Nonlinear Beltrami equations: Families of quasiconformal maps

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based on joint works with Kari Astala, Albert Clop, Daniel Faraco, Aleksis Koski and László Székelyhidi Jr.

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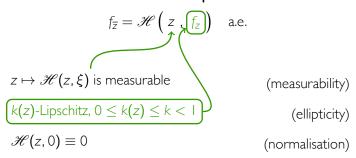
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 a.e.

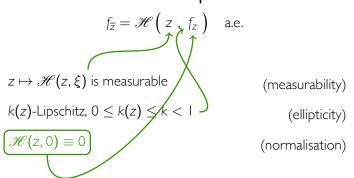
$$f_{\overline{z}} = \mathcal{H}\left(\overline{z}, f_z\right) \quad \text{a.e.}$$

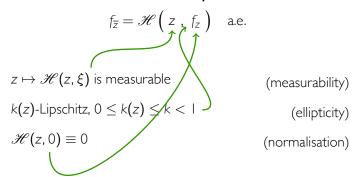
$$z \mapsto \mathcal{H}(z, \xi) \text{ is measurable} \qquad \qquad \text{(measurability)}$$

$$k(z)\text{-Lipschitz, } 0 \leq k(z) \leq k < 1 \qquad \qquad \text{(ellipticity)}$$

$$\mathcal{H}(z, 0) \equiv 0 \qquad \qquad \text{(normalisation)}$$







The domain of definition for solutions $f:\Omega\to\mathbb{C}$ is Sobolev space $\mathscr{W}_{loc}^{1,2}(\Omega,\mathbb{C})$, where $\Omega\subset\mathbb{C}$ is a domain.

Examples

C-linear Beltrami equation

$$f_{\overline{z}} = \mu(z) f_z, \qquad |\mu(z)| \le k < 1.$$

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R-linear Beltrami equation

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R-linear Beltrami equation

$$f_{\overline{z}} = \mu(z) f_z + \nu(z) \overline{f_z}, \qquad |\mu(z)| + |\nu(z)| \le k < 1.$$

Let Γ be a curve in the complex plane. Set

$$f_{\overline{z}} = \mu(z) \operatorname{dist}(f_z, \Gamma), \qquad |\mu| \le k < 1.$$

The above Beltrami equation is a key to the solution of Tartar's conjecture by Faraco-Székelyhidi |r. (2008).

Quasiconformal maps

Solutions to Beltrami equation $f_{\overline{z}} = \mathcal{H}(z, f_z)$ satisfy distortion inequality

$$|f_{\overline{z}}| \le k |f_z|$$
 or $||Df||^2 \le K J_f$, where $K = \frac{1+k}{1-k}$,

since

$$|f_{\overline{z}}| = |\mathcal{H}(z, f_z) - \mathcal{H}(z, 0)| \le k |f_z|.$$

Thus homeomorphic $\mathcal{W}_{loc}^{1,2}$ -solutions are quasiconformal. General solutions of such an equation are called quasiregular mappings.

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Thus homeomorphic $\mathcal{W}_{loc}^{1,2}$ -solutions are quasiconformal. General solutions of such an equation are called quasiregular mappings.

An important result in the theory of the Beltrami equation is the *Stoïlow* factorisation: Every solution to the Beltrami equation $f_{\overline{z}} = \mu f_z$ may be written in the form

$$f = \Phi \circ h$$
,

Where Φ is holomorphic and h is a quasiconformal mapping that solves the same equation.

A key aspect of quasiregular mappings and Beltrami equations is their strong connection to other elliptic PDEs.

There is one-to-one correspondence between

$$f_{\overline{z}} = \mathcal{H}(z, f_z)$$
 and div $\mathcal{A}(z, \nabla u) = 0$

Here $u = \operatorname{Re} f \in \mathscr{W}_{loc}^{1,2}(\Omega, \mathbb{R})$.

