Set-theoretic methods in model theory

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Models i.e. structures

- Relational structure (M,R,...).
- A set with relations, functions and constants.
- Partial orders, trees, linear orders, lattices, groups, semigroups, fields, monoids, graphs, hypergraphs, directed graphs.

Models and topology

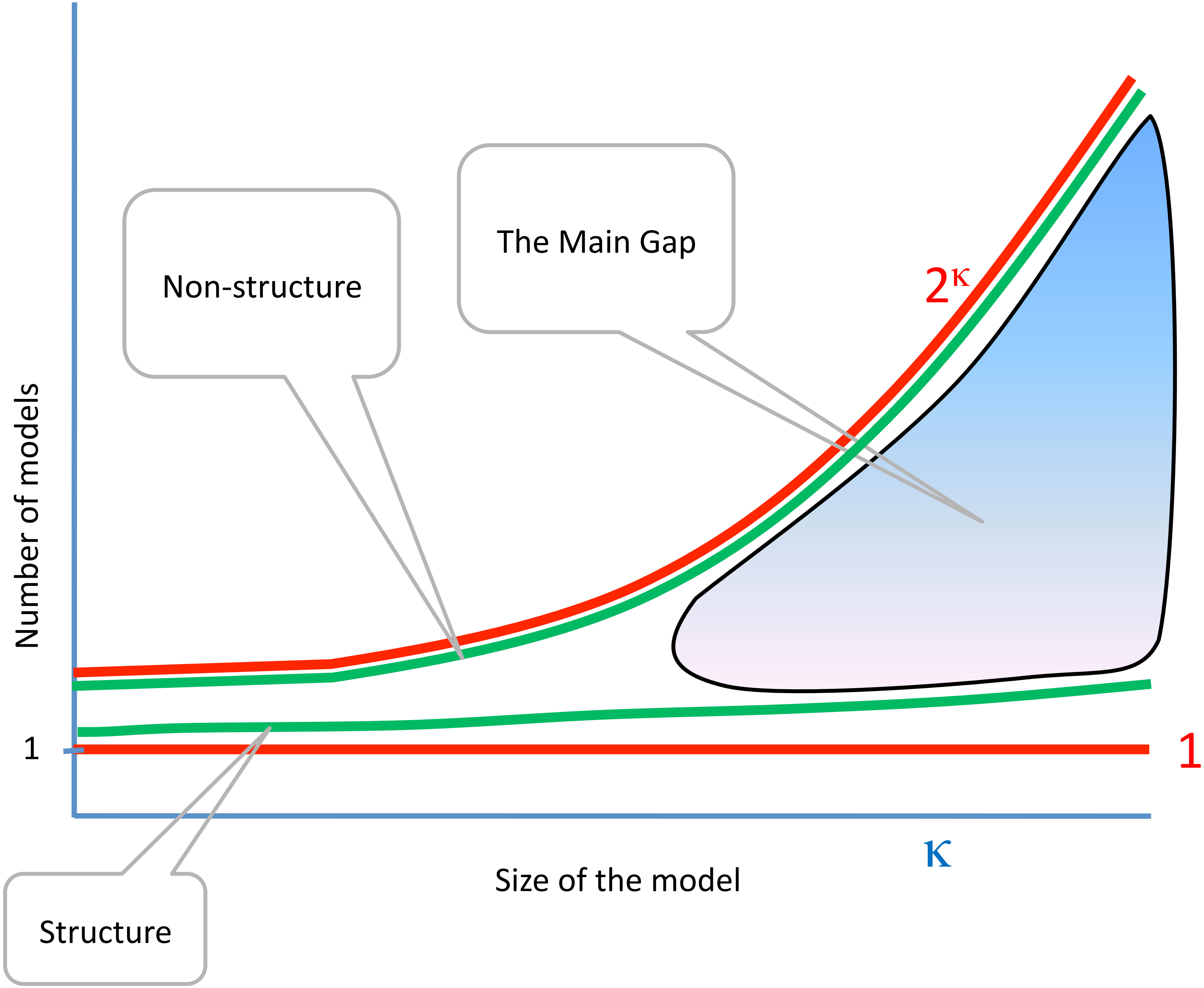
- A countable model is a point in 2^{ω} (mod ≅).
- A model of size κ is a point in 2^{κ} (mod ≅).
- Properties of models ~ subsets of 2^{κ}.
- Isomorphism of models: ``analytic" subset of $2^{\kappa} \times 2^{\kappa}$.

The basic question

- How to identify a structure?
- Relevant even for finite structures.
- Can infinite structures be classified by invariants?

Shelah's Main Gap

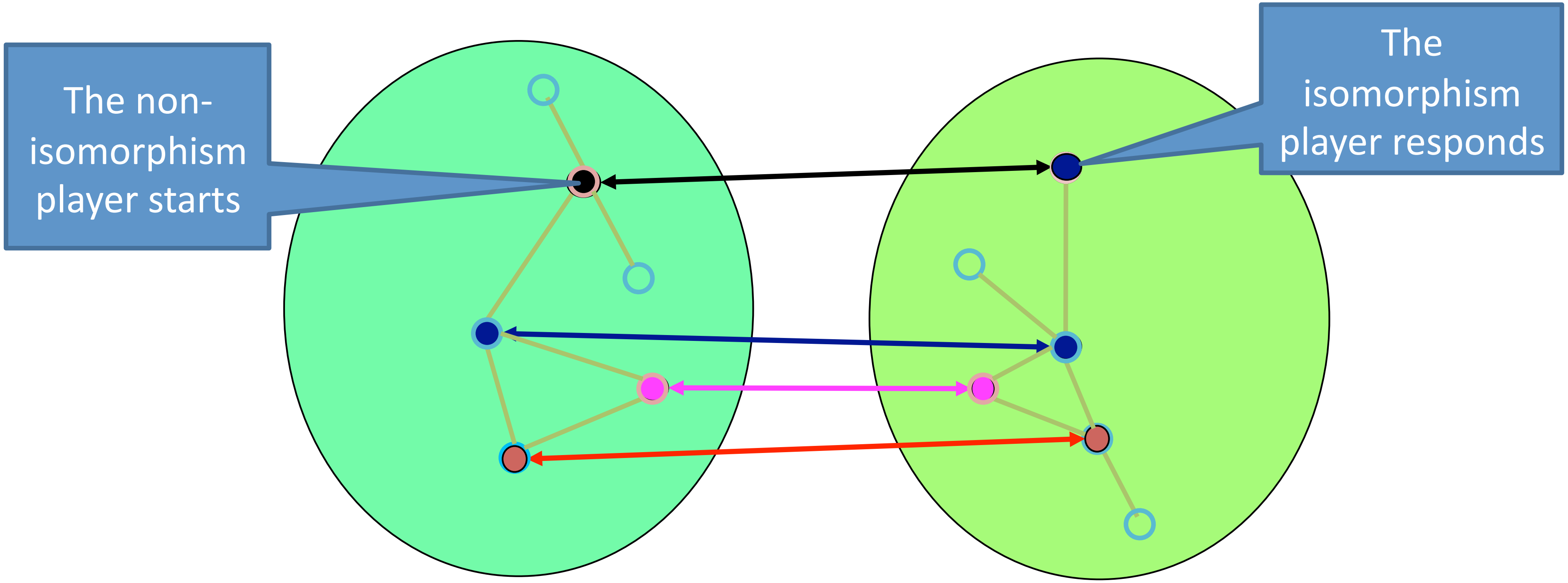
- M any structure.
- The first order theory of M is either of the two types:
  - Structure Case: All uncountable models can be characterized in terms of dimension-like invariants.
  - Non-structure case: In every uncountable cardinality there are non-isomorphic models that are "extremly" difficult to distinguish from each other by means of invariants.


