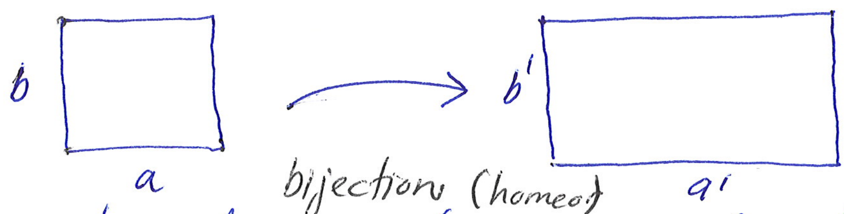
MANIFOLDS OF QC MAPS AND THE NONLINEAR BELTRAMI EQUATION

joint work w/ Astala, Clop, Faraco

1⁰ QC MAPS

Notion introduced by Grötzsch (1928)



No conformal map (preserves angles locally) that maps vertices to vertices (reflection → to $\mathbb{C}$ $\alpha z+\beta$)

What is most nearly conformal map?

One needs measure of approximate conformality:

$$K(z) = \frac{|f_z| + |f_{\bar z}|}{|f_z| - |f_{\bar z}|} \geq 1$$

(Complex notion, $z = x+iy$

formal adjoint → $f_z = \frac{1}{2}(f_x - i f_y)$

CR → $f_{\bar z} = \frac{1}{2}(f_x + i f_y)$)

affine $az + b\bar z = Az$

$A_z = a$

$A_{\bar z} = b$

Definition $f \in W^{1,2}_{loc}(\mathbb{C}, \mathbb{C})$ (A homeomorphism) is K-quasiconformal if $K(z) \leq K \in [1, \infty)$

name by Ahlfors '35

Equivalently,

$$\|Df\|^2 \leq K \det(Df)$$

$$|\partial_{\bar z} f| \leq k\, |\partial_z f|$$

$$\frac{K-1}{K+1}$$

$$\partial_{\bar z} f = \mu(z)\, \partial_z f, \quad \|\mu\|_\infty \leq k < 1$$

measurable

classical Beltrami equation

Solution to G: most nearly conformal if $\sup K(z)$ is as small as possible; $f(z) = \frac{1}{2}\left(\frac{a'}{a} + \frac{b'}{b}\right) z + \frac{1}{2}\left(\frac{a'}{a} - \frac{b'}{b}\right) \bar z$

Example: $z|z|^{K-1}$, $z|z|^{1/K-1}$

Measurable Riemann mapping theorem (Morrey 1938)

(classical Riemann mapping theorem: simply connected open $\subsetneq \mathbb{C}$ $\xrightarrow{\text{conformal bijection (homeo?)}}$ $\mathbb{D}$)

There is a unique homeomorphic $W^{1,2}_{loc}(\mathbb{C}, \mathbb{C})$-solution to classical Beltrami equation with $0 \mapsto 0$, $1 \mapsto 1$.

measurably set beforehand the eccentricity and angle of infinitesimal ellipses

QS

Why QC? Superficial: natural generalisation of conformal maps

wide-ranging applications: elliptic PDEs, dynamics, holomorphic motion, fluid mechanics

Closely related QUASISYMMETRY (global behaviour)

f is qs if

$$\frac{|f(z)-f(w_1)|}{|f(z)-f(w_2)|} \leq \eta\left(\frac{|z-w_1|}{|z-w_2|}\right)$$

increasing homeomorphism $(0,\infty) \to (0,\infty)$

Global qc $\Leftrightarrow$ qs $\quad \eta = C(n,K)\max\{t^K, t^{1/K}\}$

2° FAMILIES OF QC MAPS

$$\boxed{\partial_{\bar z} f = \mu(z)\, \partial_z f} \quad (*)$$

$\exists!\ \varphi_a : \mathbb{C} \to \mathbb{C}$ K-qc s.t.

φ_a solves $(*)$ and $\varphi_a(0)=0$, $\varphi_a(1)=a$.

Namely, $a\varphi_1$

exists and is unique by measurable Riemann mapping theorem

μ generates $\{\varphi_a\}_{a\in\mathbb{C}}$ (family of qc maps)

$\varphi_0 \equiv 0$; φ_a K-qc $0\mapsto 0$, $1\mapsto a$, stable under $\mathbb{C}$-linear combinations

Is $\mathbb{C}$-linearity important? No.

