false in N?"

Really just one game. Essential to logic. Distinguishes logic from algebra, topology, analysis, etc.


$G(M, \phi)$
$\mathrm{MEG}(\phi)$
$\mathrm{EF}_n(M, N)$

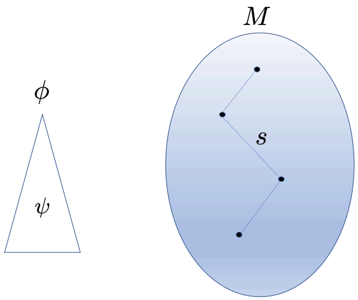

Truth
$M \models \phi$
$G(M, \phi)$
Φ
Θ
Ψ
Ξ_E
Ξ_A
$\exists M(M \models \phi)$
$\mathrm{MEG}(\phi)$
Consistency
$\exists \phi(M \models \phi$
$\& \ N \not\models \phi)$
$\mathrm{EF}_n(M, N)$
Formula

Games vs. inductive methods

- Games suggest generalizations.
- Example: transfinite game, trees as generalized ordinals, generalized Baire spaces, higher descriptive set theory.
- Question: If Eloise has a winning strategy in the EF-game of length ω_1, is there a σ-closed dense set?
- Example: Partially ordered quantifiers [Henkin, 1961], inductive definition led to team semantics.

Evaluation (a.k.a. semantic) Game $G(M, \phi)$

- Two players Abelard and Eloise.
- M a model, ϕ a sentence.
- s an assignment.
- Pairs (ψ, s) are positions.
- Starting position is $(\phi, \emptyset)$.



Evaluation (a.k.a. semantic) Game $G(M, \phi)$

- Suppose s is an assignment. $\mathrm{Diag}_M(s)$ = the set of all literals i.e. atomic and negated atomic formulas that s satisfies in M.
- $\neg, \wedge, \vee, \forall, \exists$.
- Negation Normal Form (for simplicity!).
- Intuitively, Eloise defends the proposition that ϕ is (informally) true in M and Abelard doubts it.

The **rules** in position (ψ, s) are:

(1) If ψ is a literal, the game ends and Eloise wins if $\psi \in \mathrm{Diag}_M(s)$. Otherwise Abelard wins.

(2) If ψ is $\psi_0 \wedge \psi_1$, then Abelard chooses whether the next position is (ψ_0, s) or (ψ_1, s).

(3) If ψ is $\psi_0 \vee \psi_1$, then Eloise chooses whether the next position is (ψ_0, s) or (ψ_1, s).

(4) If ψ is $\forall x\theta$, then Abelard chooses $a \in M$ and the next position is $(\theta, s(a/x))$.

(5) If ψ is $\exists x\theta$, then Eloise chooses $a \in M$ and the next position is $(\theta, s(a/x))$.

- We say that ϕ is true in M if Eloise has a winning strategy in $G(M, \phi)$.
- This is the game-theoretical meaning of truth in a model.
- We can go further and say that the *game* $G(M, \phi)$ is the meaning of ϕ in M. Here meaning would be a broader concept than the mere truth or falsity of ϕ.
- [Wittgenstein, 1953], [Henkin, 1961], [Hintikka, 1968]

- The game $G(M, \phi)$ reflects the syntactical structure of ϕ.
- The game $G(M, \phi \wedge \psi)$ is intimately related to the two games $G(M, \phi)$ and $G(M, \psi)$.
- The same with $G(M, \phi \vee \psi)$, $G(M, \exists x \phi)$ and $G(M, \forall x \phi)$.
- This phenomenon is a manifestation of the broader concept of *compositionality*.
- The games $G(M \times N, \phi)$, $G(M + N, \phi)$, and $G(\Pi_i M_i / F, \phi)$ are intimately related to the games $G(M, \phi)$, $G(N, \phi)$ and $G(M_i, \phi)$ [Feferman, 1972].

- If ϕ is **propositional** i.e. has only zero-place relation symbols and no constant or function symbols, and no quantifiers, then only moves (1)-(3) occur in $G(M, \phi)$, and the assignments can be forgotten.
- If ϕ is **universal**, the game $G(M, \phi)$ has no moves of type (5).
- If it is **existential**, the game has no moves of type (4).
- If **universal-existential**, then all type (5) moves come before type (4) moves.
- If we add **new logical operations** to our logic, such as infinite conjunctions and disjunctions, generalized quantifiers or higher order quantifiers, it is clear how to modify the game $G(M, \phi)$ to accommodate the new logical operations.