Non-structure
The Main Gap
2^κ
Number of models
1
1
κ
Size of the model
Structure

The program

- To analyze further the non-structure case.
  - We replace isomorphism by a game.
  - We develop topology of 2^κ.

Ehrenfeucht-Fraïssé game



Two players: The non-isomorphism player and the isomorphism player.

Approximating isomorphism

- M,N countable (graphs, posets,...)
- $M \ncong N$
- The non-isomorphism player wins the EF game of length ω with the enumeration strategy τ
- T(M,N)=the countable tree of plays against τ, where the isomorphism player has not lost yet.
- T(M,N) has no infinite branches, well-founded

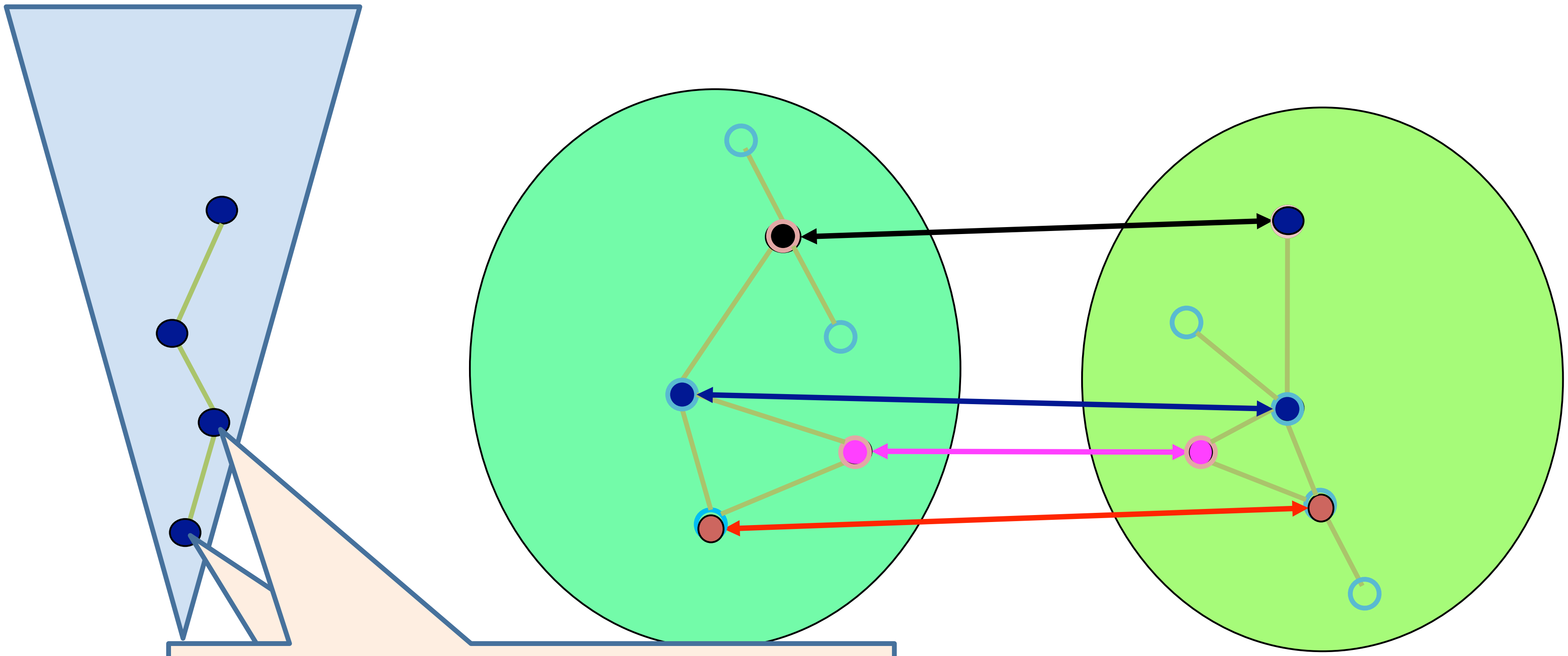
Approximating isomorphism (contd.)

- T(M,N) has a rank $\alpha<\omega_1$.
- $\sigma_M=\sup_{\mathbf{a},\mathbf{b}}\{\mathrm{rank}(T((M,\mathbf{a}),(M,\mathbf{b})) : (M,\mathbf{a}) \not\cong M,\mathbf{b})\}$
- Scott rank of M.
- Scott ranks put countable models into a hierarchy, calibrated by countable ordinals.
- The orbit of M is a Borel subset of ω^ω.
- 60's and 70's: Scott, Vaught invariant topology
- 90's and 00's: Kechris, Hjorth, Louveau: Borel equivalence relations

Game with a clock

- The isomorphism player loses the EF game of length ω, but maybe she can win if the non-isomorphism player is forced to obey a clock.

