ON THE RADÓ-KNESER-CHOQUET THEOREM

Syracuse Mathematics Colloquium
02/23/2018

$\boxed{\text{Question}}$ (Aufgabe 41), Tibor Radó 1926 (3/17)

If $V \subset \mathbb{C}$ is convex, and bounded, is the harmonic extension of a homeomorphism $\phi : \partial\mathbb{D} \to \partial V$ an injective (univalent) mapping from the unit disk $\mathbb{D}$ onto V?

(the annual report of the German Mathematical Society, Jahresbericht d. Deutschen Mathem.-Vereinigung)

The map $f = u + iv : \mathbb{D} \to \mathbb{C}$ is harmonic, if

$$\Delta u(x_1, x_2) = 0 = \Delta v(x_1, x_2).$$

Here the Laplacian $\Delta = \dfrac{\partial^2}{(\partial x_1)^2} + \dfrac{\partial^2}{(\partial x_2)^2}$.

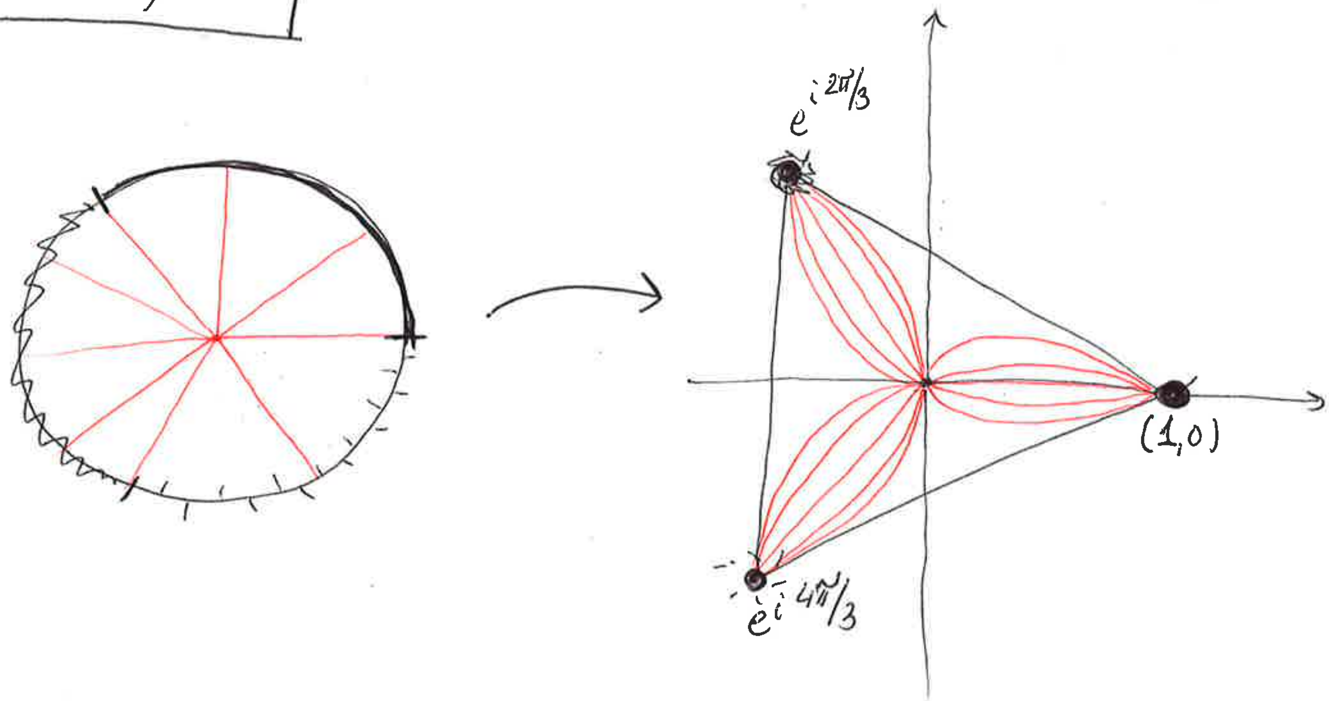
A harmonic map differs from an analytic (holomorphic) map in that its real and imaginary parts are not necessary connected by the Cauchy-Riemann equations.