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 Here $u=\operatorname{Re} f\in\mathscr{W}^{1,2}_{\operatorname{loc}}(\Omega,\mathbb{R}).$ The ellipticity of \mathscr{H} , that is, k-Lipschitz

$$|\mathcal{H}(z,\xi_1) - \mathcal{H}(z,\xi_2)| \le k|\xi_1 - \xi_2|, \qquad K = \frac{1+k}{1-k}$$

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$$|\xi_1-\xi_2|^2+|\mathscr{A}(z,\xi_1)-\mathscr{A}(z,\xi_2)|^2\leq \left(K+\frac{1}{K}\right)\langle \xi_1-\xi_2,\mathscr{A}(z,\xi_1)-\mathscr{A}(z,\xi_2)\rangle\ .$$

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Linearity and autonomity are preserved.

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Remarks

$$f_{\overline{z}} = \mathcal{H}(z, f_z)$$

Introduced by Bojarski and Iwaniec in 1970s.

 L^p -theory by Astala-Iwaniec-Saksman (2001), that is, there is a solution with $Df \in L^p(\mathbb{C})$ to inhomogeneous equation

$$f_{\overline{z}} = \mathscr{H}(z, f_z) + \psi(z)$$

where
$$\psi \in L^p(\mathbb{C})$$
, $p \in \left(1 + k, 1 + \frac{1}{k}\right)$.

$$f_{\overline{z}} = \mu f_z$$

$$L \uparrow_Z$$

 $f_{\overline{z}} = \mu f_z + \nu \overline{f_z}$ $f_{\overline{z}} = \mathcal{H}(z, f_z)$

Homeomorphic $\mathscr{W}^{1,2}_{loc}(\mathbb{C},\mathbb{C})$ solutions $0 \mapsto 0$,

- $|\mapsto|$
- · Uniqueness

Existence

$$f_{\overline{z}} = \mu f_z \qquad \qquad f_{\overline{z}} = \mu f_z + \nu f_{\overline{z}} \qquad f_{\overline{z}} = \mathscr{H}(z, f_z)$$
• Existence —Yes, Morrey

 $\mathscr{W}^{1,2}_{loc}(\mathbb{C},\mathbb{C})$ solutions $0 \mapsto 0$, · Uniqueness

(1938)

Homeomorphic

 $\vdash \mapsto \vdash$

Measurable Riemann mapping theorem

Nonlinear Beltrami equations

	$f_{\overline{z}} = \mu f_z$	$f_{\overline{z}} = \mu f_z + \nu \overline{f_z}$	$f_{\overline{z}} = \mathscr{H}(z, f_z)$
· Existence	-Yes, Morrey	–Yes	-Yes

 $\mathcal{W}_{loc}^{1,2}(\mathbb{C},\mathbb{C})$ solutions $0 \mapsto 0$, $1 \mapsto 1$

Homeomorphic

· Uniqueness

Measurable Riemann mapping theorem

There is a good existence theory even for $f_{\overline{z}} = \mathcal{H}(z, f, f_z)$. One needs so-called Lusin-measurability of the structure field \mathcal{H} , Astala-Iwaniec-Martin (2009).

(1938)

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Nonlinear Beltrami equations

$$f_{\overline{z}} = \mu f_z$$
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Homeomorphic $\mathscr{W}^{1,2}_{loc}(\mathbb{C},\mathbb{C})$ solutions $0 \mapsto 0$, $I \mapsto I$

· Existence

-Yes, Morrey (1938)

-Yes

-Yes

· Uniqueness

		$f_{\overline{z}}=\muf_z$	$f_{\overline{z}} = \mu f_z + \nu \overline{f_z}$	$f_{\overline{z}} = \mathscr{H}(z, f_z)$	_
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Homeomorphic $\mathscr{W}_{loc}^{1,2}(\mathbb{C},\mathbb{C})$ -solutions $0\mapsto 0$, $l\mapsto l$

Uniqueness

 Yes, by Stoilow factorisation,
 Bojarski (1950s)

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Stoïlow factorisation: Every solution to the Beltrami equation $f_{\overline{z}}=\mu\,f_z$ may be written in the form

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	<u> </u>			
Homeomorphic	 Existence 	–Yes, Morrey	–Yes	–Yes
$\mathscr{W}^{1,2}_{loc}(\mathbb{C},\mathbb{C})$ -		(1938)		

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solutions $0 \mapsto 0$, $\vdash \mapsto \vdash$

 $f_{\overline{z}} = u_1 f_{\overline{z}}$

 $f = \Phi \circ h$. Where Φ is holomorphic and h is a quasiconformal mapping that solves the

Reduction to $f_{\overline{z}} = \mu \operatorname{Im} f_{\overline{z}}$.