Ehrenfeucht-Fraïssé game with a clock



The non-isomorphism player goes up the clock-tree

The clock gives a chance

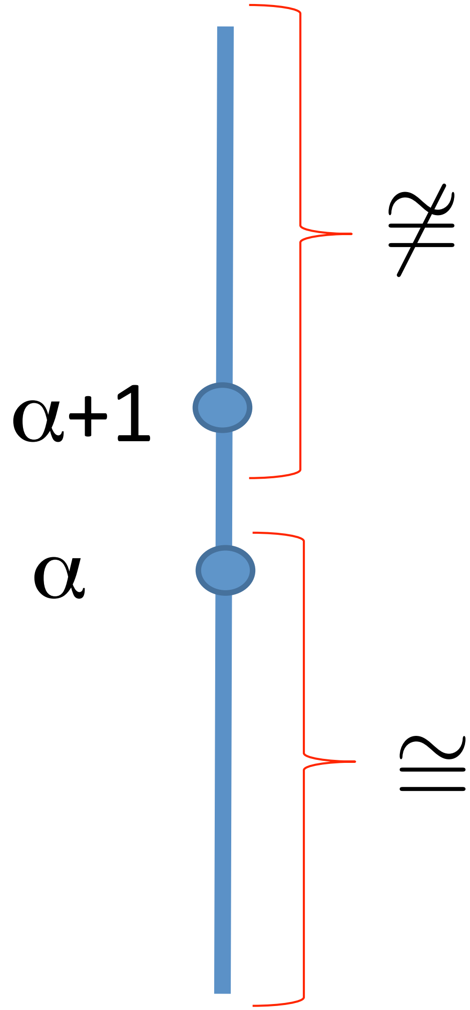
- Although the isomorphism player loses the EF game of length ω, she wins the game which has T(M,N) as the clock.
- T(M,N)=the tree of plays against τ, where the isomorphism player has not lost yet.

A well-founded clock

- The tree B_α of descending sequences of elements of α is the canonical well-founded tree of rank α

For countable M and N:

- TFAE:
  - $M \cong N$
  - The isomorphism player wins the EF game clocked by B_α for all $\alpha<\omega_1$.
- TFAE:
  - $M \cong N$
  - The isomorphism player wins the EF game clocked by B_α for some $\alpha<\omega_1$ such that the non-isomorphism player wins with clock $B_{\alpha+1}$



An ordering of trees, motivated by games

- $T \leq T'$ if there is $f: T \rightarrow T'$ such that

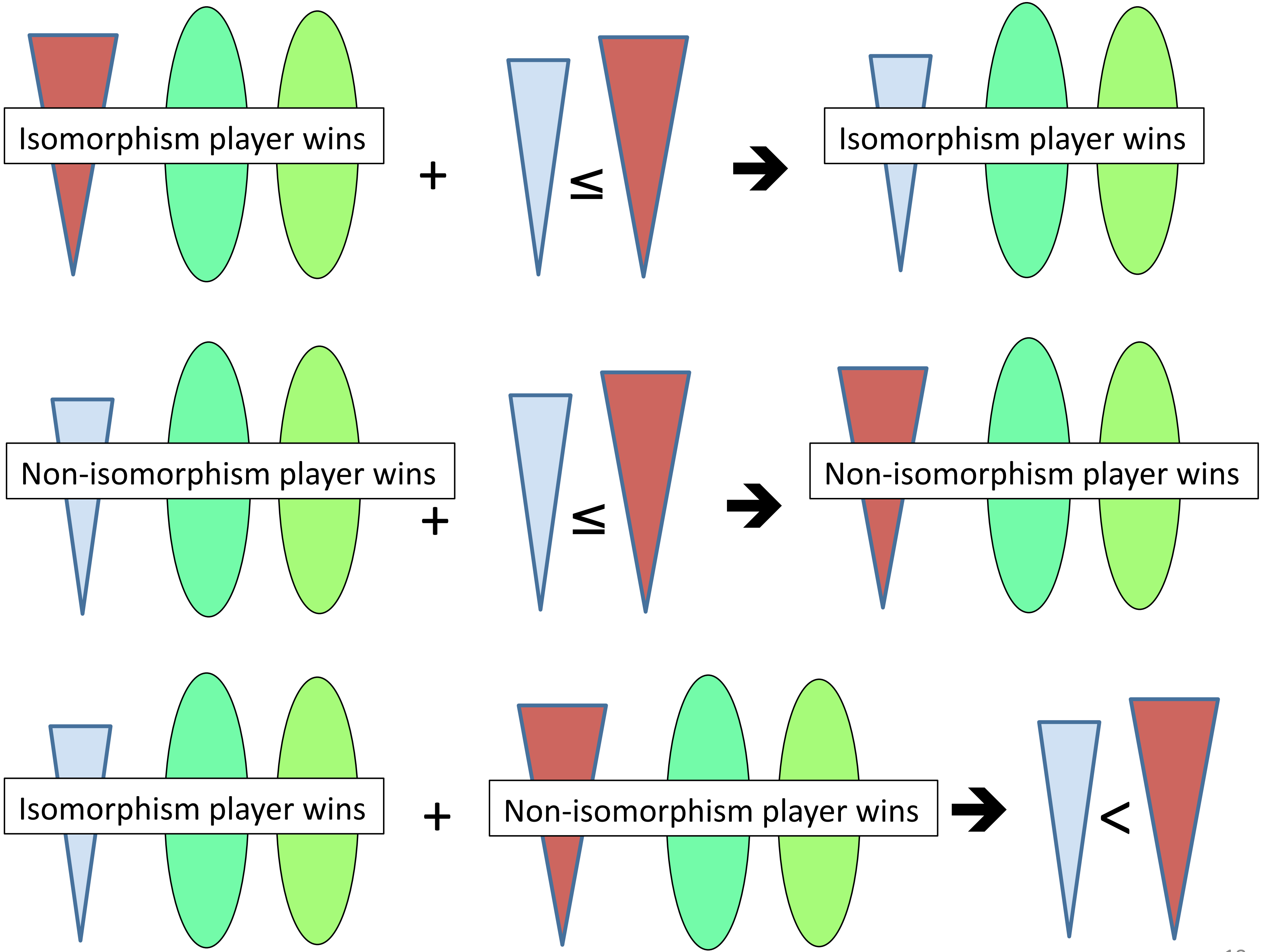
  $$x <_T y \rightarrow f(x) <_T f(y).$$

- If T and T' do not have infinite branches, then $T \leq T'$ iff $\mathrm{rank}(T) \leq \mathrm{rank}(T')$.
- Fact: $T \leq T'$ iff II wins a comparison game on T and T'.

