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IMPROVED HÖLDER REGULARITY FOR STRONGLY ELLIPTIC PDES

JOINT WORK WITH ASTALA, CLOP, FARACO, KOSKI

$\Omega \subset \mathbb{C}$ is a domain in the plane

B. $\boxed{\text{Leray-Lions}}$

$$\operatorname{div} A(z, \nabla u) = 0$$

distributional solutions $u \in W^{1,2}_{loc}(\Omega, \mathbb{R})$, i.e.,

$$\int_\Omega \langle A(z, \nabla u), \nabla \varphi \rangle = 0$$

for all $\varphi \in C_0^\infty(\Omega, \mathbb{R})$.

ELLIPTICITY:

$$|\xi|^2 + |A(z,\xi)|^2 \leq (K + \tfrac{1}{K}) \langle A(z,\xi), \xi \rangle$$

A. $\boxed{\text{Beltrami}}$ $f \in W^{1,2}_{loc}(\Omega, \mathbb{C})$

$$f_{\bar z} = \mathcal{H}(z, f_z) \quad \text{a.e.}$$

measurable in z

uniform ellipticity: k-Lipschitz in f_z

homogeneity: $\mathcal{H}(z,0) = 0$

– f is K-qr, $K = \frac{1+k}{1-k}$,

$$|f_{\bar z}| = |\mathcal{H}(z, f_z)| = |\mathcal{H}(z, f_z) - \mathcal{H}(z,0)| \leq k|f_z| \quad \text{a.e.}$$

where $f_z = \frac{1}{2}(f_x - i f_y)$, $f_{\bar z} = \frac{1}{2}(f_x + i f_y)$

Basic result improving regularity beyond $W^{1,2}$:

Solutions (u, f) are $\frac{1}{K}$-Hölder continuous (locally) (Morrey 1938, Piccinini–Spagnolo 1972),

(or $|Df|^2 \leq K J_f(z)$)

i.e., $|f(z) - f(\zeta)| \leq C|z - \zeta|^{\frac{1}{K}}$

(This shows that qr maps solve Beltrami eq $f_{\bar z} = \mu(z) f_z$ a.e. where $|\mu| \leq k < 1$ a.e.)

D.

Connection in linear case (Leonetti-Nesi (1997)) with the same ellipticity constant

$\mu \in W^{1,2}_{loc}(\Omega, \mathbb{R})$ $\qquad\qquad$ $f \in W^{1,2}_{loc}(\Omega, \mathbb{C})$

$\operatorname{div} A(z)\nabla u = 0$ $\qquad\qquad$ $f_{\bar z} = \mu(z) f_z$

if u solves (L-L), there exists v s.t. $u+iv = f$ that solves (B). Conversively, if f solves (B), $\operatorname{Re} f$ solves (L-L).

($A(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mu \equiv 0$; $\operatorname{div}\nabla u = \Delta u = 0$

u harmonic function

there exists a harmonic conjugate of u s.t. $u + iv = f$ is analytic, i.e., $f_{\bar z} = 0$.)

1° Gradient field $E = \nabla u$ has zero curl

2° divergence-free field $B = A(z)\nabla u$ has zero divergence

Hodge star $*$ operator transforms curl-free $\longleftrightarrow$ divergence-free

$* = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = i$

Now, $*A(z)\nabla u$ is curl-free and in simply connected domains it is a gradient field, i.e. $*A(z)\nabla u = \nabla v$, (Poincaré lemma)

for some $v \in W^{1,2}_{loc}(\Omega, \mathbb{R})$

Let $f = u + iv$

$$\|Df\|_{HS} = |\nabla u|^2 + |\nabla v|^2$$

$$\leq \left(K + \frac{1}{K}\right) \langle A(z) \nabla u, \nabla u \rangle$$

$$= \left(K + \frac{1}{K}\right) \langle -* \nabla v, \nabla u \rangle$$

$$= \left(K + \frac{1}{K}\right) (u_x v_y - u_y v_x)$$

$$= \left(K + \frac{1}{K}\right) J(z, f)$$

$$\Rightarrow |f_{\bar z}| \leq k |f_z| .$$

C.