 $f_{-} - \mu f_{-} + \mu \overline{f_{-}}$ $f_{-} - \mathcal{H}(z, f_{-})$

–Yes. Astala-

(2009)

Iwaniec-Martin

		$f_{\overline{z}} = \mu f_z$	$f_{\overline{z}} = \mu f_z + \nu \overline{f_z}$	$f_{\overline{z}} = \mathcal{H}(z, f_z)$
Homeomorphic $\mathscr{W}^{1,2}_{loc}(\mathbb{C},\mathbb{C})$ - solutions $0 \mapsto 0$,	• Existence	–Yes, Morrey (1938)	–Yes	–Yes
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Theorem. (ACFJS, IMRN 2012)

Normalised homeomorphic $\mathscr{W}_{\text{loc}}^{1,2}(\mathbb{C},\mathbb{C})$ -solutions are unique, if

$$\limsup_{|z|\to\infty} k(z) < 3 - 2\sqrt{2}.$$

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Question: Do nonlinear Beltrami equations always have a unique normalised homeomorphic solution?

$\mathscr{W}^{1,2}_{loc}(\mathbb{C},\mathbb{C})$ - solutions $0 \mapsto 0$,	(1938)		
$\square \mapsto \square$ Uniqueness	–Yes, by Stoïlow factorisation, Bojarski (1950s)	–Yes, Astala- Iwaniec-Martin (2009)	–Yes, when lim sup $k(z)$

 $f_{\overline{z}} = \mu f_z$

-Yes, Morrey

-Yes

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 $\limsup k(z) < 3 - 2\sqrt{2}$. $|z| \rightarrow \infty$

• Existence

Homeomorphic

homeomorphic solution?

NO, above bound is sharp.

Theorem. (ACFJS, IMRN 2012)

Question: Do nonlinear Beltrami equations always have a unique normalised

 $f_{\overline{z}} = \mu f_z + \nu \overline{f_z}$ $f_{\overline{z}} = \mathcal{H}(z, f_z)$

-Yes

 $3-2\sqrt{2}$, ACFJS

(2012)

Let, f and g solve the same \mathscr{H} -equation, $f_{\overline{z}} = \mathscr{H}(z, f_z)$. Then

$$f_{\overline{z}} - g_{\overline{z}} = \mathcal{H}(z, f_z) - \mathcal{H}(z, g_z)$$

Hence,

$$|(f-g)_{\overline{z}}| = |\mathcal{H}(z,f_z) - \mathcal{H}(z,g_z)| \le k |(f-g)_z|$$

that is, f - g is quasiregular.

Suppose f and g are homeomorphic solutions, both fixing 0 and 1. Then:

$$\deg(f-g) \le K^2$$
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$$|\Phi \circ h(z)| = |f - g| \le C|z|^K \le C|h(z)|^{K^2}$$

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$$\limsup_{|z|\to\infty} K(z) = \frac{k(z)-1}{k(z)+1} < \sqrt{2}$$
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If $K^2 < 2$, then $\deg(f - g) = 1$. Thus f = g.

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Sharpness: Example by construction.

Counterexamples

Let

$$F_t(z) = \begin{cases} (1+t)z|z| - tz^2, & \text{for } |z| > 1, \\ (1+t)z - tz^2, & \text{for } |z| \le 1, \end{cases}$$

$$G_t(z) = \begin{cases} (1+t)z|z| - tz, & \text{for } |z| > 1, \\ z, & \text{for } |z| \le 1. \end{cases}$$
Then $F_t - G_t \equiv t(z-z^2)$ and both F_t and G_t fix 0 and 1.

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Set

$$\mathscr{H}(z,0) = 0$$
, $\mathscr{H}(z,\partial_z F_t(z)) := \partial_{\overline{z}} F_t(z)$, $\mathscr{H}(z,\partial_z G_t(z)) := \partial_{\overline{z}} G_t(z)$.

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Then $F_t - G_t \equiv t(z - z^2)$ and both F_t and G_t fix 0 and 1.