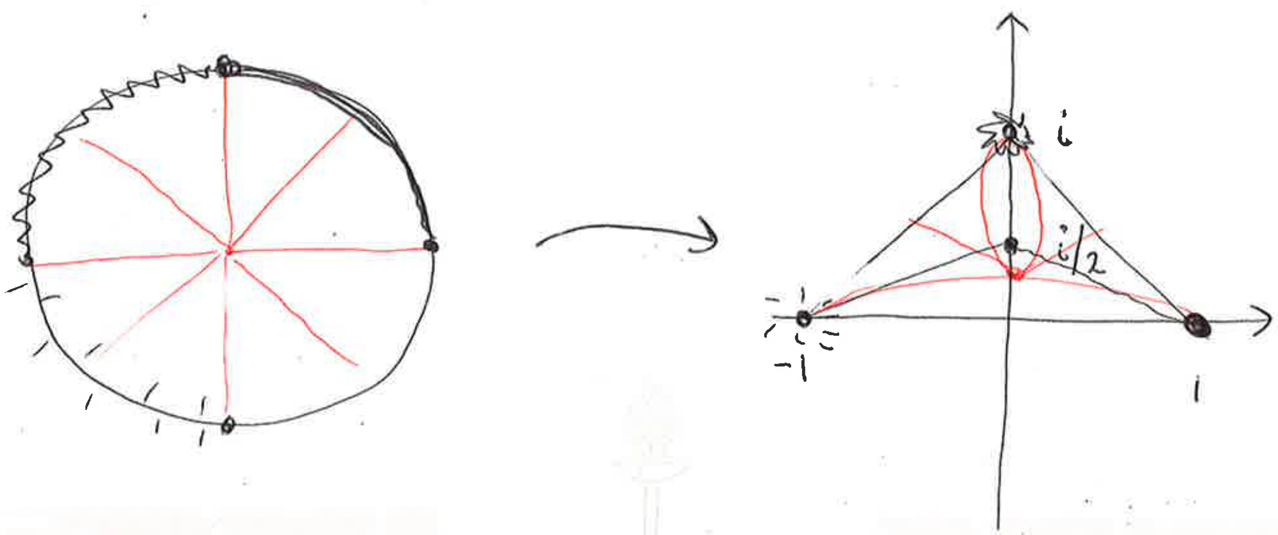
The Poisson integral formula, $z \in \mathbb{D}$,

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-|z|^2}{|e^{is}-z|^2} \, \phi(e^{is}) \, ds$$

defines the harmonic extension of a boundary map ϕ.

Examples





Each homeomorphism $\phi : \partial\mathbb{D} \to \partial V$ has the unique harmonic extension to the unit disk $\mathbb{D}$ (defined by the Poisson integral formula).

The values of this extension must lie in the closed convex hull of V in view of the averaging property of the Poisson integral.

$\boxed{\text{Theorem}}$ Radó-Kneser-Choquet

If $V \subset \mathbb{C}$ is convex, and bounded, a harmonic map $f : \mathbb{D} \to V$ which extends continuously as a monotone map homeomorphism onto the boundary is injective.
a C^∞-diffeomorphism $\mathbb{D} \xrightarrow{\text{onto}} V$

Lewy

Helmut Kneser 1926
Gustave Choquet 1945

monotone = continuous & the preimage of any point is a connected set

Remarks

- A conformal map $\mathbb{D} \xrightarrow{\text{onto}} \mathbb{D}$ is uniquely determined by the correspondence of three boundary points. RKC tells that there are many harmonic homeomorphisms with a given correspondence of three boundary points.

- RKC does not hold for nonconvex targets, however, if $f(\mathbb{D}) \subset V$, then RKC is true.
  (nonconvex versions w/ parametrization of the boundary map)

- RKC does not hold for unbounded convex domains.

- There is a version of RKC for multiply connected domains.

- RKC is not true for $\mathbb{R}^n$, $n \geq 3$.
  Richard Laugesen 1996 ($\exists\, \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ homeomorphism s.t. the Poisson extension is not injective)

  (Liu & Liao
  Melas, not diffeomorphism)

  as close to the identity in $|\cdot|$-norm as one wishes

$\boxed{\text{Notes}}$ on history & motivation

- Finding injective solutions to a given PDE has a long history. E.g., Eugenio Beltrami introduced in 1867 a complex PDE (1. order) whose homeomorphic solutions have come to be known in GFT as quasiconformal mappings.

