HOMEOMORPHIC SOBOLEV EXTENSIONS OF PARAMETRIZATIONS OF JORDAN CURVES

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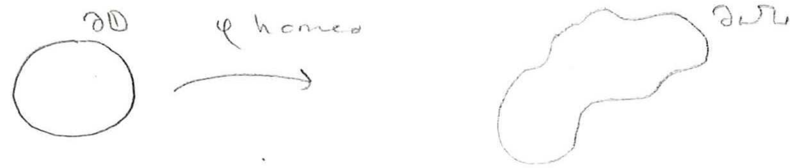


Let $\varphi : \partial\mathbb{D} \to \mathbb{C}$ be a homeomorphism. Then $\varphi(\partial\mathbb{D})$ bounds an interior Jordan domain Ω.

Jordan-Schoenflies theorem: φ extends to a homeomorphism $\overline{\mathbb{D}} \to \overline{\Omega}$

(Carathéodory: conformal map $\mathbb{D} \to \Omega$ extends continuously to a homeomorphism $\partial\mathbb{D} \to \partial\Omega$)

Question: What is the optimal regularity of such an extension $\overline{\mathbb{D}} \to \overline{\Omega}$?

Ω CONVEX

If Ω is convex, then the complex-valued Poisson extension h of φ is a diffeomorphism, by Radó-Kneser Choquet thm (harmonic)

Proc Am Math Soc

Verchota (2007): in the case $\Omega = \mathbb{D}$, derivatives of h may fail to be L^2, but they are $L^p(\mathbb{D})$ for any $p < 2$.

Discrete Contin. Dyn. Syst. (2009)

Iwaniec-Martin-Sbordone: derivatives of h belong to weak-L^2, also sharp estimate and optimal Orlicz conditions $|\{z : |h(z)| > t\}| \le C/t^2$

London Math Soc

Astala-Iwaniec-Martin-Onninen (2005): $\partial\Omega \in C^1$: sufficient and necessary conditions for $Dh \in L^2$

$$\iint_{\partial\Omega\,\partial\Omega} |\log|\varphi^{-1}(\xi) - \varphi^{-1}(\eta)|| \, |d\xi||d\eta| < \infty$$

Theorem Zhang (2019) Ann. Acad. Sci. Fenn Math

There is a homeomorphic parametrization $\varphi : \partial\mathbb{D} \to \partial\Omega$ of a Jordan curve that does not admit a $W^{1,1}$-homeomorphic extension to the unit disk $\mathbb{D}$.

Parametrization φ can be chosen to be the boundary value of a conformal map.

Question: What one needs to know to extend in some Sobolev class?

One can pose

- a property to φ or
- a property to a Jordan curve $\partial\Omega$

⊕

J Eur Math Soc

Thm (Koski - Onninen 2021) Let $\varphi: \partial\mathbb{D} \to \partial\Omega$ be a homeomorphic parametrization of a Jordan curve. If $\partial\Omega$ is rectifiable, then φ has a $W^{1,p}$-homeomorphic extension to the unit disk $\mathbb{D}$ for all $1 \le p < 2$.

JFA

Thm (Koskela - Koski - Onninen 2020) —"—

If Ω is a John disk, then —"—

quasi + ⊖

"locally bi-Lip image of quasidisk"

NE, GFT

⊕ e.g. pure displacement questions for infimum for stored energy:

Are there admissible homeos: Sob-homeos $\mathbb{D} \xrightarrow{\text{onto}} \Omega$ with boundary data φ?

Question 1.5 in (KKO 2020)
Under which conditions on the simply connected interior Jordan domain $\Omega \subset \mathbb{C}$ does there exist a quasiconformal mapping $\mathbb{D} \xrightarrow{\text{onto}} \Omega$ in the Hölder class $C^{\alpha}(\mathbb{D}, \mathbb{C})$ with $\alpha > 1/2$?