Set

$$\mathscr{H}(z,0)=0, \quad \mathscr{H}(z,\partial_z F_t(z)):=\partial_{\overline{z}} F_t(z), \quad \mathscr{H}(z,\partial_z G_t(z)):=\partial_{\overline{z}} G_t(z).$$

Extend by Kirzbraun's theorem. Here $K \rightarrow 2$

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Extend by Kirzbraun's theorem. Here $K \rightarrow 2$

To get $K \to \sqrt{2}$ compose F_t and G_t with

$$\phi(z) = \begin{cases} z|z|^{\frac{1}{\sqrt{2}}-1} & \text{, when } |z| > 1, \\ z & \text{, when } |z| \le 1 \end{cases}$$

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		$f_{\overline{z}} = \mu f_z$	$f_{\overline{z}} = \mu f_z + \nu \overline{f_z}$	$f_{\overline{z}} = \mathcal{H}(z, f_z)$
Homeomorphic $\mathscr{W}^{1,2}_{loc}(\mathbb{C},\mathbb{C})$ -solutions $0 \mapsto 0$,	· Existence	–Yes, Morrey (1938)	-Yes	–Yes
family of solutions $\mathscr{F}_{\mathscr{H}} = \{ \varphi_a : a \in \mathbb{C} \}$, where	· Uniqueness	–Yes, by Stoïlow factorisation, Bojarski (1950s)	-Yes, Astala- Iwaniec-Martin (2009)	-Yes, when $\limsup_{ z \to\infty} k(z) < 3-2\sqrt{2}$, ACFJS (2012)
homeomorphisms φ_a satisfy $\varphi_a(0) = 0$, $\varphi_a(1) = a$, when	· Structure of homeomorphic solutions			
$a \neq 0; \varphi_0 \equiv 0$				

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	-a complex line in $\mathscr{W}^{1,2}_{loc}(\mathbb{C},\mathbb{C})$			

$$\mathscr{F}_{\mu} = \{a\varphi_{\perp} : a \in \mathbb{C}\}$$

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family of solutions $\mathscr{F}_{\mathscr{H}} = \{\varphi_a : a \in \mathbb{C}\}, \text{ where }$		Bojarski (1950s)	(2009)	$ z \to \infty$ 3-2 $\sqrt{2}$, ACFJS (2012)
homeomorphisms φ_a satisfy $\varphi_a(0) = 0$, $\varphi_a(1) = a$, when	· Structure of homeomorphic solutions	-a complex line in $\mathscr{W}^{1,2}_{loc}(\mathbb{C},\mathbb{C})$	-an \mathbb{R} -linear 2D-subspace of $\mathscr{W}_{loc}^{1,2}(\mathbb{C},\mathbb{C})$	
$a \neq 0; \varphi_0 \equiv 0$				
$igg(\mathscr{F}_{\mu}=\{a\pmb{arphi}_{oldsymbol{arphi}}:a$	$\in \mathbb{C}\}$			

$$\mathscr{F}_{\mu} = \{a\varphi_{1} : a \in \mathbf{C}\}$$

$$\mathscr{F}_{\mu,\nu} = \{ \mathsf{s}\, \varphi_1 + \mathsf{t}\, \varphi_i : \mathsf{s}, \mathsf{t} \in \mathbb{R} \}$$

		$f_{\overline{z}} = \mu f_z$	$f_{\overline{z}} = \mu f_z + \nu \overline{f_z}$	$f_{\overline{z}} = \mathcal{H}(z, f_z)$
Homeomorphic $\mathscr{W}^{1,2}_{loc}(\mathbb{C},\mathbb{C})$ -solutions $0\mapsto 0$,	· Existence	–Yes, Morrey (1938)	–Yes	–Yes
family of solutions	. Uniqueness	–Yes, by Stoilow factorisation, Bojarski (1950s)	–Yes, Astala- Iwaniec-Martin (2009)	-Yes, when $\limsup_{ z \to\infty} k(z) < 1$ $3-2\sqrt{2}$, ACFJS
\mathbb{C} }, where homeomorphisms φ_a satisfy $\varphi_a(0) = 0$, $\varphi_a(1) = a$, when $a \neq 0$; $\varphi_0 \equiv 0$	Structure of homeomorphic solutions	–a complex line in $\mathscr{W}^{1,2}_{loc}$ (ℂ, ℂ)	-an \mathbb{R} -linear 2D-subspace of $\mathscr{W}^{1,2}_{loc}(\mathbb{C},\mathbb{C})$	(2012)

