NONLINEAR BELTRAMI EQUATIONS AND POSITIVE JACOBIANS

JOINT WORK WITH ASTALA, CLOP, FARACO, KOSKI

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the term nonlinear Beltrami equation refers to a nonlinear elliptic system of the form

$$\boxed{f_{\bar z} = \mathcal{H}(z, f_z) \qquad \text{a.e.}}$$

for $f \in W^{1,2}_{loc}(\Omega, \mathbb{C})$, where

- $z \mapsto \mathcal{H}(z, \xi)$ is measurable
- the ellipticity is coded in the k-Lipschitz property, $k<1$,

$$|\mathcal{H}(z, \xi_1) - \mathcal{H}(z, \xi_2)| \le k\,|\xi_1 - \xi_2|$$

$$\mathcal{H}(z, 0) \equiv 0$$

- introduced by Bojarski and Iwaniec in 70s

- L^p-theory Astala-Iwaniec-Saksman 2001

- Astala-Iwaniec-Martin 2009, $\mathcal{H}(z, f, f_z) = f_{\bar{z}}$ governs effectively all planar elliptic systems; the existence theory with weak assumptions (Lusin-measurability)

- the ellipticity + $W^{1,2}_{loc}$ give the distortion inequality

$$|f_{\bar{z}}| \leq k|f_z|,$$

i.e., a solution is quasiregular

- an application, e.g., in the differential inclusions (a key in the proof of Tartar's conjecture, Faraco-Székelyhidi 2008)

Theorem A (Schauder estimates)

Let $f \in W^{1,2}_{loc}(\Omega, \mathbb{C})$ be a solution to

$$f_{\bar z} = \mathcal{H}(z, f_z) \quad \text{a.e.}$$

k-Lipschitz

Then $f \in C^{1,\gamma}_{loc}(\Omega, \mathbb{C})$,

Hölder continuous, i.e.,

$$z \mapsto \mathcal{H}(z, \xi) \in C^{\alpha}_{loc}(\Omega, \mathbb{C})$$

$\gamma < \min\{\alpha, \underset{\displaystyle \frac{1}{K}}{\underset{\shortparallel}{\frac{1-k}{1+k}}}\}$

[$\gamma = \alpha$, if $\alpha < \frac{1}{K}$]

Schauder in 30s (elliptic equations of second order)

Theorem B (Positive Jacobian)

A homeomorphic solution $f \in W^{1,2}_{loc}(\Omega, \mathbb{C})$ to

$$f_{\bar z} = \mathcal{H}(z, f_z) \quad \text{a.e.}$$

Hölder continuous

k-Lipschitz;
C^1 in the ξ-variable (uniformly),
i.e.,
$\mathcal{H}_{\xi}(z, \xi)$, $\mathcal{H}_{\bar\xi}(z, \xi)$
continuous in (z, ξ)

has a positive Jacobian

$J(z, f) > 0$ everywhere.

- Linear case: $\mathcal{H}(z,\xi) = \mu(z)\xi + \nu(z)\bar{\xi}$, $|\mu|+|\nu| \leq k < 1$, $\mu, \nu \in C^1_{loc}(\Omega, \mathbb{C})$; a homeomorphic solution has $J(z,f) > 0$, the Bers-Nirenberg representation theorem or studying the inverse $g = f^{-1}$ (see below)

- $J(z,f) > 0 \Rightarrow$ local-homeomorphism

- $f(z) = z|z|^{K-1}$, $K \geq 1$ (Radial stretching)

$f_{\bar{z}} = \mu(z) f_z$, $\mu(z) = \frac{K-1}{K+1}\frac{z}{\bar{z}}$ (NOT Hölder continuous at $z=0$)

$J(z,f) = |f_z|^2 - |f_{\bar{z}}|^2 = \left|\frac{K+1}{2}|z|^{K-1}\right|^2 - \left|\frac{K-1}{2}|z|^{K-1}\frac{z}{\bar{z}}\right|^2$

thus $J(0,f) = 0$

Proof of Theorem A in the case $F_{\bar z} = \mathcal{H}(F_z)$

Claim: $F \in W^{2,p}_{loc}(\Omega, \mathbb{C}) \subset C^{1,\gamma}_{loc}(\Omega, \mathbb{C})$ $\quad (\gamma = 1 - \frac{2}{p})$

