RESTORING THE COMMONS: OPTIMIZING THE RESTORATION OF NATURAL ASSETS

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Abstract. This paper analyzes the optimal restoration of natural assets, including, for example, degraded ecosystems and biodiversity, contaminated sites, and lost carbon stocks. The focus is on dynamic allocation of a fixed budget, where the decision maker chooses the timing and level of restoration investments for each stock. The decision maker is able to rely on both costly restoration investments and on natural regeneration processes in restoration planning. We characterize the properties of any optimal solution to the dynamic allocation problem, including related waiting and investment rules, and derive conditions for when it is optimal to rely on both natural restoration processes and active, but costly, restoration. We also illustrate the model by applying it to the allocation of biodiversity restoration funds in Europe.

Keywords: Biodiversity; Climate change; Conservation; Envelope theorem; Hazardous waste; Optimal timing; Restoration.

JEL codes: C61, H50, Q54, Q57.

1. Introduction

After decades of overexploitation, the restoration of Earth's ecosystems, habitats, and more generally, natural assets, is a pressing problem. The United Nations has declared the 2020s as the decade of ecosystem restoration (United Nations, 2023) and the European Commission has proposed a Biodiversity Strategy for 2030, containing the Nature Restoration Law to tackle biodiversity loss and climate change (European Commission, 2023b). Both the UN declaration and the EC strategy aim, in particular, to prevent species extinction and to increase carbon stocks by channeling effort to ecosystem restoration. While it has been estimated that the benefits of restoration far exceed the costs, the available restoration

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budgets should still be used efficiently and in the most suitable locations. These budgets are large and typically counted in billions of euros.[1]

The purpose of our paper is to study the relevant economic trade-offs in restoring multiple damaged stocks with limited public funds. The stocks can be degraded ecosystems or habitats, or lost carbon stocks or pollution stocks, and our aim is to develop a general model for the restoration of these stocks as opposed to "merely" conserving what is left, as has been done in relation to biodiversity in Weitzman (1998), Harstad and Mideksa (2017) and Harstad (2023), amongst others. We create a dynamic model that allows decision makers to integrate costly restoration investments and natural regeneration processes into their restoration planning.[2] This model features a fixed budget that can be allocated to one-time restoration investments either at the start of the planning period or delayed, with the aim of maximizing overall benefits through savings.

Policy problems and natural assets. Many natural assets are currently in need of conservation and restoration (Dasgupta, 2021). At the same time, countries face a problem with contaminated sites, which causes adverse effects on human health and damages ecosystems. Our paper and model covers multiple natural assets and the related restoration policy.

First, the model is applicable to biodiversity and ecosystem restoration and conservation planning when there are multiple stocks or sites to be restored. Human actions have caused major changes to different ecosystems, including for example forests, wetlands and coral reefs, which have had significant effects on biodiversity (IPBES, 2019). The benefits, including the various ecosystem services of healthy ecosystems, are great, but restoration and conservation are required to gain them.

Second, it is well known that the world's forests and other systems are often low-cost carbon sinks and therefore play a major role in climate change mitigation and adaptation (Pan *et al.*, 2011; IPCC, 2019). The restoration of forests, wetlands, and grasslands is important for carbon storage (Griscom *et al.*, 2017; Lewis *et al.*, 2019) and valuable for societies as cheap alternatives to other abatement investments. As is the case with biodiversity benefits, this low-cost option for carbon storage requires investments in restoration.

Third, our model can be applied to the cleanup of contaminated sites. Of particular interest is the US EPA Superfund program and the groundwater contamination caused by pollutants in Superfund sites. This type of contamination is common in these sites, and cleanup options

[1]The financing of the EU's biodiversity strategy, for example, includes annual 20 billion euros for restoration and for the Natura 2000 network (European Commission, 2020), and an additional 30 percent of climate action funds for biodiversity (European Commission, 2023a). While these budgets are large, there nevertheless exists an annual biodiversity funding gap between the current spending and funding needs of around 900 billion dollars (Barbier, 2022).

[2]Natural regeneration is called natural attenuation for pollution stocks.

include multiple technological solutions, including natural processes (U.S. Environmental Protection Agency, 2023).[3]

These sites, whether they contain damaged natural assets or pollution, share certain key common properties: typically, no responsible party exists who could be made liable for restoration without payment; investing in restoration can be very costly; and the adverse effects may dissipate due to natural regeneration even if the restoration investment is postponed. Common properties such as these are relevant to understanding the trade-offs related to the restoration of damaged stocks when the available budget is limited. The decision maker, using limited public funds, is faced with the problem of choosing which sites to restore, when, and to what extent. How, then, should limited restoration and conservation budgets be used to finance this effort? How should the budget be allocated between different assets?

Results. We answer these questions through the observation that the developed restoration model has an alternative formulation with the same optimal value and solution. The alternative involves the minimization of losses under a simple budget constraint, in which the initial restoration budget equals the total discounted restoration costs. After we show this equivalence, we use the alternative model formulation to characterize the optimal waiting and investment rules, and to find a sufficient condition for the restoration of any damaged stock.