T≤T' ranks game clocks

- If T≤T' then a game clocked by T is
  - easier for the isomorphism player
  - harder for the non-isomorphism player

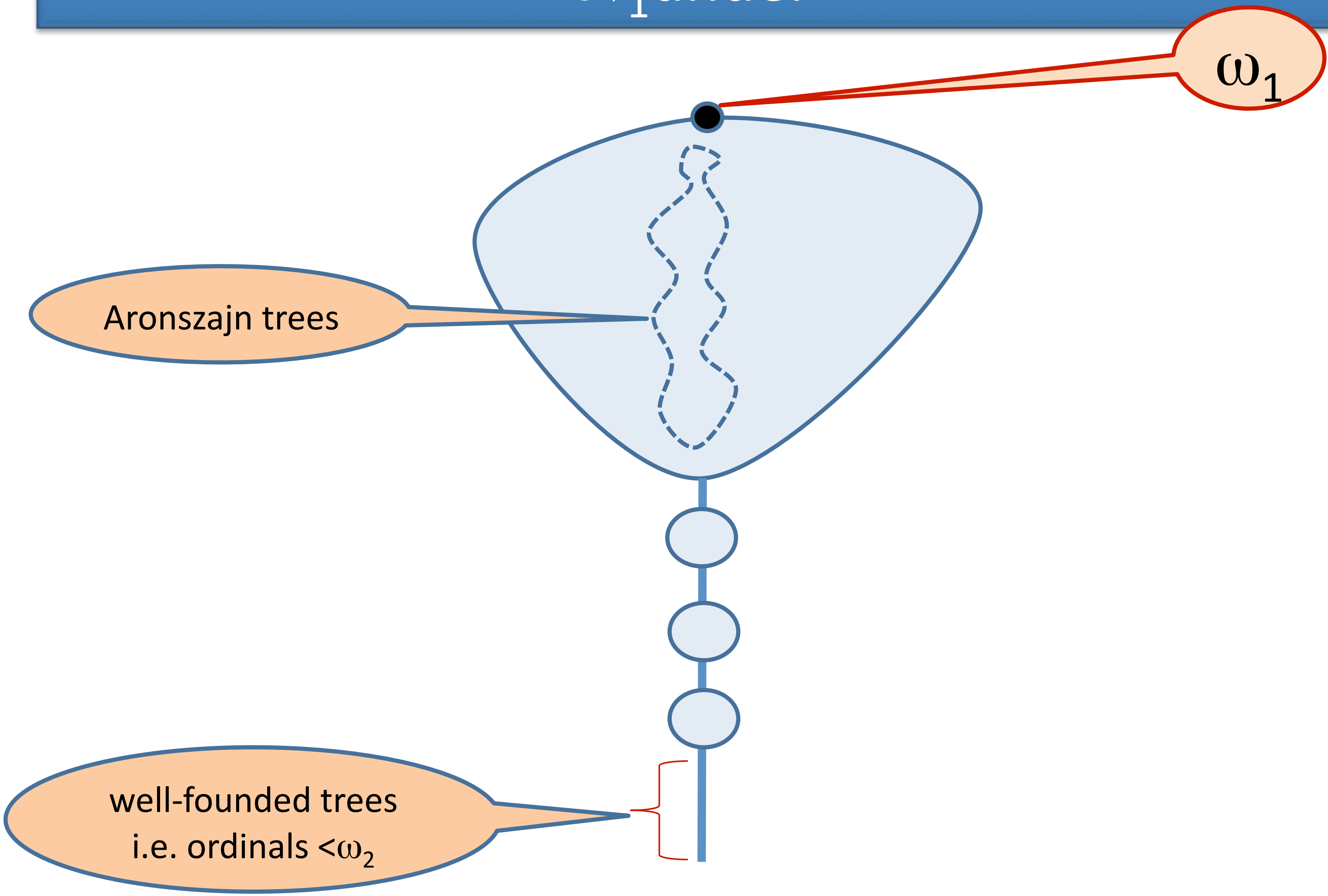
  than the same game clocked by T'.


Isomorphism player wins
+
≤
Isomorphism player wins
Non-isomorphism player wins
+
≤
Non-isomorphism player wins
Isomorphism player wins
+
Non-isomorphism player wins
<

There are incomparable trees

- (Todorcevic) There are incomparable Aronszajn trees.
- A tree is a bottleneck if it is comparable with every other tree.
- (Mekler-V., Todorcevic-V.) It is consistent that there are no non-trivial bottlenecks.
- (Todorcevic) PFA➔coherent Aronszajn trees are all comparable, and there is a canonical family of coherent Aronszajn trees that are bottlenecks in the class of trees of size $\aleph_1$ (Aronszajn trees).

The structure of trees of size and height $\aleph_1$ under $\leq$



A ``successor” operator on trees

- T a tree
- σT = the tree of ascending chains in T
- $T < \sigma T$
- $\sigma B_{\alpha} \equiv B_{\alpha+1}$