$\uparrow$ Sobolev embedding $\quad p_k > 2$

Let $U \Subset \Omega$, [redacted]

Differential quotients $\quad F^h(z) = \dfrac{F(z+h) - F(z)}{h}$

$$|F^h_{\bar z}| = \left| \frac{F_{\bar z}(z+h) - F_{\bar z}(z)}{|h|} \right| = \frac{1}{|h|} \left| \mathcal{H}(F_z(z+h)) - \mathcal{H}(F_z(z)) \right|$$

$$\leq k \, \frac{|F_z(z+h) - F_z(z)|}{|h|} = k \, |F^h_z| \quad , \quad z \in U'$$

$\uparrow$ $U \Subset U' \Subset \Omega$

Thus $F^h : U' \to \mathbb{C}$ is quasiregular.

There is $p = p(k) > 2$ s.t.

$$\int_U |D_z F^h|^p \leq C(U, U', p) \left[\int_{U'} |F^h|^2 \right]^{\frac{p}{2}}$$

(Caccioppoli-type estimate)

Hence

$$F_z, F_{\bar z} \in W^{1,p}(U)$$

$\square$ Claim

General case by freezing the coefficients.

Idea of the proof of Theorem B

Let us study the inverse $g = f^{-1}$: $f_{\bar z} = \mathcal{H}(z, f_z)$,

$$f_{\bar z} = -\frac{g_{\bar w}}{J_g}, \quad f_z = \frac{\overline{g_w}}{J_g}, \quad w = f(z)$$

$$(*) \quad -g_{\bar w} = J_g \, \mathcal{H}\Big(g, \frac{\overline{g_w}}{J_g}\Big) \quad \text{a.e.}$$

- in the linear case J_g cancels out and we can use Theorem A to get $g \in C^{1,\gamma'}_{loc}(\Omega, \mathbb{C})$ (Note that g is Hölder continuous as quasiconformal map). Thus

$$J(z,f)\, \underbrace{J(w,g)}_{\in\, C^{\gamma'}_{loc} \text{ hence locally bounded}} \equiv 1$$

$\Rightarrow J(z,f) > 0$, e.g.,

in Bojarski–D'Onofrio–Iwaniec–Sbordone 2005

We want to use similar idea in the nonlinear setting.

Set $G = \{z \in \Omega : J(z,f) = 0\}$.

Assume $G \neq \emptyset$. For each $r > 0$ we define r-neighbourhood

$$G_r = \{z \in \Omega : 0 < d(z, G) < r\}.$$

Lemma For $r > 0$ small enough

$$(**) \qquad g_{\bar{w}} = \mathcal{H}^*(g, g_w) \quad \text{in} \quad f(G_r)$$

k-Lipschitz

Hölder continuous

Now $g \in W^{1,2}_{loc}(f(\Omega), \mathbb{C})$ as quasiconformal map and g solves $(**)$ a.e. in the open

set $f(G_r) \cup f(G) = U$

of zero measure (f is qc, Lusin property)

By Theorem A, $g \in C^{1,\gamma'}_{loc}(U, \mathbb{C})$

and

$$J(z,f)\, J(w, g) \equiv 1$$

$\lightning$

About the proof of Lemma

(*) defined everywhere in $f(G_r)$

Plug $\zeta = g\bar{w}$, $\xi = gw$ in (*) and study

$$\psi(\zeta) = (|\xi|^2 - |\zeta|^2)\,\mathcal{H}\Big(g, \underbrace{\frac{\xi}{|\xi|^2 - |\zeta|^2}}_{=\tau}\Big)$$

$\zeta + \psi(\zeta) = 0$

Solve ζ as a function of ξ.

C^1-assumption is needed for the uniqueness and the regularity of $\mathcal{H}^*$.

One of the key points: for

$$\psi_\zeta(\zeta) = -\bar{\zeta}\big(\mathcal{H}(g,\tau) - \tau\mathcal{H}_\tau(g,\tau) - \bar{\tau}\mathcal{H}_{\bar{\tau}}(g,\tau)\big),$$

$$|\psi_\zeta(\zeta)| \le \varepsilon \quad \text{(uniformly)}$$