		$_{ z}-\mu_{ z}$	$ z - \mu z + \nu z$	z-3c(z, z)
Homeomorphic $\mathcal{W}_{loc}^{1,2}(\mathbb{C},\mathbb{C})$ - solutions $0 \mapsto 0$,	• Existence	-Yes, Morrey (1938)	-Yes	-Yes
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$a \neq 0$; $\varphi_0 \equiv 0$				

 $f_{\overline{z}} = u_1 f_{\overline{z}}$

Theorem. (ACFI, I Anal Math 2017)

 $a \mapsto \varphi_a : \mathbb{C} \to \mathscr{W}^{1,2}_{loc}(\mathbb{C},\mathbb{C})$ is bi-Lipschitz. If $\xi \mapsto \mathscr{H}(z,\xi)$ is \mathcal{C}^{\perp} , $\mathscr{F}_{\mathscr{H}}$ is a \mathcal{C}^{\perp} -embedded submanifold of $\mathscr{W}_{loc}^{\perp,2}(\mathbb{C},\mathbb{C})$.

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Nonlinear Beltrami equations

 $f_{\overline{z}} = \mu_1 f_z + \nu_1 \overline{f_z}$ $f_{\overline{z}} = \mathcal{H}(z, f_z)$

What is the tangent plane of $\mathcal{F}_{\mathcal{H}}$?

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The tangent plane at $arphi_{\scriptscriptstyle d}$ is given by the solutions to an ${
m I\!\!R}$ -linear equation,

$$T_{oldsymbol{arphi}_a}\mathscr{F}_{\mathscr{H}}=\mathscr{F}_{\mu_a,
u_a},$$

where

$$\mu_{\alpha}(z) = \partial_{\varepsilon} \mathscr{H}(z, \partial_{z} \varphi_{\alpha}(z))$$

and

$$\nu_{\scriptscriptstyle G}(z) = \partial_{\overline{\xi}} \mathscr{H}(z, \partial_z \varphi_{\scriptscriptstyle G}(z))$$
.

	-	$f_{\overline{z}} = \mu f_z$	$f_{\overline{z}} = \mu f_z + \nu \overline{f_z}$	$f_{\overline{z}} = \mathcal{H}(z, f_z)$
Homeomorphic $\mathcal{W}_{loc}^{1,2}(\mathbb{C},\mathbb{C})$ -solutions $0 \mapsto 0$,	• Existence	-Yes, Morrey (1938)	-Yes	-Yes
family of solutions $\mathscr{F}_{\mathscr{H}} = \{\varphi_a : a \in \mathbb{R} \}$	· Uniqueness	–Yes, by Stoïlow factorisation, Bojarski (1950s)	–Yes, Astala- Iwaniec-Martin (2009)	-Yes, when $\limsup_{ z \to\infty} k(z) < \sup_{ z \to\infty} 3-2\sqrt{2}$, ACFJS (2012)
\mathbb{C} }, where homeomorphisms φ_a satisfy $\varphi_a(0) = 0$, $\varphi_a(1) = a$, when $a \neq 0$; $\varphi_0 \equiv 0$	Structure of homeomorphic solutions Schauder estimates	-a complex line in $\mathscr{W}^{1,2}_{loc}(\mathbb{C},\mathbb{C})$	-an \mathbb{R} -linear 2D-subspace of $\mathscr{W}_{loc}^{1,2}(\mathbb{C},\mathbb{C})$	-an embedded submanifold of $\mathscr{W}_{loc}^{1,2}(\mathbb{C},\mathbb{C})$, ACFJ (2017)
$ \begin{array}{c} \mu,\nu\in\mathscr{C}^{\alpha}_{loc}(\Omega,\mathbb{C}),\\ z\mapsto\mathscr{H}(z,\xi)\in\\ \mathscr{C}^{\alpha}_{loc}(\Omega,\mathbb{C}) \end{array} $				