- On the other hand the injectivity of energy minimizers / minimal deformations (which are also solutions of variational PDEs, 2. order) have come to the core of nonlinear elasticity.

Historically → next page

- RKC is the injectivity criterion for the (uncoupled) pair / system of Laplace equations.

Laplace Δ is the Euler–Lagrange equation (a variational PDE) for the Dirichlet energy (harmonic maps = critical points of the energy)

$$\int |\nabla u|^2 \, dV = \mathbb{E}(u)$$

$$\Delta u = 0 \iff 0 = \frac{d}{ds}\Big|_{s=0} \mathbb{E}(u + s\varphi)$$

first variation $\qquad C_0^\infty$

Connected to
- strings
- minimal surfaces (smallest area) (e.g. catenoid)

- Generalizations: for 1) a pair of planar elliptic equations
Bauman-Marini-Nesi 2001

2) Nonlinear systems of p-harmonic maps (degeneracy in ellipticity)
Alessandrini-Sigalotti 2001
Iwaniec-Onninen 2016 (simply connected domain!)

Coordinate maps satisfy

$$\operatorname{div} |\nabla \phi|^{p-2} \nabla \phi = 0,$$

for $\phi \in W^{1,p}_{loc}$

($p = 2 \Rightarrow$ harmonic maps / Δ)

continuous up to the boundary
agrees with

$\begin{cases} \phi \in C(\bar{\Omega}) \\ \text{homeo } \Omega \to \end{cases}$

$\oplus \int E(x, D\Phi(x))\, dV$

for $\Phi \in C^\infty$ subject to given boundary values

Two ways in NE Dirichlet problem:
1) Is the minimiser injective?
2) Minimize among homeos

Historically Calculus of Variations grew out of the task to minimize a given energy functional $\oplus$

3) p-harmonic energy, $f \in W^{1,p}$

$$\int |Df|^p \, dV, \quad 1 < p < \infty$$

has coupled (!) Euler-Lagrange system

$$\operatorname{div} |Df|^{p-2} Df = 0.$$

RKC by Iwaniec-Koski-Onninen 2016

- Nonlinear elasticity & Ball-Evans problem

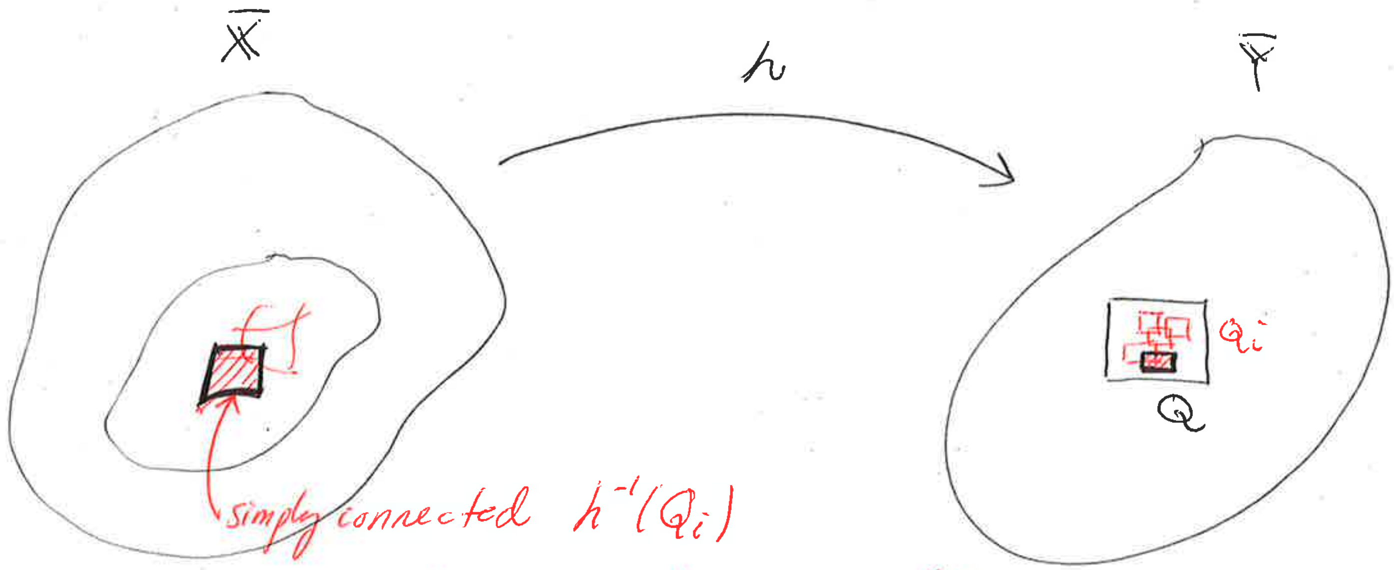
approximate $W^{1,p}$-map that is invertible by diffeomorphism in $W^{1,p}$-norm
Iwaniec-Kovalev-Onninen papers