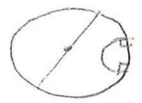
Maybe the uniform Hölder continuity of a conformal map $\mathbb{D} \xrightarrow{\text{onto}} \Omega$ would already be enough for homeo Sobolev extendability... ($C^{\varepsilon}(\partial\mathbb{D})$, $\varepsilon > 0$)

From the point of the hyperbolic metric, a conformal map $f : \mathbb{D} \xrightarrow{\text{onto}} \Omega$ is uniformly Hölder continuous to some exponent $\alpha \in (0, 1]$ precisely when there is a constant c so that

$$(*) \qquad h_{\Omega}(w, f(0)) \leq c \log\left(\frac{c}{d(w, \partial\Omega)}\right)$$

Hyperbolic Metric

GEODESICS



$$h_{\mathbb{D}}(z,w) = \inf_{\gamma} \int_{\gamma} ds_{hyp}$$

$$ds_{hyp}(z) = \frac{2|dz|}{1-|z|^2}$$ Riemannian metric

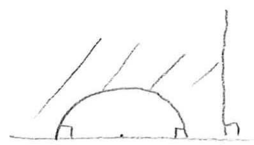
inf over rectifiable curves

$$\Rightarrow \quad h_{\mathbb{D}}(0,z) = \log \frac{1+|z|}{1-|z|}$$

and $$h_{\mathbb{D}}(z,w) = \log \frac{|1-\bar{z}w| + |z-w|}{|1-\bar{z}w| - |z-w|}$$

$z \mapsto i\,\frac{1-z}{1+z} : \mathbb{D} \to \mathbb{H}^+$ conformal isometry

GEODESICS



$$ds_{hyp} \longrightarrow \frac{|dz|}{\operatorname{Im} z}$$

hyperbolic distance is conformally invariant
(simply connected domain ... : define through conformal map)

$\mathbb{H}_+$ $\quad \int \frac{1}{\operatorname{Im} z}|dz|$ "grows at most like log"

Geometric interpretation of (*) in simply connected domains

Jones-Makarov (Annals 1995):

h_Ω integrable to any power,
even exponentially integrable, i.e.,

$$u(w) = \exp(\varepsilon\, h_\Omega(w, f(0))); \quad \mu \in L^{\varepsilon(C)}(\Omega)$$

power

(IDEA TAKE exp of (*) $\Rightarrow \left(\frac{1}{\mathrm{dist}(w, \partial\Omega)}\right)^{\text{power}}$

& prove dim - bound)

In this case Minkowski dimension of $\partial\Omega$
(Box counting $\frac{\log N(\varepsilon)}{\log(1/\varepsilon)}$)
< 2

(Jones-Makarov, Koskela-Rohde (1997) Math Ann., Smith-Stegenga (1991) Ann Acad. Sci. Fenn. Math.)

For Question 1.5 it is enough to prove homeomorphic Sobolev extendability in this class.

Much more is true

Thm (BJKXZ) Let $\varphi : \partial\mathbb{D} \to \partial\Omega$ be a homeomorphic parametrization of a Jordan curve.

Suppose that there is an exponent $q > 1$ st that $u(w) = h_\Omega(w, w_0) \in L^q(\Omega)$, where Ω is the interior Jordan domain, w_0 a base point, h_Ω is the hyperbolic distance in Ω.

Then φ has a $W^{1,p}$-homeomorphic extension to the unit disk $\mathbb{D}$ for all $1 \le p < 2$.

All or nothing

Thm (BJKXZ) There is a homeomorphic parametrization $\varphi : \partial\mathbb{D} \to \partial\Omega$ of a Jordan curve so that φ does not have a homeomorphic $W^{1,1}$-extension, but $u(w) = h_\Omega(w, w_0) \in L^1(\Omega)$.

$h_\Omega(w, w_0)$ is comparable to the quasihyperbolic distance given by the path metric with the density $d(w, \partial\Omega)^{-1}$. (We have sufficient geometric condition that gives the optimal extendability)

Note 2: regularity fails "badly": we show that there is a set of positive length on $\partial \mathbb{D}$ such that for every homeomorphic extension $\Phi : \overline{\mathbb{D}} \to \overline{\Omega}$ and any curve γ connecting an interior point to a point $z \in C$, we have that Euclidean length of image curve $\Phi(\gamma_z) = \infty$.

Note 3: Sharp criteria under L^p-integrability; refined scales, e.g., $h_{\Omega} \log^{\alpha}(e + h_{\Omega})$.

Note [illegible]: For our condition the boundary $\partial\Omega$ can have full dimension 2.