		$f_{\overline{z}}=\muf_z$	$f_{\overline{z}} = \mu f_z + \nu \overline{f_z}$	$f_{\overline{z}} = \mathcal{H}(z, f_z)$
Homeomorphic $\mathscr{W}_{loc}^{1,2}(\mathbb{C},\mathbb{C})$ -solutions $0\mapsto 0$,	· Existence	–Yes, Morrey (1938)	-Yes	–Yes
family of solutions $\mathcal{F}_{\mathscr{H}} = \{ \varphi_a : a \in \mathbb{C} \}, \text{ where}$ homeomorphisms $\varphi_a \text{ satisfy}$ $\varphi_a(0) = 0,$ $\varphi_a(1) = a, \text{ when}$	· Uniqueness	–Yes, by Stoilow factorisation, Bojarski (1950s)	–Yes, Astala- lwaniec-Martin (2009)	-Yes, when $\limsup_{ z \to\infty} k(z) < 1$ ACFJS (2012)
	· Structure of homeomorphic solutions	-a complex line in $\mathscr{W}_{loc}^{1,2}(\mathbb{C},\mathbb{C})$	-an \mathbb{R} -linear 2D-subspace of $\mathcal{W}_{loc}^{1,2}(\mathbb{C},\mathbb{C})$	-an embedded submanifold of $\mathcal{W}_{loc}^{1,2}(\mathbb{C},\mathbb{C})$, ACFJ (2017)
$\varphi_{a}(1) = 0, \text{ when}$ $a \neq 0; \varphi_{0} \equiv 0$ $\mu, \nu \in \mathscr{C}^{\alpha}_{loc}(\Omega, \mathbb{C}),$ $z \mapsto \mathscr{H}(z, \xi) \in$ $\mathscr{C}^{\alpha}_{loc}(\Omega, \mathbb{C})$	• Schauder estimates	$f \in \mathscr{C}^{1,\alpha}_{loc}(\Omega, \mathbb{C}),$ Ladyzhenskaya- Uralt'seva, (1968)	$f \in \mathscr{C}^{1,\alpha}_{loc}(\Omega, \mathbb{C}),$ Ladyzhenskaya- Uralt'seva, (1968)	
	/			

Nonlinear Beltrami equations

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		$f_{\overline{z}}=\muf_z$	$f_{\overline{z}} = \mu f_z + \nu \overline{f_z}$	$f_{\overline{z}} = \mathscr{H}(z, f_z)$
Homeomorphic $\mathcal{W}_{loc}^{1,2}(\mathbb{C},\mathbb{C})$ -solutions $0 \mapsto 0$,	• Existence	–Yes, Morrey (1938)	–Yes	–Yes
family of solutions $\mathscr{F}_{\mathscr{H}} = \{\varphi_a : a \in \mathbb{C}\}$, where homeomorphisms φ_a satisfy $\varphi_a(0) = 0$, $\varphi_a(1) = a$, when	. Uniqueness	–Yes, by Stoïlow factorisation, Bojarski (1950s)	–Yes, Astala- Iwaniec-Martin (2009)	-Yes, when $\limsup_{ z \to\infty} k(z) < 3-2\sqrt{2}$, ACFJS (2012)
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$a \neq 0; \varphi_0 \equiv 0$ $\mu, \nu \in \mathscr{C}^{\alpha}_{loc}(\Omega, \mathbb{C}),$ $z \mapsto \mathscr{H}(z, \xi) \in$ $\mathscr{C}^{\alpha}_{loc}(\Omega, \mathbb{C})$	• Schauder estimates	$f \in \mathcal{C}^{1,\alpha}_{loc}(\Omega, \mathbb{C}),$ Ladyzhenskaya- Uralt'seva, (1968)	$f \in \mathcal{C}^{1,\alpha}_{loc}(\Omega, \mathbb{C}),$ Ladyzhenskaya- Uralt'seva, (1968)	$f \in \mathscr{C}^{1,\gamma}_{loc}(\Omega, \mathbb{C}),$ $\gamma < min \left\{ \alpha, \frac{1}{K} \right\},$ ACFJK (Ann I H Poincare-An 2016)
Question: Is it r on α ?	necessary for the	e Hölder expone	nt to depend on	K and not only

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Dziękuję! Kiitos! Thank you!