For example, for ϕ in $L_{\infty\omega}$, we modify above (2) and (3) as follows:

(2') If ψ is $\bigwedge_{i\in I}\psi_i$, then Abelard chooses $i \in I$ and the next position is (ψ_i, s).

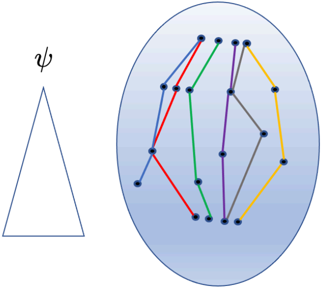
(3') If ψ is $\bigvee_{i\in I}\psi_i$, then Eloise chooses $i \in I$ and the next position is (ψ_i, s).

Similarly for generalized quantifiers.

- We can also extend the game to new atomic formulas.
- If ψ is $\boldsymbol{=(x, y)}$, then Eloise wins.
- A winning strategy of Eloise has to guarantee that if a play ends with $(=(x, y), s)$ and another play ends with $(=(x, y), s')$, then $s(x) = s'(s)$ implies $s(y) = s'(y)$.
- Similarly for $\boldsymbol{x \perp y}$. A winning strategy of Eloise has to guarantee that if a play ends with $(x \perp y, s)$ and another play ends with $(x \perp y, s')$, then there is a "crossing over" play which ends with $(x \perp y, s'')$, where $s''(x) = s(x)$ and $s''(y) = s'(y)$.

Evaluation game in team semantics

- A team is a **set** of assignments.
- A position is a pair (ψ, X), where X is a team.



Evaluation game in team semantics

- Suppose the position is (ψ, X).
- If ψ is $=(x, y)$, then the game ends, and Eloise wins if $s(x) = s'(x)$ implies $s(y) = s'(y)$ for all $s, s' \in X$.
- The original game $G(M, \phi)$ is a projection of the new team game.
- Arrow's Theorem, No-Go results of Quantum Mechanics.
- Dependence logic [Väänänen, 2007].

Modal logic

Finally, if M is a Kripke-model and ϕ a sentence of modal logic, the game $G(M, \phi)$ is entirely similar. The assignments have a singleton domain $\{x_0\}$ and values in the frame of M. The moves corresponding to $\Diamond$ and $\Box$ are like (4) and (5):

(4') If ψ is $\Box\theta$, then Abelard chooses a node b accessible from $s(x_0)$ and the next position is $(\theta, s(b/x_0))$.

(5') If ψ is $\Diamond\theta$, then Eloise chooses a node b accessible from $s(x_0)$ and the next position is $(\theta, s(b/x_0))$.

- The game $G(M, \phi)$ is useful in finding a **countable** submodel N of M with desired properties.
- For any strategy τ of Eloise in $G(M, \phi)$ let $T(M, \tau)$ be the set of countable submodels N of M such that N is closed under τ i.e. if Abelard plays in (4) always $a \in N$, then also Eloise plays in (5) always $b \in N$.
- Note that if $N \in T(M, \tau)$, then τ is a strategy of Eloise also in $G(N, \phi)$. Moreover, if τ is a winning strategy in $G(M, \phi)$, then it is also a winning strategy in $G(N, \phi)$.
- The **Löwenheim-Skolem Theorem** of $L_{\omega_1\omega}$ is essentially the statement that $T(M, \phi) \neq \emptyset$, when $\phi \in L_{\omega_1\omega}$.

- In the *Club Game* ([Kueker, 1977]) there are two players Abelard and Eloise, a set X and a set T of countable subsets of X.
- During the game, which we denote G_T, the players choose alternatingly elements $a_n \in X$, Abelard being the first to move.
- After ω moves a set $\{a_0, a_1, \ldots\}$ has been produced.
- We say that Eloise is the winner if this set is in T, otherwise Abelard is the winner. This game need not be determined.
- If Eloise has a winning strategy in G_T, then T contains a so-called club (closed unbounded) set of countable subsets of X.

A strong form of a Löwenheim-Skolem Theorem for $L_{\omega_1\omega}$.

Theorem ([Kueker, 1977])

Suppose M is a model in a countable vocabulary and ϕ is a sentence of $L_{\omega_1\omega}$. There is a mapping Υ such that if τ is a strategy of Eloise in $G(M, \phi)$, then $\Upsilon(\tau)$ is a strategy of Eloise in $G_{T(M,\phi)}$. If τ is a winning strategy, then so is $\Upsilon(\tau)$.

Proof.

