Homeomorphic Sobolev extensions of parametrizations of Jordan curves

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Let $\varphi\colon \partial\mathbb{D} \to \mathbb{C}$ be a homeomorphism. Then $\varphi(\partial\mathbb{D})$ bounds an interior Jordan domain Ω and we know that φ extends to a homeomorphism from $\overline{\mathbb{D}}$ to $\overline{\Omega}$, by the Jordan-Schoenflies theorem.

Question

What is the optimal regularity of such extension from $\overline{\mathbb{D}}$ to $\overline{\Omega}$?

Quasi-connection

Jordan curves need not be quasicircles. Are they images of $\partial\mathbb{D}$ under global homeomorphism $f\colon \mathbb{C} \to \mathbb{C}$ of an integrable distortion

$$K(z) = \frac{\|Df\|^2}{J(z,f)} \in L^p_{\mathrm{loc}}, \qquad p < 1\,?$$

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If Ω is convex, then the complex-valued Poisson extension h of φ is a diffeomorphism, by the Radó-Kneser-Choquet theorem.

Verchota (2007): in the case $\Omega = \mathbb{D}$, derivatives of h may fail to be L^2, but they are $L^p(\mathbb{D})$ for any $p < 2$.

Iwaniec-Martin-Sbordone (2009): derivatives of h belong to weak-L^2, also sharp estimates and optimal Orlicz conditions.

Astala-Iwaniec-Martin-Onninen (2005): in the case for $\partial\Omega$ in C^1-regular: sufficient and necessary condition when derivatives of h are in L^2

Theorem (Zhang (2019))

There is a homeomorphic parametrization $\varphi\colon \partial\mathbb{D} \to \partial\Omega$ of a Jordan curve that does not admit a $W^{1,1}$-homeomorphic extension to the unit disk $\mathbb{D}$.

Parametrization φ can be chosen to be the boundary value of a conformal map.

Question

What one needs to know to extend in some Sobolev class? One can pose

- a property to φ or
- a property to a Jordan curve.

Theorem

Let $\varphi\colon \partial\mathbb{D} \to \partial\Omega$ be a homeomorphic parametrization of a Jordan curve.

- (Koski-Onninen (2021)) If the Jordan curve $\partial\Omega$ is rectifiable,
- (Koskela-Koski-Onninen (2020)) If the interior Jordan domain Ω is a John disk,

then φ has a $W^{1,p}$-homeomorphic extension to the unit disk $\mathbb{D}$ for all $1 \leq p < 2$.

Question 1.5 in Koskela-Koski-Onninen (2020)

Under which conditions on the simply connected interior Jordan domain $\Omega \subset \mathbb{C}$ does there exist a quasiconformal mapping $f\colon \mathbb{D} \xrightarrow{\text{onto}} \Omega$ in the Hölder class $C^{\alpha}(\mathbb{D}, \mathbb{C})$ with $\alpha > 1/2$?

Maybe the uniform Hölder continuity of a conformal map $\mathbb{D} \xrightarrow{\text{onto}} \Omega$ would already be enough.

From the point of view of the hyberbolic metric, a conformal map $f\colon \mathbb{D} \xrightarrow{\text{onto}} \Omega$ is uniformly Hölder continuous to some exponent $\alpha \in (0, 1]$ precisely when there is a constant C so that

$$h_\Omega(w, f(0)) \leq C \log\left(\frac{C}{d(w, \partial\Omega)}\right)$$

The condition implies that h_Ω is integrable to any power and even exponentially integrable;

$$u(w) = \exp(\epsilon h_\Omega(w, f(0)))$$

is integrable over Ω for some $\epsilon = \epsilon(C)$; note that the Minkowski dimension of $\partial\Omega$ is < 2 (Jones-Makarov (1995), Koskela-Rohde (1997), Smith-Steganga (1991)).

It would suffice to establish the extendability under the exponential integrability condition of $h_\Omega(w, f(0))$.

Much more is true.

Theorem (Bouchala-J-Koskela-Xu-Zhou)

Let $\varphi\colon \partial\mathbb{D} \to \partial\Omega$ be a homeomorphic parametrization of a Jordan curve.

Suppose that there is an exponent $q > 1$ so that $u(w) = h_{\Omega}(w, w_0) \in L^q(\Omega)$, where Ω is the interior Jordan domain, $w_0 \in \Omega$ and h_{Ω} is the hyperbolic distance in Ω.

Then φ has a $W^{1,p}$-homeomorphic extension to the unit disk $\mathbb{D}$ for all $1 \leq p < 2$.

All... or nothing

Theorem (Bouchala-J-Koskela-Xu-Zhou)

There is a homeomorphic parametrization $\varphi : \partial\mathbb{D} \to \partial\Omega$ of a Jordan curve so that φ does not have a homeomorphic $W^{1,1}$-extension, but $u(w) = h_{\Omega}(w, w_0) \in L^1(\Omega)$, for some $w_0 \in \Omega$, where Ω is the interior Jordan domain and h_{Ω} is the hyperbolic distance in Ω.

Theorem (Koski-Onninen (2023))

Suppose that Ω is a Jordan domain and $\varphi\colon \partial\mathbb{D} \to \partial\Omega$ is a boundary homeomorphism. Suppose that for some $n_0 \in \mathbb{N}$ there is a dyadic family $I = \{I_{n,j}\}$ of closed arcs in $\partial\mathbb{D}$ such that the following hold:

- For each $I_{n,j}$ with $n \geq n_0$ there exists a crosscut $\Gamma_{n,j}$ (i.e., a curve in Ω) connecting the endpoints of the boundary arc $\varphi(I_{n,j}) \subset \partial\Omega$ and such that the estimate
$$\sum_{n=n_0}^{\infty} 2^{(p-2)n} \sum_{j=1}^{2^n} \ell(\Gamma_{n,j})^p < \infty \tag{1}$$
holds. Here $\ell(\Gamma)$ is the Euclidean length of a curve Γ.
- The crosscuts $\Gamma_{n,j}$ for $n \geq n_0$ are all pairwise disjoint apart from their endpoints at the boundary

Then φ admits a homeomorphic extension from $\overline{\mathbb{D}}$ to $\overline{\Omega}$ in the class $W^{1,p}(\mathbb{D}, \mathbb{C})$.

As we need to sum up lengths of crosscuts $\Gamma_{n,j}$, the key is to have a good upper bound for the tail of the series, i.e., for croscuts corresponding "short" dyadic intervals.

Key estimate

Suppose that Ω is a Jordan domain and $f\colon \overline{\mathbb{D}} \to \overline{\Omega}$ is a homeomorphism that is conformal on $\mathbb{D}$. Let $\xi_1, \xi_2 \in \partial\mathbb{D}$ be such that $|\xi_1 - \xi_2| \leq \dfrac{4\pi}{1+\pi^2} \approx 1.156$ and γ a hyperbolic geodesic connecting ξ_1 and ξ_2. Then we have, for the crosscut $\Gamma = f(\gamma)$ joining $f(\xi_1)$ and $f(\xi_2)$, that

$$\Big(\ell(\Gamma)\Big)^2 \leq C(q) \int_\Delta \big(h_\Omega(z, f(0))\big)^q \, dz,$$

where Δ is the region bounded by Γ and the shorter-length part of the boundary connecting points $f(\xi_1)$ and $f(\xi_2)$.