BLACK BOX

Adv Math

Theorem (KO 2023) Ω Jordan

$\varphi : \partial\mathbb{D} \to \partial\Omega$ homeo

For some $n_0 \in \mathbb{N}$ there is a dyadic family $I = \{I_{n,j}\}$ of closed arcs in $\partial\mathbb{D}$ such that

- for each $I_{n,j}$, $n \geq n_0$, there is a crosscut $\Gamma_{n,j}$ (curve in Ω) connecting end points of the boundary arc $\varphi(I_{n,j})$ and such that

$$\sum_{n=n_0}^{\infty} 2^{(p-2)n} \sum_{j=1}^{2^n} \ell(\Gamma_{n,j})^p < \infty$$

- crosscuts are pairwise disjoint (apart from the end points)

Then φ admits a homeomorphic extension from $\overline{\mathbb{D}}$ to $\overline{\Omega}$ in the class $W^{1,p}(\mathbb{D}, \mathbb{C})$.

"proof" Dyadic decomposition of $\mathbb{D}$ with $U_{n,j}$

Similarly in the target with conformal images $V_{n,j}$

$\partial V_{n,j} \approx \Gamma_{n,j}$

Lip-homeo $U_{n,j} \to V_{n,j}$

$$\iint_{\partial\mathbb{D}\,\partial\mathbb{D}} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^p} < \infty$$

USED : 1) QUASICIRCLE : p-Douglas $\Rightarrow$ homeomorphic ext

2) QUASICONVEX & PIECEWISE SMOOTH

p-Douglas $\Leftrightarrow$ homeomorphic extension with finite Dirichlet energy

SKETCH (consult the paper)

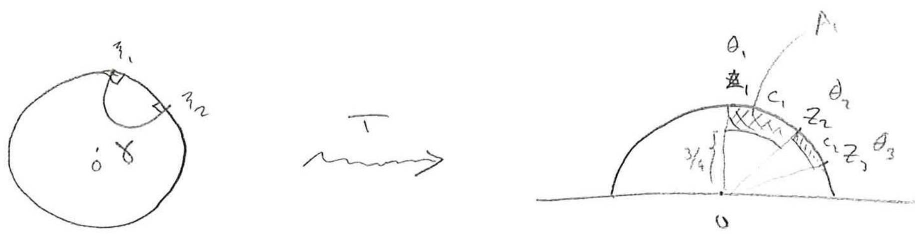
IDEA:

Key is a good upper bound for the tail of the series, i.e., "short" dyadic intervals

<u>Lemma 2.4</u> $f: \overline{\mathbb{D}} \xrightarrow{\text{homeo}} \overline{\Omega}$ (conformal on $\mathbb{D}$ Riemann mapping / universal covering Ω simply connected)

$\Gamma = f(\gamma)$, γ hyperbolic geodesic

crosscut



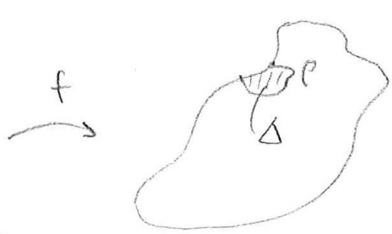
$$h_{\mathbb{H}^+}(\omega, T(0)) \geq |m| \log 2 \qquad (2.4)$$

$\vdots$

$$l(\Gamma) = C \left(\sum_{m=1}^{\infty} m^{-q} \right)^{1/2} \left(\sum_{m=1}^{\infty} \int_{B_m} (h_{\Omega}(w, f(0)))^q \, dw \right)^{1/2}$$

$$\leq C \, (\zeta(q))^{1/2} \left(\int_{\Delta} h_{\Omega}(w, f(0))^q \, dw \right)^{1/2}$$

Riemann zeta-function



Counterexample

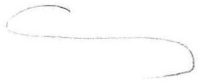
Jordan domain & boundary homeo

"Infinite tree-like fingers" based on

$\frac{1}{4}$-Cantor set

core / skeleton

$\frac{1}{2}$ - 2



$\frac{1}{4}$ - 4

$\frac{1}{8}$ - 8

fattening

+ $h_{\Omega} \in L^1$
"distance of boundary getting small fast"

Construction φ continuous & bijective
$\Rightarrow$ homeo

There is a Cantor like set C' of ~~...~~ positive length on ∂D such that radia ending there have infinite length image!