(Sketch) Using bookkeeping Eloise makes sure during the game $G_{T(M,\phi)}$ that her moves $a_{2n+1} \in M$ render the final set $\{a_0, a_1, \ldots\}$ such that it is both the domain of a submodel of M and if Abelard plays his $\forall$-moves in $\{a_0, a_1, \ldots\}$ then so does Eloise, using τ, in her moves of type (5). $\square$

In conclusion, the game $G(M, \phi)$ is a versatile tool for understanding the meaning of a logical sentence ϕ in a mathematical structure M, or even in V.

Model Existence Game MEG(ϕ)

- We have a sentence and we ask whether the sentence has a model. Thus this is about *consistency* and its opposite, *contradiction*.
- Is there some model M such that Eloise can win $G(M, \phi)$?
- Suppose ϕ is a first order sentence. Logical operations: $\neg, \wedge, \vee, \forall$ and $\exists$.
- We assume ϕ is in Negation Normal Form.

- The game MEG(ϕ) has two players Abelard and Eloise.
- Intuitively, Eloise defends the proposition that ϕ has a model and Abelard doubts it. Abelard expresses his doubt by asking questions.
- We let $C = \{c_0, c_1, \ldots, c_n, \ldots\}$ be a set of new distinct constant symbols. Intuitively these are names of elements of the supposed model.

Model Existence Game

A position is a finite set S of pairs (ψ, s), where s is an assignment into C. Starting position is $\{(\phi, \emptyset)\}$. Abelard chooses a pair $(\psi, s) \in S$.

(1) $(\psi_0 \wedge \psi_1, s)$: Next position is $S \cup \{(\psi_0, s)\}$ or $S \cup \{(\psi_1, s)\}$ (Abelard decides which).

(2) If $(\psi_0 \vee \psi_1, s)$: Next position is $S \cup \{(\psi_0, s)\}$ or $S \cup \{(\psi_1, s)\}$ (Eloise decides which).

(3) If $(\forall x \theta, s)$: Next position is $S \cup \{(\theta, s(c/x))\}$ (Abelard chooses $c \in C$).

(4) If $(\exists x \theta, s)$: Next position is $S \cup \{(\theta, s(c/x))\}$ (Eloise chooses $c \in C$).

If $(\psi, s), (\neg\psi, s') \in S$, where $s(x) = s'(x)$ for all free x in ψ, Abelard wins.

[Beth, 1955], [Hintikka, 1955], [Smullyan, 1963], [Makkai, 1969].

- Gentzen's natural deduction,
- [Beth, 1955],
- [Hintikka, 1955],
- [Smullyan, 1963],
- [Makkai, 1969].
- Craig Interpolation Theorem.
- Completeness Theorem.
- Preservations Theorems.

Truth $\Rightarrow$ consistency

Theorem

Every strategy τ of Eloise in $\boldsymbol{G(M, \phi)}$ determines a strategy $\Phi(\tau)$ of Eloise in $\mathrm{MEG}(\phi)$. If τ is a winning strategy, then so is $\Phi(\tau)$.

(We assume the vocabulary of M is countable.)

- There is a countable submodel N of M such that τ is a strategy of Eloise in $G(N, \phi)$. Let $\pi : C \to N$ be an onto map.
- A pair (ψ, s) is a τ-position if there is there is some sequence of positions in $G(N, \phi)$, following the rules of $G(N, \phi)$ starting with $(\phi, \emptyset)$, Eloise using τ, which ends at (ψ, s).
- A C-translation of the τ-position (ψ, s) is a pair (ψ, s') where s' is a C-assignment with $\pi(s'(x)) = s(x)$.
- The strategy $\Phi(\tau)$ of Eloise in MEG(ϕ) is to make sure that at all times the position S consists only of C-translations of τ-positions.

$$
\begin{tikzcd}
C \arrow[r, dash] \arrow[d, dash, "\Phi(\tau)"'] & N \arrow[r, phantom, "\subseteq"] & M \arrow[d, dash, "\tau"] \\
\phi & & \phi
\end{tikzcd}
$$

Figure: From model to model existence.

1. If $(\psi_0 \vee \psi_1, s') \in S$, then Eloise can decide that the next position is $S \cup \{(\psi_0, s')\}$ or $S \cup \{(\psi_1, s')\}$. There is a sequence α of moves in $G(N, \phi)$ in which Eloise uses τ, which ends at $(\psi_0 \vee \psi_1, s)$, a C-translation of which is $(\psi_0 \vee \psi_1, s')$. We continue α by letting Eloise move whatever τ tells her to move, say ψ_0. Now $S \cup \{(\psi_0, s')\}$ consists of C-translations of τ-moves. So we let Eloise choose $S \cup \{(\psi_0, s')\}$ as the next position.