$\exists!\ \varphi_a$ s.t.

$$\partial_{\bar z} \varphi_a = \mu(z)\, \partial_z \varphi_a + \nu(z)\, \overline{\partial_z \varphi_a}$$

$$|\mu| + |\nu| \leq k < 1$$

and $\varphi_a(0) = 0$, $\varphi_a(1) = a$ $(K\text{-qc})$.

(uniqueness in Astala-Iwaniec-Martin 2009)

(G-compactness questions (Giannetti, Iwaniec, Kovalev, Moscariello, Sbordone 2003))

μ, ν generate $\mathbb{R}$-linear family

$$\{\varphi_a\}_{a \in \mathbb{C}} = \{\alpha \varphi_1 + \beta \varphi_i\}_{\alpha, \beta \in \mathbb{R}}$$

What about nonlinear Beltrami equation?

$$\boxed{\partial_{\bar z} f = \mathcal{H}(z, \partial_z f)}$$

$\mathcal{H}(z, 0) = 0$

(Bojarski, Iwaniec 70s)

$z \mapsto \mathcal{H}(z, w)$ measurable

$w \mapsto \mathcal{H}(z, w)$ k-Lipschitz $\to$ difference is K-QR (not necessary injective)

There is ~~one~~ a solution mapping $0 \mapsto 0$, $1 \mapsto a$

(Bojarski, Iwaniec 70s, Astala-Iwaniec-Martin 2009 (more general setting with measurable f-dependence))

Is it unique?

Rem. No uniqueness for general system w/ f-dependence

$\mu(z,\zeta) = \begin{cases} k \\ \frac{|\zeta - z|}{|\zeta - \bar{z}|}, & 0 \le |\zeta - z| \le k|\zeta - \bar{z}| \end{cases}$

$H(z,\zeta,w) = \mu(z,\zeta)w$

$z \mapsto t\bar{z} + (1-t)z$
$0 < t \le \frac{ik}{1+k}$

Yes, if

$$\limsup_{|z|\to\infty} k(z) < 3 - 2\sqrt{2} = 0.17157\ldots .$$

Moreover, the bound is sharp (Astala, Clop, Faraco, J, Székelyhidi) 2012

We have a nonlinear family of qc maps generated by $\mathcal{H}$ (that has the (above) uniqueness property)

$F_{\mathcal{H}} = \{ \varphi_a \}_{a \in \mathbb{C}}$

$\varphi_a(0) = 0$, $\varphi_a(1) = a$, φ_a K qc, $a \neq 0$

$\varphi_0 \equiv 0$, and

$\varphi_a - \varphi_b$ is K-qc $(a \neq b)$!

(by the uniqueness) homeomorphism

$z_0 \mapsto w_0$
$z_1 \mapsto w_1$

$\longrightarrow$

3° MANIFOLD STRUCTURE

Thm (ACFJ) Let $\mathcal{H} \in C^1(\omega)$. Then $\mathcal{F}_{\mathcal{H}}$ is a C^1-embedded submanifold of $W^{1,2}_{loc}(\mathbb{C}, \mathbb{C})$.

Notes

- $W^{1,2}_{loc}(\mathbb{C}, \mathbb{C})$ Fréchet space
  (seminorms $\|\cdot\|_{W^{1,2}(\mathbb{D}_R, \mathbb{C})}$; metric $\sum_{R=0}^{\infty} \frac{\|x-y\|_R}{1+\|x-y\|_R}$)
- Fréchet ~~dimensional~~ manifolds (infinite dimensional)
  $\{U_\alpha\}$ open cover "look like Fréchet space"
  $\Phi_\alpha : U_\alpha \to F_\alpha$ homeos s.t.
  transitions $\Phi_\alpha \circ \Phi_\beta^{-1} \Big|_{\Phi_\beta(U_\beta \cap U_\alpha)}$ are smooth
- C^1-embedded submanifold "submanifold topology = subspace topology"
  $\mathbb{C} \to W^{1,2}_{loc}(\mathbb{C}, \mathbb{C})$
  $a \mapsto \varphi_a$ ← C^1?
  image of $\mathbb{C}$ is $\mathcal{F}_{\mathcal{H}}$ | topological embedding?
  immersion?