Youngs 1948: Continuous map between compact oriented surfaces is monotone $\iff$ it is uniform limit of homeomorphism
on

In NE as weak limits of energy minimizing sequences of Sobolev ~~maps~~ homeomorphisms one gets as a limit a monotone map

Iwaniec-Onninen 2016: $W^{1,p}$ monotone maps can be approximated uniformly and strongly by homeomorphisms in $W^{1,p}$-norm.

$h : \overline{\mathbb{X}} \to \overline{\mathbb{Y}} \in W^{1,p}$ monotone



φ_i: p-harmonic homeomorphism as a replacement

Does not increase energy

$\mathbb{E}[h_j] \leq \mathbb{E}[h] \Rightarrow$ weak convergence

Lower semicontinuity & "uniform convexity" $\Rightarrow$ strong

- Riemannian surfaces (locally look like the complex plane; globally can be quite different, like a sphere or a torus)

→ harmonic maps RKC holds — nonpositive curvature on the target
Schoen-Yau 1978
Jost 1981
(minimize among "small maps")

- compact surface without boundary
- Riemannian surface with C^2 boundary, $K \leq 0$
- $N \subset N'$ geodesically convex with $C^{1,\alpha}$-boundary & small $N \subset B_r$
- Gaussian curvature $K' \geq 0$ on N

$\boxed{\text{Thm.}}$ Adamowicz - J - Koski

Let $u: \bar{M} \to N'$ be a minimizer of the p-harmonic energy with the homeomorphic boundary data $\phi_0 : \partial M \to \partial N$.
→ $C^{1,\alpha}$
→ nonvanishing tangential derivative

Assume $u \in C^{1,\alpha}(\bar{M}, N')$. Then u is in fact a $C^{1,\alpha}$-diffeomorphism from $\bar{M} \to \bar{N}$ with nonvanishing Jacobian in M.

- Euler-Lagrange equation has nonlinear term coming from the curvature (cannot use Bauman-Marini-Nesi for uniform energies $\int (\varepsilon^2 + |Df|^2)^{\frac{p}{2}}$)

Some smallness needed:

- Energy minimal maps need not be a priori small, i.e., one needs not to minimize among "small maps" ($u(M) \subset B_r$); because WE HAVE A MAXIMUM PRINCIPLE, i.e., $u(M) \subset N$.
(NEW)

Idea of the Proof(s)

Classical case: The main difficulty is to establish the local injectivity of f in $\mathbb{D}$ or equivalently $J_f \neq 0$.

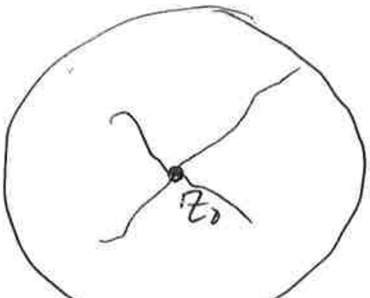
$J_f \neq 0 \Rightarrow J_f > 0$ or $J_f < 0$ as it approaches a boundary map with the property.

Argument principle $\Rightarrow$ global injectivity.

Kneser:

Local injectivity: counterassumption $J_f(z_0) = 0$
$(f = u + iv)$

$\Rightarrow \exists a, b \in \mathbb{R}$ s.t. real-valued harmonic function $\psi = au + bv$ has a critical point at z_0



Study level set $\psi(z) = c \overset{\text{def}}{=} \psi(z_0)$

Iwaniec-Onninen 2014: Homotopy argument between uniform p-harmonic & conformal one
Continuous path from the given harmonic map to a conformal one; ← positive Jacobian everywhere

along the path maps retain positive Jacobians by a so-called Minimum Principle.

← infimum is attained on the boundary of any compact set

Follows by the subharmonicity result

$\Delta \log J_f \leq 0$, when $J_f > 0$.

also for our thm for "Jacobian-type" expression (NEW)