- If Eloise follows this strategy and τ happens to be a winning strategy, then she wins because if at some point there are (ψ, s') and $(\neg\psi, t')$ in the set S, such that ψ is atomic and $s'(x) = t'(x)$ for variables x in ψ, then these would be C-translations of (ψ, s) and $(\neg\psi, t)$, where $s(x) = t(x)$ for x in ψ, respectively, and s (and t) would satisfy both ψ and $\neg\psi$ in N because τ is a winning strategy, a contradiction.

Consistency $\Rightarrow$ model and truth

Theorem

Every strategy τ of Eloise in $\mathrm{MEG}(\phi)$ *determines a model $M(\tau)$ and a strategy $\Psi(\tau)$ of Eloise in* $\boldsymbol{G(M(\tau), \phi)}$. *If τ is winning, then so is $\Psi(\tau)$.*

[Beth, 1955]

$$
\begin{tikzcd}
C \arrow[r, dash] \arrow[d, dash, "\tau"'] & M(\tau) \arrow[d, dash, "\Psi(\tau)"] \\
\phi & \phi
\end{tikzcd}
$$

Figure: From model existence to a model.

Let σ_0 be the following enumeration strategy of Abelard in MEG(ϕ): During the game Abelard makes sure that if S is the position, then:

1. If $(\psi_0 \wedge \psi_1, s) \in S$, then during the game he will at some position $S' \supseteq S$ decide that the next position is $S' \cup \{(\psi_0, s)\}$ and at some further position $S'' \supseteq S'$ he will decide that the next position is $S'' \cup \{(\psi_1, s)\}$.
2. If $(\psi_0 \vee \psi_1, s) \in S$, then at some position $S' \supseteq S$ Abelard asks Eloise to choose whether the next position is $S' \cup \{(\psi_0, s)\}$ or $S' \cup \{(\psi_1, s)\}$.
3. If $(\forall x\theta, s) \in S$, then for all n during the game he will at some position $S' \supseteq S$ decide that the next position is $S' \cup \{(\theta, s(c_n/x)\}$.
4. If $(\exists x\theta, s) \in S$, then at some position $S' \supseteq S$ Abelard will ask Eloise to choose n after which the next position is $S' \cup \{(\theta, s(c_n/x))\}$.

- Let us play MEG(ϕ) while Abelard uses this strategy and Eloise plays τ.
- Let $\mathcal{S} = \langle S_n : n < \omega \rangle$ be the (unique) infinite sequence of positions during this play. Note that $S_n \subseteq S_{n+1}$ for all n. Let $\mathbf{\Gamma}$ be the union of all the positions in $\mathcal{S}$.
- We build a model $M = M(\tau)$ as follows[1] : The domain of the model is $\{c_n : n \in \mathbb{N}\}$. If R is a relation symbol, then we let $R(c_{n_0}, \ldots, c_{n_k})$ hold in M if $(R(x_{n_0}, \ldots, x_{n_k}), s) \in \mathbf{\Gamma}$ for some s such that $s(x_i) = c_i$ for $i = n_0, \ldots, n_k$.
- The strategy $\boldsymbol{\Psi(\tau)}$ of Eloise in $G(M, \phi)$ is the following: She makes sure that if the position in $G(M, \phi)$ is (ψ, s), then $(\psi, s) \in \mathbf{\Gamma}$. Let us see that she can follow the strategy throughout the game:

[1] We assume ϕ has a relational vocabulary and does not contain the identity symbol.

1. If ψ is $\exists x\theta$, then Eloise should choose for which n the next position is $(\theta, s(c_n/x))$.
2. We know $(\exists x\theta, s) \in S$ for some position S during the game, because $(\exists x\theta, s) \in \mathbf{\Gamma}$.
3. By how σ_0 was defined, Abelard has at some later position $S' \supseteq S$ asked Eloise to choose n for which the next position would be $S' \cup \{(\theta, s(c_n/x))\}$.
4. The strategy τ has directed Eloise to indeed choose an n leading to the new position $S' \cup \{(\theta, s(c_n/x))\}$.
5. Thus $(\theta, s(c_n/x)) \in \mathbf{\Gamma}$ and she can safely play $(\theta, s(c_n/x))$ in $G(M, \phi)$.

