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NONLINEAR BELTRAMI EQUATIONS

I: Families of quasiconformal maps

joint works with

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homeomorphic solution		$f_{\bar z} = \mu f_z$ $z \mapsto \mu(z)$ measurable $\lvert\mu\rvert \le k < 1$ a.e.	$f_{\bar z} = \mu f_z + \nu \overline{f_z}$ $z \mapsto \mu(z)$, $z \mapsto \nu(z)$ measurable $\lvert\mu\rvert + \lvert\nu\rvert \le k < 1$ a.e.	$f_{\bar z} = \mathcal{H}(z, f_z)$ $z \mapsto \mathcal{H}(z, \xi)$ measurable $\lvert\mathcal{H}(z,\xi_1) - \mathcal{H}(z,\xi_2)\rvert \le k\lvert\xi_1 - \xi_2\rvert$; $\mathcal{H}(z,0) \equiv 0$	$f_{\bar z} = \mathcal{H}(z, f, f_z)$ Lusin measurable k-Lip & homogeneity on f_z
Existence		YES Measurable Riemann mapping theorem, Morrey 1938	YES	YES	YES Astala-Iwaniec-Martin 2009
Uniqueness $f: \mathbb{C} \to \mathbb{C}$ $0 \mapsto 0$ $1 \mapsto 1$		YES Stoilow factorisation, Bojarski 1950s $f = \Phi \circ h$ (holomorphic $\circ$ qc)	YES Astala-Iwaniec-Martin 2009 $f = \Phi \circ h$ $\Phi_{\bar z} = \lambda(z) \, \mathrm{Im}\, \Phi_z$, h qc	if $\limsup_{\lvert z\rvert \to \infty} k(z) < 3 - 2\sqrt{2}$ ($K(z) < \sqrt{2}$), then YES Astala-Clop-Faraco-J-Székelyhidi Jr 2012 bound is sharp (NO >)	NO, no matter how small $k(z)$ is near ∞ (has unique principal solution $z + O(\frac{1}{z})$)
Structure of Solutions $\{\varphi_a\}_{a \in \mathbb{C}} = \mathcal{F}_{\mathcal{H}}$	$\varphi_a: \mathbb{C} \to \mathbb{C}$ $\varphi_a(0) = 0$ $\varphi_a(1) = a$	line in $W^{1,2}_{loc}$ $\varphi_a = a\varphi_1$	a 2-dimensional plane in $W^{1,2}_{loc}$ $\varphi_a = (\mathrm{Re}\, a)\varphi_1 + (\mathrm{Im}\, a)\varphi_i$	embedded submanifold of $W^{1,2}_{loc}$; tangent plane $T_{\varphi_a}\mathcal{F}_{\mathcal{H}}$ given by solutions to $\mathbb{R}$-linear $\mu = \mathcal{H}_z$, $\nu = \mathcal{H}_{\bar z}$ Astala-Clop-Faraco-J 2014	
C^α Hölder	$C^{1/k}_{loc}$ Morrey 1938	✓	✓	✓	✓
	Schauder estimates γ-Hölder continuous coeffients C^{γ}_{loc}	$C^{1,\gamma}_{loc}$ Ladyzhenskaya-Ural'tseva 1968	$C^{1,\gamma}_{loc}$ Ladyzhenskaya-Ural'tseva 1968	$C^{1,\alpha}_{loc}$ $\alpha = \min\{\gamma, \gamma(k)\}$ $\gamma(k) > \frac{1}{K}$ genuine dependence on the ellipticity K! Astala, Clop, Faraco, J, Koski 2015–	
Gradient L^p	$W^{1,p}_{loc}$ $\forall p < \frac{2K}{K-1}$ Astala 1994	✓	✓	✓	✓
integrability Invertability	$1+k < p < 1+\frac{1}{k}$	Astala-Iwaniec-Saksman 2001	Astala-Iwaniec-Saksman 2001	Astala-Iwaniec-Saksman 2001	
	VMO-coefficients $p \in (1,\infty)$		Koski 2011 Clop-Cruz 2011 (weighted L^p)	"plausible"	
One-to-one correspondence of $\mathcal{F}_{\mathcal{H}}$ and $\mathcal{H}$		YES	Giannetti-Iwaniec-Kovalev-Moscariello-Sbordone 2002; Bojarski-D'Onofrio-Iwaniec-Sbordone 2005; Alessandrini-Nesi; Astala-J 2009	under extra smoothness assumptions Astala-Clop-Faraco-J 2014	
Non-vanishing of the Jacobian when coefficients are Hölder		YES	Renelt 1988 (representation theorem) Bojarski-D'Onofrio-Iwaniec-Sbordone 2005 (Schauder estimates for inverse)	Astala-Clop-Faraco-J-Koski 2015	

NOTES & REMARKS

Elliptic PDEs that evolve from the first order system of Cauchy-Riemann equations for $f = u + iv : \Omega \to \mathbb{C}$, $\Omega \subset \mathbb{C}$ domain

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \quad \text{or} \quad f_{\bar z} = 0.$$

1) The $\mathbb{C}$-linear Beltrami equation

A homeomorphism $f : \Omega \to f(\Omega)$ is called quasiconformal if it is in $W^{1,2}_{loc}(\Omega, \mathbb{C})$ and solves

(*) $\quad f_{\bar z} = \mu(z) f_z \quad$ where $|\mu(z)| \leq k < 1$ a.e.