The uncountable case

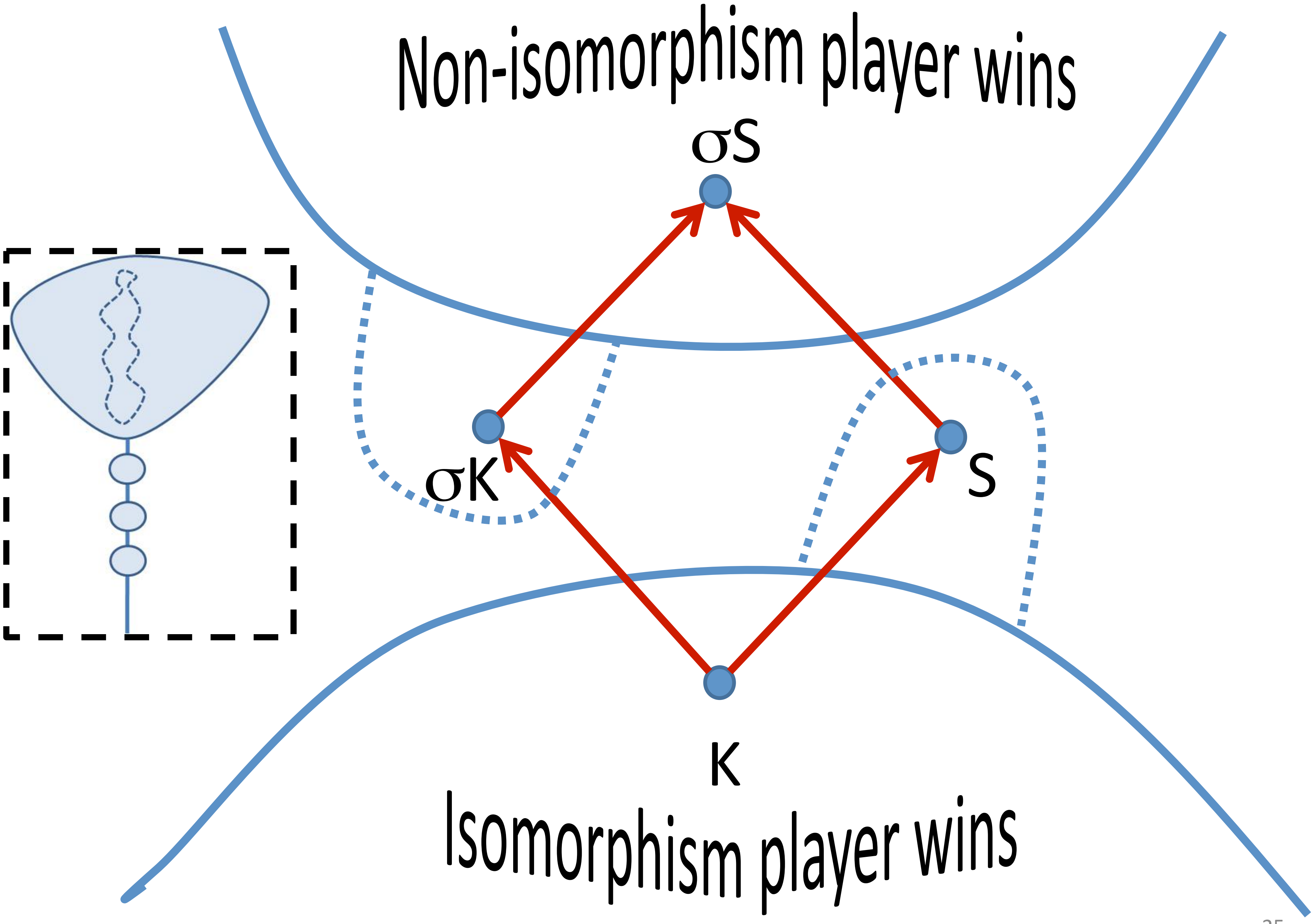
- M,N of size κ (graphs, posets,...)
- M $\not\cong$ N
- The non-isomorphism player wins the EF game of length κ with the enumeration strategy τ.
- T(M,N)=the tree of plays against τ, where the isomorphism player has not lost yet.
- T(M,N) has no branches of length κ, ``bounded".
- The cardinality of T(M,N) is $\kappa^{<\kappa}$.

The uncountable case

- For M and N of cardinality κ TFAE:
  - $M \cong N$
  - The isomorphism player wins the EF game clocked by T for all trees T w/o κ-branches, $|T| \leq 2^{\kappa^{<\kappa}}$.
  - The non-isomorphism player loses the EF game clocked by T for all trees T w/o κ-branches, $|T| \leq \kappa^{<\kappa}$.

Watershed

- For M and N of cardinality κ TFAE:
  – $M \not\cong N$
  – The isomorphism player wins the EF game clocked by K for some tree K w/o κ-branches, $|K| \leq 2^{\kappa^{<\kappa}}$, but does not win the game clocked by σK
  – The non-isomorphism player does not win the EF game clocked by S for some tree S w/o κ-branches, $|S| \leq \kappa^{<\kappa}$, but wins if clocked by σS.


Non-isomorphism player wins
σS
σK
S
K
Isomorphism player wins

Non-determinacy of the EF game

- Determinacy of the EF game of length ω_1 in the class of models of size $\aleph_2$ is equiconsistent with the existence of a weakly compact cardinal. (Hyttinen-Shelah-V.)

Generalized Baire space

- $\omega_1^{\omega_1}$, models of size $\aleph_1$
  - G_δ-topology.
  - ω_1-metrizable, ω_1-additive.
  - meager ($\bigcup_{\alpha<\omega_1} A_\alpha$, A_α nowhere dense), Baire Category Theorem holds: B_α dense open ➔ $\bigcap_{\alpha<\omega_1} B_\alpha \neq \emptyset$.
  - dense set of continuum size.
  - Sikorski, Todorcevic, Shelah, Juhasz &Weiss, …
- κ^κ, models of size κ
- λ^κ, κ=cof(λ), models of size λ, which are unions of chains of length κ of smaller models.

Descriptive Set Theory in $\omega_1^{\omega_1}$

- A set $A \subseteq \omega_1^{\omega_1}$ is analytic if it is the projection of a closed set $\subseteq \omega_1^{\omega_1} \times \omega_1^{\omega_1}$.
- Equivalently, there is a tree $T \subseteq \omega_1^{<\omega_1} \times \omega_1^{<\omega_1}$ such that

