Nonlinear Beltrami equations: Families of quasiconformal maps

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based on joint works with
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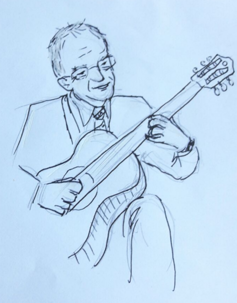
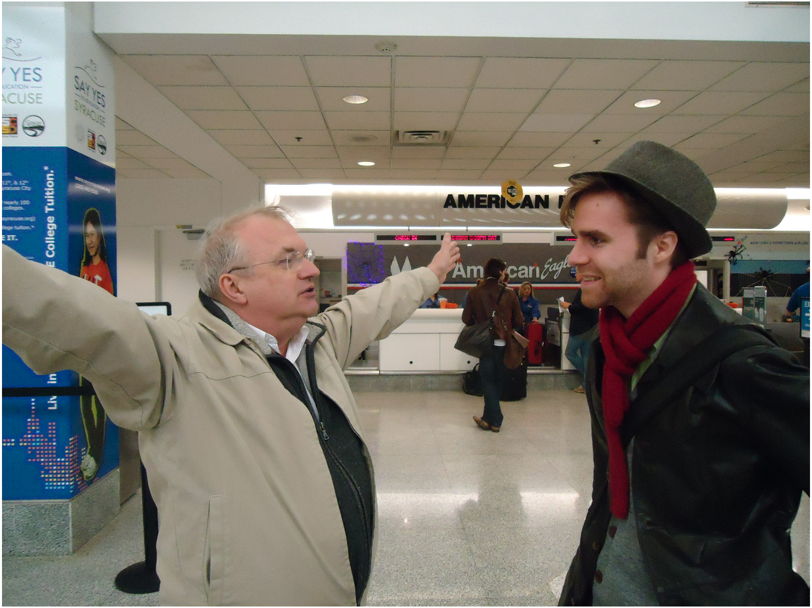
18th July 2017, Będlewo

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The domain of definition for solutions $f : \Omega \to \mathbb{C}$ is Sobolev space $\mathscr{W}^{1,2}_{\text{loc}}(\Omega, \mathbb{C})$, where $\Omega \subset \mathbb{C}$ is a domain.

Examples

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Let Γ be a curve in the complex plane. Set

$$f_{\bar z} = \mu(z)\, \mathrm{dist}(f_z, \Gamma), \qquad |\mu| \le k < 1.$$

The above Beltrami equation is a key to the solution of Tartar's conjecture by Faraco-Székelyhidi Jr. (2008).

Quasiconformal maps

Solutions to Beltrami equation $f_{\bar{z}} = \mathscr{H}(z, f_z)$ satisfy distortion inequality

$$|f_{\bar{z}}| \le k\,|f_z| \quad \text{or} \quad \|Df\|^2 \le K\,J_f, \qquad \text{where } K = \frac{1+k}{1-k},$$

since

$$|f_{\bar{z}}| = |\mathscr{H}(z, f_z) - \mathscr{H}(z, 0)| \le k\,|f_z|.$$

Thus homeomorphic $\mathscr{W}^{1,2}_{\mathrm{loc}}$-solutions are *quasiconformal*. General solutions of such an equation are called *quasiregular mappings*.

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An important result in the theory of the Beltrami equation is the *Stoïlow factorisation*: Every solution to the Beltrami equation $f_{\bar{z}} = \mu\, f_z$ may be written in the form

$$f = \Phi \circ h\,,$$

Where Φ is holomorphic and h is a quasiconformal mapping that solves the same equation.

Connection to div-type equations

A key aspect of quasiregular mappings and Beltrami equations is their strong connection to other elliptic PDEs.

There is one-to-one correspondence between

$$f_{\bar{z}} = \mathscr{H}\Big(z \; , \; f_z \Big) \qquad \text{and} \qquad \operatorname{div} \mathscr{A}(z, \nabla u) = 0$$

Here $u = \operatorname{Re} f \in \mathscr{W}^{1,2}_{\text{loc}}(\Omega, \mathbb{R})$.

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Linearity and autonomity are preserved.

Remarks

$$f_{\bar{z}} = \mathscr{H}(z, f_z)$$

Introduced by Bojarski and Iwaniec in 1970s.