- Suppose τ is winning for Eloise. We show that then $\Psi(\tau)$ is winning.
- Suppose the game $G(M,\phi)$ ends at a position (ψ, s) where ψ is atomic. Then (ψ, s) is in Γ whence s satisfies ψ in M.
- Suppose, on the other hand, the game $G(M,\phi)$ ends at a position $(\neg\psi, s)$ where ψ is atomic. Then $(\neg\psi, s)$ is in S_n for some n. It suffices to show that s does not satisfy ψ in M. Suppose it does. Then there is $(\psi, s') \in S_m$ for some m and some s' such that $s(x) = s'(x)$ for variables x in ψ. Then S_{n+m} has both $(\neg\psi, s)$ and (ψ, s'), contrary to the assumption that τ is a winning strategy of Eloise.

- A winning strategy of Eloise in MEG(ϕ) can be conveniently given in the form of a so-called *consistency property*, which is just a set of finite sets of sentences satisfying conditions which essentially code a winning strategy for Eloise in MEG(ϕ).
- Sometimes it is more convenient to use a consistency property than Model Existence Game. But as far as strategies of Eloise are concerned, the two are one and the same thing.
- Consistency properties have been successfully used to prove interpolation and preservations results in model theory, especially infinitary model theory [Makkai, 1969].

- Suppose now Abelard has a winning strategy in MEG(ϕ).
- We can form a tree, a Beth Tableaux, of all the positions when Abelard plays his winning strategy and we stop playing as soon as Abelard has won.
- Every branch of the tree is finite and ends in a position which includes a contradiction.
- We can make the tree finite. We can then view this tree as a proof of $\neg\phi$. In this sense the Model Existence Game builds a bridge between proof theory and model theory.
- Strategies of Abelard direct us to proof theory, while strategies of Eloise direct us to model theory.

Apart from first order and infinitary logic, the Model Existence Game can be used in the proof theory and model theory of

- propositional and modal logic.
- logic with generalized quantifiers and higher order logic.
- weak models, which have to be transformed to real models by a model theoretic argument [Keisler, 1970].
- general models for higher order logics [Henkin, 1950].
- infinitary logic $L_{\kappa\lambda}$,
- chain models, rather than real models.

EF (Ehrenfeucht-Fraïssé) game

- In the EF game we have a model but no sentence.
- The sentence should be true in one but false in the other. It may be that no such sentence can be found, i.e. the models are *elementarily equivalent*.
- In the EF game strategies of one player track possibilities for elementary equivalence and the strategies of the other player track possibilities for a separating sentence.
- [Fraïssé, 1954], [Ehrenfeucht, 1961]
- M and N are two structures for the same vocabulary L.

Definition

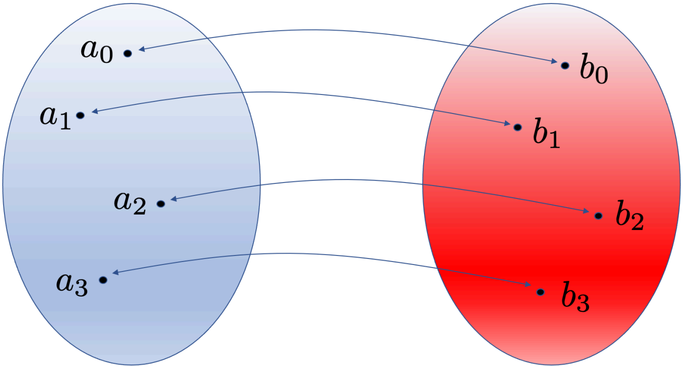
The game $\mathrm{EF}_m(M, N)$ has two players Abelard and Eloise and m moves. A position is a set

$$s = \{(a_0, b_0), \ldots, (a_{n-1}, b_{n-1})\} \tag{1}$$

of pairs of elements such that the a_i are from A and the b_i are from B, and $n \leq m$. In the beginning the position is $\emptyset$. The rules:

1. Abelard may choose some $a_n \in A$. Then Eloise chooses $b_n \in B$ and the next position is $s \cup \{(a_n, b_n)\}$.
2. Abelard may choose some $b_n \in B$. Then Eloise chooses $a_n \in A$ and the next position is $s \cup \{(a_n, b_n)\}$.

Abelard wins if during the game the position (1) is such that $(a_0, \ldots, a_{n-1})$ satisfies some literal in M but $(b_0, \ldots, b_{n-1})$ does not satisfy the corresponding literal in N.


a_0
a_1
a_2
a_3
b_0
b_1
b_2
b_3

- Intuitively, Eloise defends the proposition that M and N are very similar.
- Abelard doubts this similarity.
- If Eloise knows an isomorphism $f : M \to N$ she can respond by playing always so that $b_n = f(a_n)$.
- Two models of (any) size $\geq m$ in the empty vocabulary.
- Two finite linear orders of (any) size $\geq 2^m$.
- This game is **determined**.
- How long games can Eloise win although $M \not\cong N$? Interesting for transfinite games.