Topological embedding

- homeomorphism

$$\mathbb{C} \longrightarrow \mathcal{F}_{\mathcal{H}} \subset W^{1,2}_{loc}(\mathbb{C}, \mathbb{C})$$

$$\left(\frac{1}{C}|a-b| \leq \| \quad \|_{W^{1,2}(\mathbb{D}_R)} \leq C|a-b| \right)$$

$L^{\infty}_{loc}(\mathbb{C}, \mathbb{C})$ - case

$g = \varphi_a - \varphi_b : \mathbb{C} \to \mathbb{C}$ K-qc

$$|\varphi_a(z) - \varphi_b(z)| = |g(z) - g(0)| \overset{qs}{\leq} \eta_k(|z|)\, |g(1) - g(0)|$$

$$\vee\!\!/ \qquad\qquad = \eta_k(|z|)\, |a-b|$$

$$\frac{1}{\eta_k(\frac{1}{|z|})} |a-b|$$

$\overset{qc}{\Longrightarrow}$ L^p-norms of derivatives ~~...~~ $W^{1,2}_{loc}(\mathbb{C}, \mathbb{C})$ - case

Immersion

- ~~...~~ the differential $D_a\varphi_a$ is injective

and it splits

$$W^{1,2}_{loc}(\mathbb{C}) \cong \left(T_a\mathcal{F}_{\mathcal{H}} \oplus (\mathbb{I} - P)(W^{1,2}_{loc}(\mathbb{C})) \right)$$

finite dimensional (2-dim)

projection $W^{1,2}_{loc} \to T_a\mathcal{F}_{\mathcal{H}}$

Hahn-Banach (Frechet space)

$$a \mapsto \varphi_a \in C^1(\mathbb{C}, W^{1,2}_{loc}(\mathbb{C},\mathbb{C}))$$

1)

$$\partial_e^a \varphi_a(z) = \lim_{t \to 0^+} \frac{\varphi_{a+te}(z) - \varphi_a(z)}{t}, \quad e \in \mathbb{C}$$

exists; and in z they are qc solving $\mathbb{R}$-linear Beltrami equation

$$\partial_{\bar z} f = \underbrace{\mu_a(z)}_{\partial_w \mathcal{H}(z, \partial_z \varphi_a(z))} \partial_z f(z) + \underbrace{\nu_a(z)}_{\partial_{\bar w} \mathcal{H}(z, \partial_z \varphi_a(z))} \overline{\partial_z f(z)}$$

2) $\partial_e^a \varphi_a$ is the e-directional derivative of

$$a \mapsto \varphi_a : \mathbb{C} \to W^{1,2}_{loc}(\mathbb{C}, \mathbb{C})$$

3) the continuity of $D_a \varphi_a$

ideas of the proofs.

1) a) converging subsequence for

$$\eta_t^e = \frac{\varphi_{a+te} - \varphi_a}{t}$$

$\uparrow$ K-qc, $0 \mapsto 0$, $1 \mapsto e$

$\Rightarrow \{\eta_t^e : t \in (0,\infty)\}$ a normal family

(qs) Montel-type thm.

$\Rightarrow \eta_{t_j}^e \to \eta^e$ locally uniformly

$\uparrow$ K-qc, $0 \mapsto 0$, $1 \mapsto e$

b) η^e solves $\mathbb{R}$-linear equation

[error term converges to 0 distributionally]

c) Another converging subsequence: limit solves the same $\mathbb{R}$-linear equation

$\Rightarrow$ the limit is the same

($\mathbb{R}$-linear equation has the unique normalized solution)

d) All converging subsequence have the same limit, thus $\eta_t^e \to \eta^e$ $t \to 0^+$.

2) L^2-theory of Beltrami operators (the invertability)

+ the error converges in L^2 to 0, too.

3) L^2-Convergence of the μ_{b_j}, ν_{b_j} and

$f = \lim_{b_j \to a} \partial^e_{b_j} \varphi_a$ solves the same $\mathbb{R}$-linear equation