L^p-theory by Astala-Iwaniec-Saksman (2001), that is, there is a solution with $Df \in L^p(\mathbb{C})$ to inhomogeneous equation

$$f_{\bar{z}} = \mathscr{H}(z, f_z) + \psi(z)$$

where $\psi \in L^p(\mathbb{C})$, $p \in \left(1 + k,\, 1 + \frac{1}{k}\right)$.

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· Uniqueness			

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There is a good existence theory even for $f_{\bar z} = \mathscr{H}(z, f, f_z)$. One needs so-called Lusin-measurability of the structure field $\mathscr{H}$, Astala-Iwaniec-Martin (2009).

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Stoïlow factorisation: Every solution to the Beltrami equation $f_{\bar{z}} = \mu f_z$ may be written in the form

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Reduction to $f_{\bar z} = \mu \, \mathrm{Im} f_z$.

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Theorem. (ACFJS, IMRN 2012)

Normalised homeomorphic $\mathscr{W}^{1,2}_{loc}(\mathbb{C},\mathbb{C})$-solutions are unique, if

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NO, above bound is sharp.

Sketch of proof

Let, f and g solve the same $\mathscr{H}$-equation, $f_{\overline{z}} = \mathscr{H}(z, f_z)$. Then

$$f_{\overline{z}} - g_{\overline{z}} = \mathscr{H}(z, f_z) - \mathscr{H}(z, g_z)$$

Hence,

$$|(f-g)_{\overline{z}}| = |\mathscr{H}(z, f_z) - \mathscr{H}(z, g_z)| \leq k\,|(f-g)_z|$$

that is, $f-g$ is quasiregular.

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Suppose f and g are homeomorphic solutions, both fixing 0 and 1. Then:

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Sharpness: Example by construction.

Counterexamples

Let

$$F_t(z) = \begin{cases} (1+t)\,z|z| - tz^2, & \text{for } |z| > 1, \\ (1+t)\,z - tz^2, & \text{for } |z| \le 1, \end{cases}$$

$$G_t(z) = \begin{cases} (1+t)\,z|z| - tz, & \text{for } |z| > 1, \\ z, & \text{for } |z| \le 1. \end{cases}$$

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Set

$$\mathscr{H}(z,0) = 0, \quad \mathscr{H}(z, \partial_z F_t(z)) := \partial_{\bar{z}} F_t(z), \quad \mathscr{H}(z, \partial_z G_t(z)) := \partial_{\bar{z}} G_t(z).$$

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Extend by Kirzbraun's theorem. Here $K \to 2$
To get $K \to \sqrt{2}$ compose F_t and G_t with

$$\phi(z) = \begin{cases} z|z|^{\frac{1}{\sqrt{2}}-1} & \text{, when } |z| > 1, \\ z & \text{, when } |z| \leq 1 \end{cases}$$

Homeomorphic $\mathscr{W}^{1,2}_{\text{loc}}(\mathbb{C},\mathbb{C})$-solutions $0 \mapsto 0$, $1 \mapsto 1$

family of solutions $\mathscr{F}_{\mathscr{H}} = \{\varphi_a : a \in \mathbb{C}\}$, where homeomorphisms φ_a satisfy $\varphi_a(0) = 0$, $\varphi_a(1) = a$, when $a \neq 0$; $\varphi_0 \equiv 0$

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If $\xi \mapsto \mathscr{H}(z,\xi)$ is $\mathcal{C}^1$, $\mathscr{F}_{\mathscr{H}}$ is a $\mathcal{C}^1$-embedded submanifold of $\mathscr{W}^{1,2}_{loc}(\mathbb{C},\mathbb{C})$.

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$$T_{\varphi_a}\mathscr{F}_{\mathscr{H}} = \mathscr{F}_{\mu_a,\nu_a},$$

where

$$\mu_a(z) = \partial_\xi \mathscr{H}\left(z, \partial_z \varphi_a(z)\right) \qquad \text{and} \qquad \nu_a(z) = \partial_{\overline{\xi}} \mathscr{H}\left(z, \partial_z \varphi_a(z)\right).$$

Homeomorphic $\mathscr{W}^{1,2}_{\mathrm{loc}}(\mathbb{C},\mathbb{C})$-solutions $0 \mapsto 0$, $1 \mapsto 1$

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Question: Is it necessary for the Hölder exponent to depend on K and not only on α?

Dziękuję!   Kiitos!   Thank you!