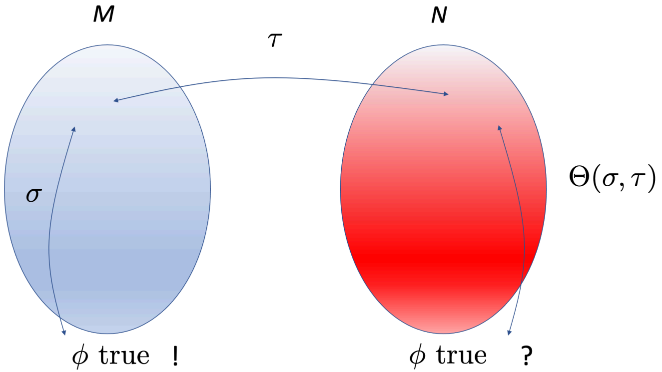
- A logician's version of isomorphism.
- A formula "is" this game.

Strategy of Eloise $\Rightarrow$ elementary equivalence

Theorem

Suppose ϕ is an $L_{\infty\omega}$-sentence of quantifier rank $\leq m$. Every strategy τ of Eloise in $EF_m(M, N)$, and every strategy σ of Eloise in $G(M, \phi)$ determine a strategy $\Theta(\sigma, \tau)$ of Eloise in $G(N, \phi)$. If τ and σ are winning strategies, then so is $\Theta(\sigma, \tau)$.

[Ehrenfeucht, 1961]


M
N
τ
σ
Θ(σ, τ)
φ true !
φ true ?

- We call a position of the game $\mathrm{EF}_m(M, N)$ a **τ-position** if it arises while Eloise is playing τ.
- We call a position of the game $G(M, \phi)$ a **σ-position**, if it arises while Eloise is playing σ.
- If the position of the game $G(N, \phi)$ is (ψ, s), the **strategy** $\Phi(\sigma, \tau)$ of Eloise is to play simultaneously $G(N, \phi)$, $\mathrm{EF}_m(M, N)$ and $G(M, \phi)$, and make sure that if

$$\pi = \{(a_0, b_0), \ldots, (a_{n-1}, b_{n-1})\}$$

is the current τ-position in $\mathrm{EF}_m(M, N)$ and $s(x) = \pi(s'(x))$ for all x in the domain of s, then (ψ, s') is the current σ-position in $G(M, \phi)$ (see Figure 4).

$$
\begin{tikzcd}
M \arrow[r, dash, "\pi"] \arrow[d, dash, "s'"'] & N \arrow[d, dash, "s"] \\
\phi & \phi
\end{tikzcd}
$$

Figure: The strategy $\Theta(\sigma, \tau)$, Let us check that it is possible for Eloise to play this strategy:

1. If the position is (ψ, s) where ψ is a literal, the game ends.
2. If the position is (ψ, s) where is ψ is$\bigwedge_i \phi_i$, then Abelard chooses i and the next position is (ψ_i, s). Whichever he chooses, we let Abelard make the respective move (ψ_i, s') in $G(M, \phi)$.
3. If the position is (ψ, s) where is ψ is $\bigvee_i \phi_i$, then Eloise chooses i as follows. Since (ψ, s') is a σ-position, the strategy σ tells Eloise which of (ψ_i, s') to play in $G(M, \phi)$. Then Eloise plays the respective (ψ_i, s) in $G(N, \phi)$.

4. If the position is (ψ, s) is $(\forall x\theta, s)$, then Abelard chooses $b_n \in N$ and the next position is (θ, t), $t = s(b_n/x)$. We continue the game $\mathrm{EF}_m(M, N)$ from the τ-position $\{(a_0, b_0), \ldots, (a_{n-1}, b_{n-1})\}$ letting Abelard play b_n. The strategy τ tells Eloise to choose $a_n \in M$ so that

$$\pi' = \{(a_0, b_0), \ldots, (a_n, b_n)\} \tag{2}$$

is again a τ-position. Now we continue the game $G(M, \phi)$ from position $(\forall x\theta, s')$ by letting Abelard play a_n. We reach the position (θ, t'), $t' = s'(a_n/x)$, which is still a σ-position, and we have $t'(y) = \pi'(t(y))$ for all y in the domain of t'.