Sharp $\frac{1}{K}$- Hölder regularity (for K-elliptic equations): $z \mapsto z|z|^{1/K - 1}$ is K-qc $\left(\mu = \frac{1-K}{1+K} \frac{z}{\bar z}\right)$

real and imaginary parts solve the divergence eq

$$\operatorname{div} A(z) \nabla u = 0,$$

where

$$A(z) = \frac{1}{K} Id + \left(K - \frac{1}{K}\right) \frac{z}{|z|} \otimes \frac{z}{|z|}$$

div case
Serrin's example
(1964)

<u>Theorem</u> Solutions to the autonomous equation

$$f_{\bar{z}} = \mathcal{H}(f_z) \quad \text{a.e.}$$

belong to C^{1,α_K}_{loc} (where

$$\alpha_K = \frac{2}{K+1+\frac{K-1}{\min(2,K)}} > \frac{1}{K}.)$$

Better regularity of the equation $\Rightarrow$ improved regularity of solutions

<u>Theorem</u> (Schauder estimates) Suppose that $f \in W^{1,2}_{loc}(\Omega, \mathbb{C})$ solves

$$f_{\bar{z}} = \mathcal{H}(z, f_z) + G(z) \quad \text{a.e.},$$

where $G \in C^{\alpha}(\Omega, \mathbb{C})$ and $\mathcal{H}$ is also α-Hölder continuous with respect to z.

Then $f \in C^{1,\gamma}_{loc}(\Omega, \mathbb{C})$, whenever $\gamma \le \alpha$ and $\gamma < \alpha_K$.

the Hölder exponent (and not just α)

Does it need to depend on K?

Yes: (linear Schauder estimates give $C^{1,\alpha}_{loc}$!)

$$z \mapsto z^2 |z|^{3/(2K+1) - 1}$$

solves an autonomous equation with $\mathcal{H}(\zeta) = \frac{1-k}{3(1+k)} \frac{\bar{\zeta}^2}{\zeta}$

the ellipticity constant $k = \frac{K-1}{K+1}$

and thus $C^{1,\gamma}_{loc}$-regularity can hold only for $\gamma \leq 3/(2K+1)$. ← strictly better than α_K)

One more degree of regularity than general qr maps:

Lemma Let f be a solution to the autonomous Beltrami equation $f_{\bar z} = \mathcal{H}(f_z)$.

Then directional derivatives are K-qr.

Proof Let $|\theta| = 1$. We consider differential quotients

$$F^h(z) = \frac{f(z+h\theta) - f(z)}{h}.$$ These are K-qr, since

$$|F^h_{\bar z}| = \left| \frac{\mathcal{H}(f_z(z+h\theta)) - \mathcal{H}(f_z)}{h} \right| \leq \frac{k\,|f_z(z+h\theta) - f_z|}{|h|} = k\,|F^h_z|.$$

By Caccioppoli estimates for qr maps

$$\int_{\mathbb{D}_\rho} |D_z F^h|^2 \leq \frac{C(K)}{(R-\rho)^2} \int_{\mathbb{D}_R} |F^h - c|^2 \quad \text{for any constant } c.$$

Thus we have locally a uniform bound for $W^{1,2}$-norm of F^h: $C(K) \int_{\mathbb{D}_R} (|D_z f|^2 + 1)$

As a weak limit of F^h, $\partial_\theta f$ belongs to $W^{1,2}_{loc}$.

By quasiregularity of F^h, the limit $h \to 0$ is also qr $\square$

Corollary $f \in C^{1,\frac{1}{K}}_{loc}(\Omega, \mathbb{C})$.