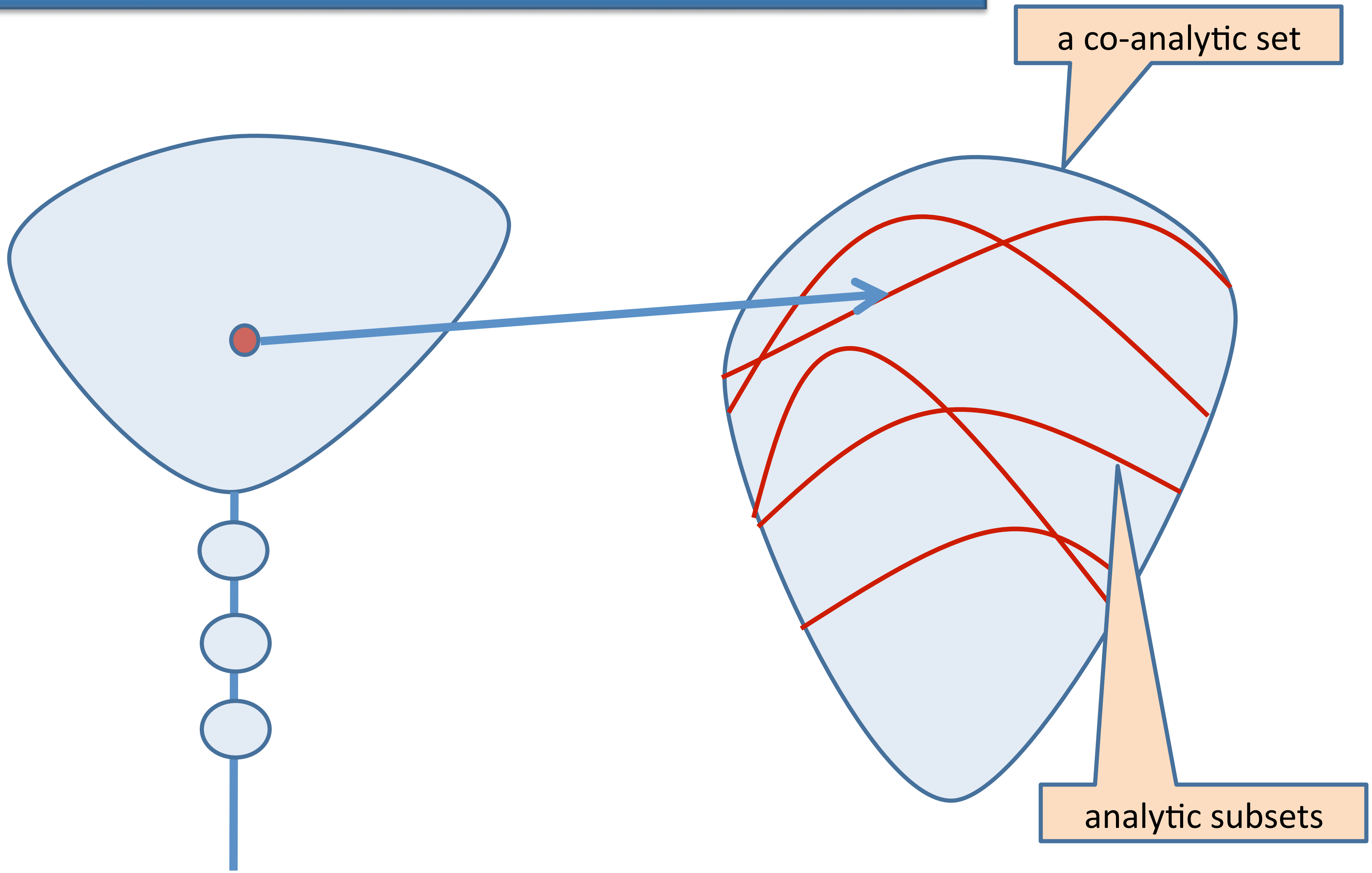
  $f \in A$ iff $T(f)$ has an uncountable branch,

  where $T(f)=\{\mathbf{g}(\alpha) : (\mathbf{g}(\alpha),\mathbf{f}(\alpha)) \in T\}$ and $\mathbf{g}(\alpha)=(g(\beta))_{\beta<\alpha}$.

A Covering Theorem

- Every co-analytic subset A of $\omega_1^{\omega_1}$ is covered by canonical sets B_T, T a tree w/o uncountable branches, such that every analytic subset of A is covered by some B_T.
- CH implies the sets B_T are analytic and the trees T are of size $\aleph_1$.

Covering Theorem under CH



Proof

- Suppose A is co-analytic and $B \subseteq A$ is analytic.
- $f \in A$ iff $T(f)$ has an uncountable branch.
- $f \in B$ iff $S(f)$ has no uncountable branches.
- Let T' be the tree of $(\mathbf{f}(\alpha),\mathbf{g}(\alpha),\mathbf{h}(\alpha))$ where $\mathbf{g}(\alpha) \in T(f)$ and $\mathbf{h}(\alpha) \in S(f)$.
- If $f \in B$, there is an uncountable branch h in $S(f)$.
- Let $F(\mathbf{g}(\alpha)) = (\mathbf{f}(\alpha),\mathbf{g}(\alpha),\mathbf{h}(\alpha))$.
- This is an order preserving mapping $T(f) \to T'$

Proof contd.

- So $T(f) \leq T'$
- Let $A_{T'} = \{f \in A : T(f) \leq T'\}$.
- Then $B \subseteq A_{T'}$.
- We have proved the Covering Theorem: If A is co-analytic, then A is the union of sets A_T such that if B is any analytic set $\subseteq A$, then there is a tree T w/o uncountable branches such that $B \subseteq A_T$.
- CH implies each A_T is analytic.

Souslin-Kleene, separation

- Souslin-Kleene: If A is analytic co-analytic, then $A=A_T$ for some T w/o uncountable branches.
- Separation: If A and B are disjoint analytic sets in $\omega_1^{\omega_1}$, then there is a set $C=(-B)_T$ which separates A and B.

Luzin Separation Theorem?