5. The position is (ψ, s) is $(\exists x\theta, s)$. Now we continue the game $G(M, \phi)$ from position $(\exists x\theta, s')$ by letting Eloise play, according to σ, an element a_n and we reach a new σ-position (θ, t'), $t' = s'(a_n/x)$. We continue the game $\mathrm{EF}_m(M, N)$ from the τ-position $\{(a_0, b_0), \ldots, (a_{n-1}, b_{n-1})\}$ letting Abelard play a_n. The strategy τ tells Eloise to choose $b_n \in N$ so that (2) is again a τ-position. We reach the position (θ, t), $t = s(b_n/x)$, and we have $t(y) = \pi(t'(y))$ for all y in the domain of t.

6. If σ is a winning strategy and the game $G(N, \phi)$ ends in the position (ψ, s), where ψ is a literal, then Eloise wins because then (ψ, s') is a σ-position meaning that s' satisfies the literal ψ in M, and τ being a winning strategy this means that s satisfies the literal ψ in N.

- There is a tight connection between σ, τ and $\Theta(\sigma, \tau)$. This is reflected in a connection between ϕ and $\mathrm{EF}_m(M, N)$.
- If the non-logical symbols of ϕ are in $L' \subset L$, then it suffices that τ is a strategy of Eloise in the game $\mathrm{EF}_m(M \restriction L', N \restriction L')$ between the reducts $M \restriction L'$ and $N \restriction L'$.
- If we know more about the syntax of ϕ, for example that it is existential, universal or positive, we can modify $\mathrm{EF}_M(M, N)$ accordingly by stipulating that Abelard only moves in M, only moves in N, or that he has to win by finding an atomic (rather than literal) relation which holds in M but not in N.
- Winning strategies for the EF game are a standard method for showing that certain kinds of sentences do not exist.

Strategy of Abelard $\Rightarrow$ separating sentence

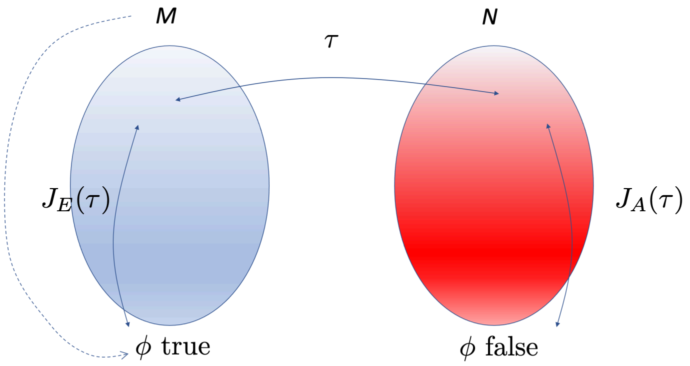
Theorem

Suppose M and N are models of the same vocabulary and $m < \omega$.

1. *There is a sentence $\phi \in L_{\infty\omega}$ of quantifier rank $\leq m$ and mappings J_E and J_A such that if τ is a strategy of Abelard in $EF_m(M, N)$, then $J_E(\tau)$ is a strategy of Eloise in $G(M, \phi)$, and $J_A(\tau)$ is a strategy of Abelard in $G(N, \phi)$.*
2. *If τ is a winning strategy, then $J_E(\tau)$ and $J_A(\tau)$ are winning strategies.*

Note: If L is finite and relational, the sentence ϕ is logically equivalent to a first order sentence of quantifier rank $\leq m$.

[Ehrenfeucht, 1961]



M
N
τ
$J_E(\tau)$
$J_A(\tau)$
ϕ true
ϕ false

Suppose s is an assignment into M with domain $\{x_0, \ldots, x_{n-1}\}$. Let

$$
\begin{aligned}
\psi^{0,n}_{M,s} &= \bigwedge_i \psi_i \\
\psi^{m+1,n}_{M,s} &= (\forall x_n \bigvee_{a \in M} \psi^{m,n+1}_{M,s(a/x_n)}) \wedge (\bigwedge_{a \in M} \exists x_n \psi^{m,n+1}_{M,s(a/x_n)}),
\end{aligned}
$$

where ψ_i lists all the literals in the variables $x_0, \ldots, x_{n-1}$ satisfied by s in M.
The sentence ϕ we need is $\boldsymbol{\psi^{m,0}_{M,\emptyset}}$.

- Clearly Eloise has a trivial strategy $J_2(\tau)$ in $G(M, \phi)$ (independently of τ), and this strategy is always a winning strategy.
- We now describe the strategy $J_1(\tau)$ of Abelard in $G(N, \phi)$.
- We call a position of the EF-game a **τ-position** if it arises while Abelard is playing τ.
- Suppose s is an assignment into M and s' an assignment into N, both with domain $\{x_0, \ldots, x_{n-1}\}$. We use $\boldsymbol{s \cdot s'}$ to denote the set of pairs $(s(x_i), s'(x_i))$, $i = 0, \ldots, n-1$. The **strategy** of Abelard is to play $G(N, \phi)$ in such a way that if the position at any point is $(\psi_{M,s}^{i,m-i}, s')$, then $s \cdot s'$ is a τ-position.