Connection in nonlinear case

The Leray-Lions equation has been associated with various versions of monotonicity; a quantified version of this was studied by Kovalev (2007), who considered the δ-monotonous maps,

$$\langle A(z,\xi_1) - A(z,\xi_2) \rangle \geq \delta |A(z,\xi_1) - A(z,\xi_2)| |\xi_1 - \xi_2|$$

Recall that ellipticity is typically expressed

$$|\xi|^2 + |A(z,\xi)|^2 \leq (K + \tfrac{1}{K}) \langle A(z,\xi), \xi \rangle$$

Combining the above gives the strong ellipticity for A, that is,

linear case $A(z)\xi$ with $\det A(z) = 1$ the bounds are same for $\delta = 2K/K^2+1$. Note δ-monotonicity is scale invariant.

$$(*) \begin{cases} |\xi_1 - \xi_2|^2 + |A(z,\xi_1) - A(z,\xi_2)|^2 \\ \qquad \leq (K + \tfrac{1}{K}) \langle \xi_1 - \xi_2, A(z,\xi_1) - A(z,\xi_2) \rangle ; \\ A(z,0) = 0 \end{cases}$$

Theorem The Leray-Lions equation with $(*)$ is equivalent with the nonlinear Beltrami equation (including the exact equivalence between the ellipticity bounds!)

"Corollary" Schauder estimates & the improved Hölder regularity for Leray-Lions

Morrey's argument for the Hölder regularity

1° Morrey-Campanato characterisation of the Hölder continuity

$$\int_{\mathbb{D}_r} |Df|^2 \leq C\left(\frac{r}{R}\right)^{2\alpha+2} \int_{\mathbb{D}_R} |Df|^2$$

for all radii $r \in (0,R)$.

2° For $g(r) = \int_{\mathbb{D}_r} J(z,f)$ prove

$$g'(r) \geq \frac{2\alpha}{r}\, g(r) \quad \text{for all } r \in (0,R).$$

Fix r.

$$-\frac{i}{2}\int_{S_r} (\bar f f_z\, dz - \bar f f_{\bar z}\, d\bar z)$$

$$\int_{\mathbb{D}_r} J(z,f) \overset{\text{Green}}{=} \frac{1}{2r}\int_{S_r} \bar f (z f_z - \bar z f_{\bar z})\, |dz|$$

f as Fourier series

$$f(re^{i\theta}) = \sum_{n\in\mathbb{Z}} c_n(r) e^{in\theta}$$

thus

$$f_\theta(re^{i\theta}) = \sum_{n\in\mathbb{Z}} i n c_n(r) e^{in\theta} = i(z f_z - \bar z f_{\bar z})$$

$$\overset{\text{Parseval}}{=} \pi \sum_{n\in\mathbb{Z}} n\, |c_n(r)|^2$$

$$\leq \pi \sum_{n\in\mathbb{Z}} n^2 |c_n(r)|^2$$

$$\overset{\text{Parseval}}{=} \frac{1}{2r} \int_{S_r} |z f_z - \bar z f_{\bar z}|^2\, |dz|$$

$$\overset{K\text{-qr}}{\leq} \frac{Kr}{2} \int_{S_r} J(z,f)\, |dz|$$

$$|z f_z - \bar z f_{\bar z}|^2 \leq r^2 K (|f_z|^2 - |f_{\bar z}|^2):$$

$$\frac{|f_z|^2}{|f_z|^2} \frac{|z - \bar z \mu|^2}{(1-|\mu|^2)} \leq \frac{r^2(1+|\mu|)^2}{1-|\mu|^2} = r^2 \frac{1+|\mu|}{1-|\mu|} \leq r^2 \frac{1+k}{1-k}$$

In our case directional derivatives are K-qr and we want to do the M-C estimate for $D^2 f$ (instead of Df)

Replace $J(z,f)$ with something one can control $D^2 f$ [redacted] (In the spirit of Piccinini-Spagnolo)

$$\int_{D_r} (k J_{f_z} + J_{f_{\bar z}})$$

K-qr

not necessary $|f_{\bar z \bar z}| \le k |f_{z\bar z}|$

$$\int_{D_r} (k J_{f_z} + J_{f_{\bar z}})\, d\mathcal{L}(z)$$

$$\le \frac{C_1 r}{2 C_2} \int_{S_r} (k J_{f_z} + J_{f_{\bar z}})\, |dz|$$

K-qr directional derivatives

improvement in Fourier

$$\alpha_K \le \frac{C_2}{C_1}$$