General solutions are called quasiregular maps.
Quasiregularity is equivalent with

$$|f_{\bar z}| \leq k |f_z| \quad \text{or} \quad |Df|^2 \leq K J(z, f), \quad K = \frac{1+k}{1-k}$$

- Measurable Riemann mapping theorem (Morrey 1938) Ω_1, Ω_2 simply connected domains ($\neq \mathbb{C}$) then for any μ there exists μ-qc $f : \Omega_1 \to \Omega_2$
  harmonic analysis in $\mathbb{C}$; boundedness of the Beurling transform $L^p \to L^p$ and continuity of its p-norms Bojarski, Ahlfors
- Stoïlow factorisation (Bojarski 1950s)
  Every solution to (*) can be written in the form
  $$f = \Phi \circ h,$$
  where Φ is analytic/holomorphic, h qc solution to (*)
  (qr open and discrete)

- Classical regularity of quasiregular maps
  $C^{1/K}_{loc}(\Omega)$ Morrey (1938)
  $W^{1,p}_{loc}(\Omega)$ $\quad p < \frac{2K}{K-1}$ Astala (1994)
  Note Sobolev embedding $C^{1/K} \subset W^{1, 2K/(K-1)}$

2) The $\mathbb{R}$-linear Beltrami equation (governs linear strongly elliptic 2D systems)

Take a prototype of nonlinear elliptic PDE, p-harmonic equation,

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \qquad p \in (1,\infty)$$

$u : \Omega \to \mathbb{R} \in W^{1,p}_{loc}$.

Now, complex gradient $f = u_z$ solves quasilinear Beltrami equation

$$f_{\bar z} = \left(\frac{1}{p} - \frac{1}{2}\right)\left(\frac{\bar f}{f} f_z + \frac{f}{\bar f}\overline{f_z}\right)$$

the inverse $g = f^{-1}$ solves the linear Beltrami equation

$$g_{\bar z} = \mu(z) g_z + \nu(z)\overline{g_z} \qquad |\mu(z)| + |\nu(z)| \leq k < 1.$$

- On uniqueness: if f is homeomorphic solution to $f_{\bar z} = \lambda(z)\,\mathrm{Im}\, f_z$ and $f(0)=0$, $f(1)=1$, study a flow

$$\frac{f(z) - tz}{1-t}$$ and show that they are qc.

3) The nonlinear Beltrami equation (governs nonlinear elliptic 2D systems)

$$f_{\bar z} = \mathcal{H}(z, f_z), \quad f \in W^{1,2}_{loc}$$

$z \mapsto \mathcal{H}(z,\xi)$ measurable

k-Lipschitz in the gradient variable, i.e.,

$$|\mathcal{H}(z,\xi_1) - \mathcal{H}(z,\xi_2)| \leq k|\xi_1 - \xi_2| \quad \text{a.e. } z$$

with homogeneity $\mathcal{H}(z,0) \equiv 0$

- introduced by Bojarski and Iwaniec 1974
- good existence theory even for (Astala-Iwaniec-Martin 2009)

$$f_{\bar z} = \mathcal{H}(z, f, f_z),$$ where

– $(z, \zeta, \xi) \mapsto \mathcal{H}(z, \zeta, \xi)$ is Lusin-measurable, i.e.,

$\mathcal{H}$ is continuous on $\square_j \times \bigcirc_j \times \triangle_j$ (that are compact and whose unions have full measure)

⌈guarantees that $z \mapsto \mathcal{H}(z, f, f_z)$ is measurable⌋

– k-Lipschitz + homogeneity on the gradient variable

Recent studies have revealed the fact that knowing some structure of the nonlinearity $\mathcal{H}$ and the use provides new information of the topology of solutions on the properties of solutions, e.g., (going beyond classical PDE techniques)

– oscillation of gradients, Tartar's conjecture for $\mathbb{R}^{2\times 2}$ (compactness properties of approximate solutions to differential inclusions), Faraco–Székelyhidi Jr 2008

$$f_{\bar z} = \mu(z)\, \mathrm{dist}(f_z, \Gamma)$$

– uniqueness: Let f, g be two solutions to the same nonlinear Beltrami equation $f_{\bar z} = \mathcal{H}(z, f_z)$. Then

$|[f-g]_{\bar z}| = |\mathcal{H}(z, f_z) - \mathcal{H}(z, g_z)| \leq k|[f-g]_z|$, i.e., $f-g$ qr

Suppose f, g are homeomorphic solutions fixing 0 and 1. Then $\deg(f-g) \leq K^2$, $K = \frac{1+k}{1-k}$. Indeed, by Stoilow, $f-g = \Phi(h)$, Φ holomorphic $0 \mapsto 0$, $1 \mapsto 1$, and now h normalised qc

$$|\Phi(h(z))| = |f(z)-g(z)| \le C|z|^K = C|h^{-1}(h(z))|^K \le C|h(z)|^{K^2}$$

Corollary If $K^2 < 2$, $\deg(f-g) = 1$. Thus $f = g$.

Questions: uniqueness when the gradient dependence is C^1 or even C^∞

- embedded submanifold of $W^{1,2}_{loc}$

topological embedding = homeomorphism onto its image

immersion

$D_a \varphi_a$ splits, that is, "the image is complemented" (need to find an isomorphism)

$D_a \varphi_a$ is injective map between tangent spaces

Questions: Is there a Frobenius theorem?
Curvature of $F_{\mathcal{U}}$ with respect to curvature of $\mathcal{U}$?
Linear to nonlinear through an exponential map?

- regularity

One-to-one correspondence is the gate to G-compactness of Beltrami operators (f^j weakly in $W^{1,2}_{loc}$ & $\mathcal{L}^j f^j$ strongly in L^2_{loc} $\Rightarrow$ $\mathcal{L}^j \xrightarrow{G} \mathcal{L}$)