1. Suppose the position in $G(N, \phi)$ is $(\psi^{i,m-i}_{M,s}, s')$, $i > 0$, and the next move for Abelard in $\mathrm{EF}_m(M, N)$ according to τ is $a \in M$.
2. The strategy of Abelard is to choose the latter conjunct of $\psi^{i,m-i}_{M,s}$. Then Abelard chooses the element $a \in M$ in the big conjunction move.
3. Now it is the turn of Eloise in $G(N, \phi)$ to choose some $b \in N$ as the value of x_{m-i} and that will be the next move of Eloise in $\mathrm{EF}_m(M, N)$. The next position in $G(N, \phi)$ is

$$(\psi^{i-1,m-i+1}_{M,s(a/x_{m-i})}, s'(b/x_{m-i})). \tag{3}$$

4. The position $s(a/x_{m-i}) \cdot s'(b/x_{m-i})$ is still a τ-position in $\mathrm{EF}_m(M, N)$.

- Suppose now τ was a winning strategy of Abelard. Then at the end of the game $s \cdot s'$ is a winning position for Abelard and therefore he is indeed able to choose a conjunct of the formula $\psi^{0,m}_{M,s}$ that is not satisfied by s' in N. He has won $G(N, \phi)$.

- If τ is a winning strategy of Abelard even in the game $\mathrm{EF}_m(M \restriction L', N \restriction L')$ for some $L' \subset L$, then the separating sentence ϕ can be chosen so that its non-logical symbols are all in L'.
- If τ is such that Abelard plays only in M, we can make ϕ existential.
- If τ is such that Abelard plays only in N, we can make ϕ universal.
- If Abelard wins with τ even the harder game in which he has to win by finding an atomic (rather than literal) relation which holds in M but not in N, then we can take ϕ to be a positive sentence.

- Strategies in $\mathrm{EF}_m(M, N)$ also reflect structural properties of M and N.
- If we know a strategy of Eloise in $\mathrm{EF}_m(M_i, N_i)$ for $i \in I$, we can construct strategies of Eloise for EF games between products and sums of the models M_i and the respective products and sums of the models N_i. This can be extended to so-called κ-local functors [Feferman, 1972]. For an example of the use of tree-decompositions, see e.g. [Grohe, 2007].
- EF games are known for infinitary logics, generalized quantifiers, and higher order logics.
- In modal logic the corresponding game is called the bisimulation game.

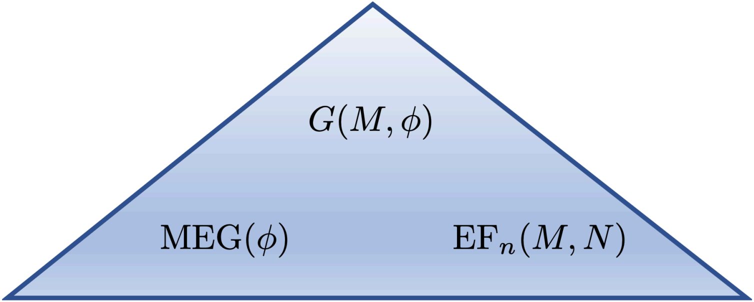
- EF game for teams: the players move and manipulate teams.
- EF game (on teams) for propositional logic [Hella and Väänänen, 2015].
- EF-game (on teams) for $L_{\omega_1\omega}$ [Väänänen and Wang, 2013].
- An EF-game with "delay" and a Lindström Theorem[2] for a new infinitary logic between $L_{\kappa\omega}$ and $L_{\kappa\kappa}$ [Shelah, 2012].

[2] A Lindström Theorem is a semantic characterization of a logic.

Summary

- The Evaluation Game, the Model Existence Game and the EF game go so deep into the essential concepts of logic such as truth, consistency, and separating models by sentences, that a lot of research in logic can be represented in terms of these games. This alone does not bring anything new.
- The translations of the strategies between the games suggest a coherent uniform approach to syntax and semantics and at the same time to model theory and proof theory.
- The Evaluation Game and the EF game are oblivious to whether the models are finite or infinite, which gives them a useful role in computer science logic, uniting CS with
- Despite the vast literature on each of the three games separately, there seems to be a lot of potential for the study of their interaction as a manifestation of the Strategic Balance of Logic.

Thank you!



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