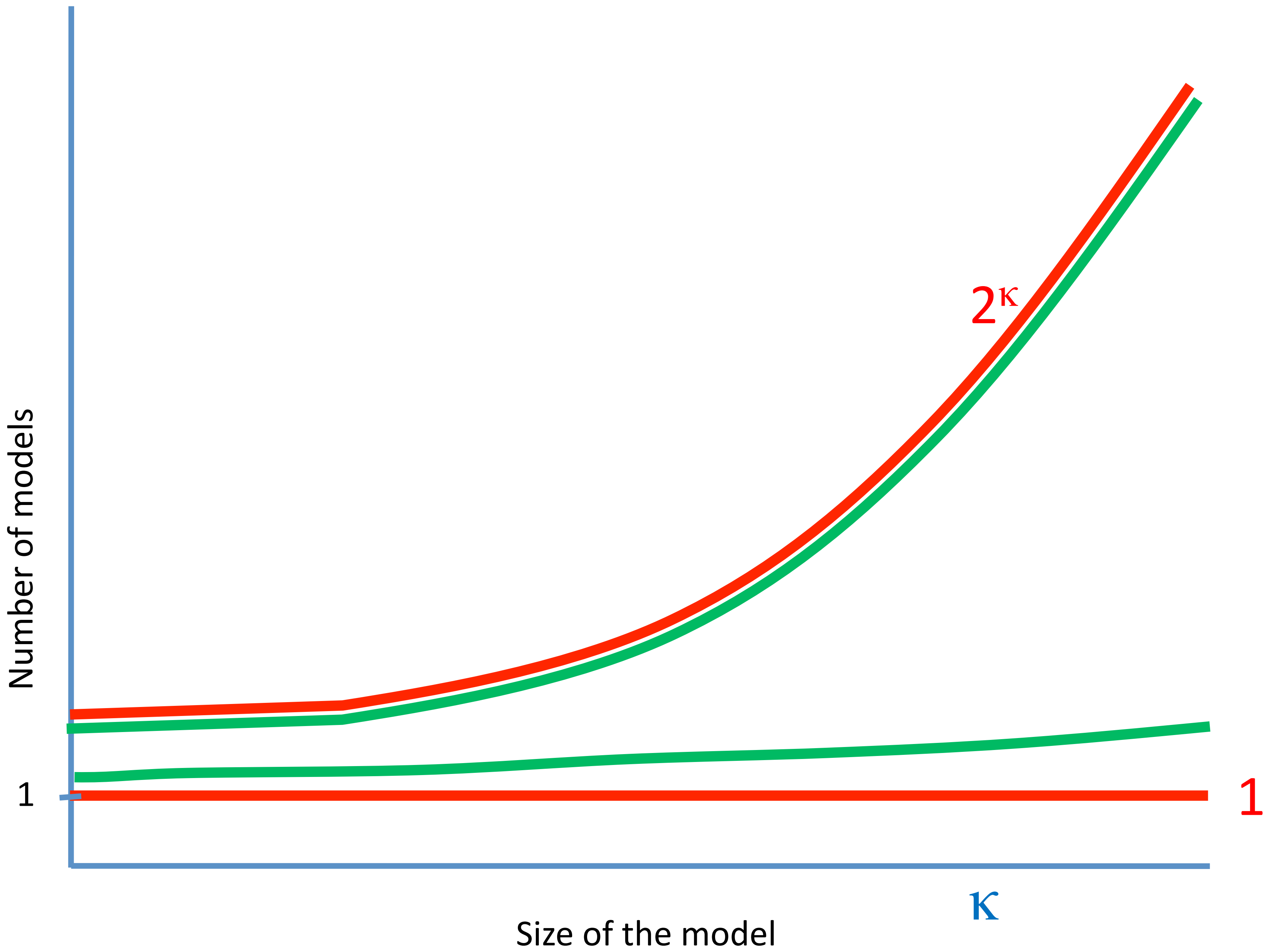
- Borel means closure of open under complements and unions of length ω_1.
- (Shelah-V.)
  - Assume CH. There are disjoint analytic sets which cannot be separated by a Borel set.
  - Assume $\neg$CH+MA. Any two disjoint analytic sets of expansions of $(\omega_1,<)$ can be separated by a Borel set.
- (Halko, Mekler, Shelah, V.)
  - CUB is not Borel, but ``CUB is analytic co-analytic" is independent of ZFC+CH, as is ``the orbit of the free group of $\aleph_1$ generators is analytic co-analytic".

Definable trees and/or models?

- (J. Steel) Assuming large cardinals,
  - If T $\subseteq R^{<\omega_1}$ is in L(R), then ``T has an uncountable branch'' is forcing absolute.
  - If M and N are in L(R) and their universe is ω_1, then M $\cong$ N is absolute with respect to forcing that preserves ω_1.

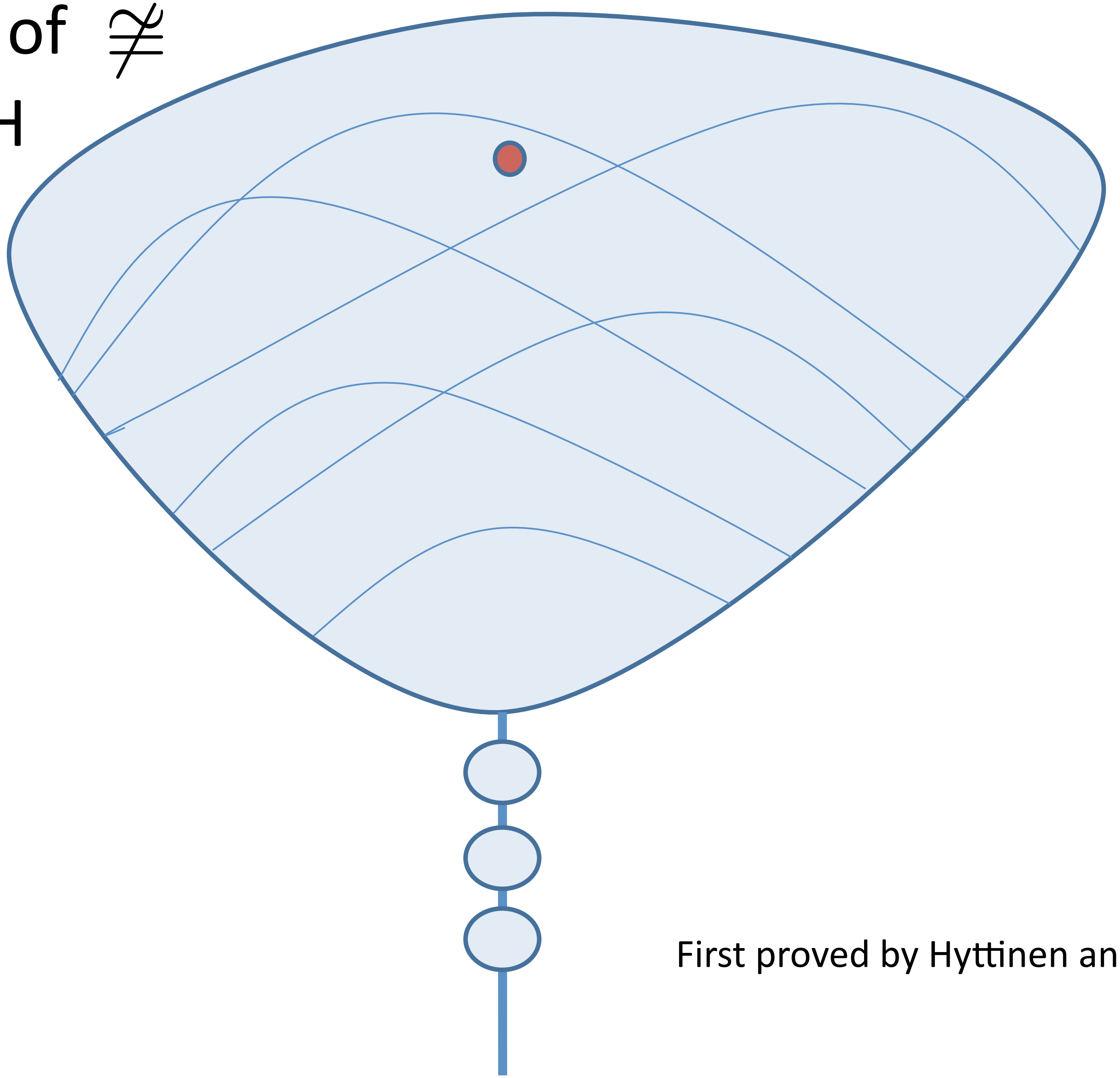
The analogy

Ordinals	Trees
No descending chains	No uncountable branches
Finite	Countable
Successor ordinal	The tree of all chains of a tree
Game clock	Clock tree
Comparison of ordinals	Order-preseving mappings
Undefinability of well-order	Undefinability of having an uncountable branch
Baire space ω^ω	Generalized Baire space $\omega_1^{\omega_1}$
Analytic union of countable ordinals is countable	Analytic union of trees with no uncountable branches is a tree with no uncountable branches



Number of models
2^κ
1
1
κ
Size of the model

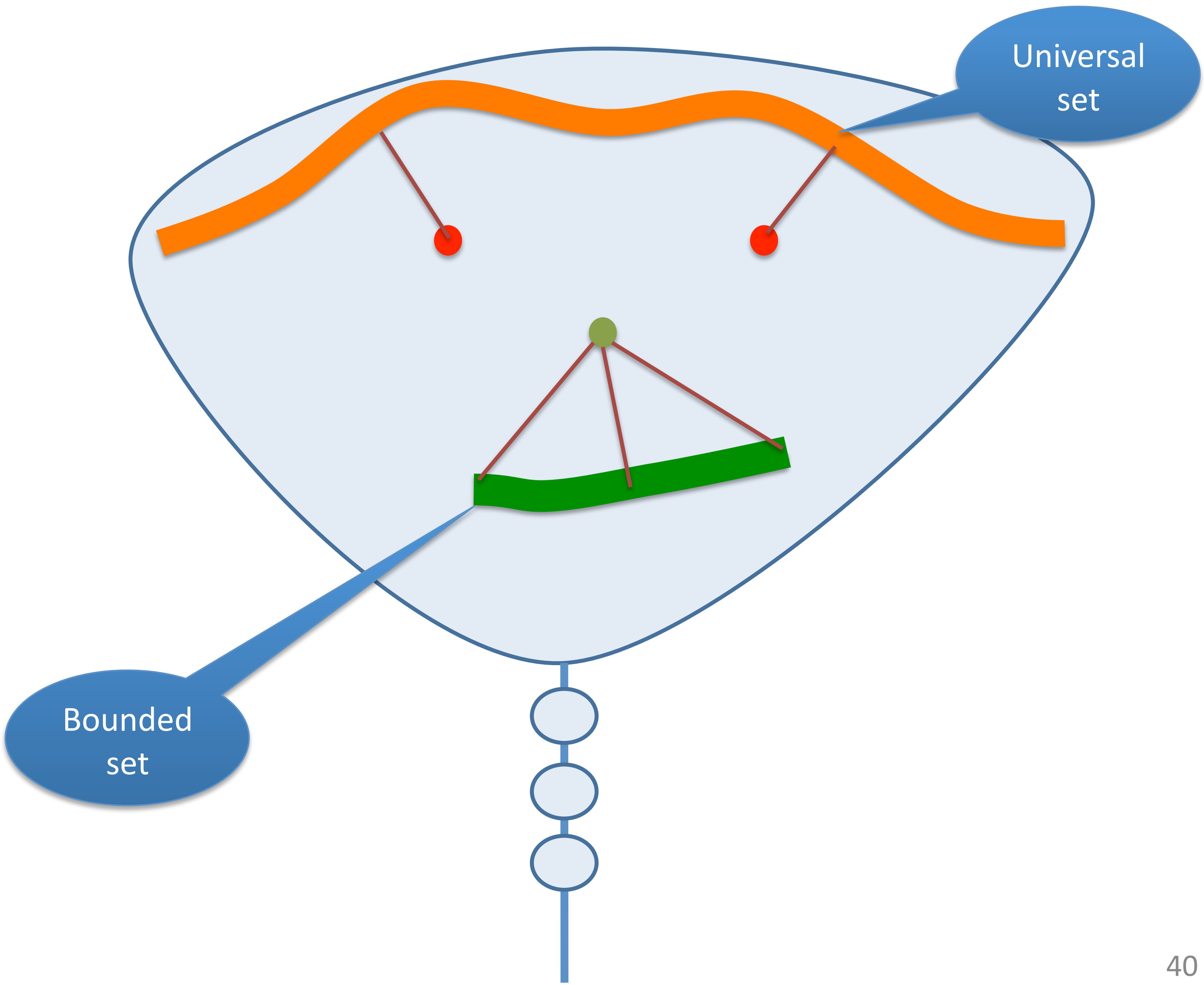
Degrees of $\not\cong$ under CH



First proved by Hyttinen and Tuuri

Cardinal invariants about trees

- U(κ) **Universality Property**: There is a family of size κ of trees of size and height $\aleph_1$ w/o branches of length ω_1 such that every such tree is ≤ one in the family.
- B(κ) **Boundedness Property**: Every family of size < κ of trees of size and height $\aleph_1$ w/o branches of length ω_1 has a tree which is ≥ each one in the family.
- C(κ) **Covering Property**: Every co-analytic subset A of $\omega_1^{\omega_1}$ is covered by κ analytic sets, such that every analytic subset of A is covered by one of them.


Universal set
Bounded set

Cardinal invariants about trees

- $U(\kappa)$ **Universality Property**
- $B(\kappa)$ **Boundedness Property**
- $C(\kappa)$ **Covering Property**
- $(U(\kappa)\&B(\lambda)) \rightarrow C(\kappa)\&\lambda\leq\kappa$, $(B(\kappa)\&\lambda<\kappa) \rightarrow \neg C(\lambda)$
- $U(\kappa)$ & $B(\kappa)$ is consistent with κ anything between $\aleph_2$ and $2^{\aleph_1}$. (Mekler-V. 1993)
- $U(\kappa^+)$ & $B(\kappa^+)$ if $\aleph_1$ replaced by a singular strong limit, of cof ω. (Dzamonja-V. 2008)

A recent result of Shelah

- There are structures M and N such that
  - The cardinality of M and N is $\aleph_1$.
  - For all $\alpha<\omega_1$, the isomorphism player wins the EF game of length α.
  - M and N are non-isomorphic.
- Note: CH not assumed.

Summary

- In the non-structure case we can get models that are very close to being isomorphic in the sense that
  - the non-isomorphism player does not win even if he is given a large clock tree.
  - the isomorphism player wins in large clock trees.
- We need to understand the structure of trees better.

Thank you!