Furthermore, we present another equivalent formulation, when the losses stemming from the stocks can be separated in the sense that a restoration investment at any of the sites only affects the losses from that site. This formulation states that the original restoration model, in which dates and investments are chosen, is equivalent to a model in which the total budget is optimally allocated between stocks given that each individual budget is used optimally. By using this formulation, we find a test result that can be used to evaluate whether a given budget allocation is optimal and derive a simplified version of the sufficient condition for restoration. We also show that, under mild conditions, one additional unit of money is equally valuable whether it is added to the total initial budget or added to the individual site's budget. Finally, we illustrate the model by applying it to the allocation of restoration funds in Europe.

[3]One is to apply a "pump and treat" option, where the contaminated groundwater is pumped out of the aquifer, treated and then released back. This operation, once started, decreases the amount of contaminants and, in time, cleans the groundwater. Another option is to apply an "in situ treatment", which means treating contaminated water without extracting it from the aquifer using, for example, chemicals, or to contain the pollutant and prevent it from contaminating drinking water supplies. (U.S. Environmental Protection Agency, 2023) All of these require costly investments. An alternative is to exploit natural attenuation processes including dilution, radioactive decay and biodegradation depending on the context at the site (U.S. Environmental Protection Agency, 2023), which allows remediation through natural processes.

Contribution to the literature. The first contribution this paper makes is to the literature on biodiversity restoration and conservation, and relatedly, on the restoration of carbon stocks. In contrast to Weitzman (1998), who formulated a theoretical framework on conservation under limited funds, we see conservation of ecosystems, habitats, and species as an essentially dynamic problem, in which the decision maker must allocate conservation funds between sites or regions. Moreover, our model's main focus is on the restoration of damaged stocks, although it can be applied, for example, to the conservation of habitats. Since Weitzman's work, the literature has, among other things, developed methods to value biodiversity (Nehring and Puppe, 2002; Brock and Xepapadeas, 2003)[4] and analyzed inefficiency in the conservation goods market (Harstad, 2016), conservation contracts (Banerjee and Shogren, 2012; Harstad and Mideksa, 2017), lobbying in a dynamic setting (Harstad, 2023), international agreements and biodiversity (e.g. Barrett, 1994; Wiszniewska-Matyszkiel and Singh, 2024) and extinctions (e.g. Mitra and Sorger, 2014; Taylor and Weder, 2023). We add to this literature by analyzing the allocation of restoration and conservation funds in a dynamic setting, when saving and natural regeneration are allowed.

There are relatively few dynamic models on restoration or conservation. An important paper is Costello and Polasky (2004), which develops a dynamic model of site selection under periodical budget constraints. In our model, the objective of the decision maker is considerably more general, and we emphasize the dynamics of the stocks, timing, and investment levels. Regarding the latter, whether restoration should rely on natural processes or more active restoration actions is currently debated and studied in relation to restoration planning (Crouzeilles *et al.*, 2017; Jones *et al.*, 2018; Reid *et al.*, 2018). By modelling the delays, and by incorporating costly restoration actions and related benefits, our results give an answer to when one should rely on active restoration and when on restoration through natural processes.

Our paper also contributes to the literature on contaminated sites and cleanups.[5] Much of this literature is empirical and has focused on estimating the benefits and costs of cleanup programs.[6] Compared to more theoretical papers on cleanups, e.g., Caputo and Wilen (1995), Keohane *et al.* (2007), Sullivan and Amacher (2009), Muehlenbachs (2015), Lyon *et al.* (2018), Lappi (2018) and LaRiviere *et al.* (2019), our paper models the dynamic allocation of a fixed budget between a given set of contaminated sites, when pollution dynamics include

[4]There also exists an empirical literature on the value of biodiversity and, in particular, on the value of specific species such as wolves (Raynor *et al.*, 2021) and vultures (Frank and Sudarshan, 2023).

[5]Examples range from abandoned and legacy mine lands, orphaned oil and gas well sites (perhaps due to failed bonding schemes, see Ho *et al.* (2018)), to sites contaminated by past industrial activity or radioactive waste (Schaffer, 2011).

[6]This part of the literature includes Gupta *et al.* (1996), Hamilton and Viscusi (1999), Greenstone and Gallagher (2008), Currie *et al.* (2011), Mastromonaco (2014), Longo and Campbell (2017) and Taylor *et al.* (2016).

natural attenuation processes. We use the model to characterize optimal waiting and cleanup investment rules and to explain when it is beneficial to wait regarding the cleanup of a site.

Furthermore, the model is connected to timing problems found in the literature on non-renewable resources, including Chakravorty and Krulce (1994), Gaudet *et al.* (2001) and Chakravorty *et al.* (2005). In particular, Gaudet *et al.* (2001) analyze the optimal exploitation of resource deposits in a spatial model, where each deposit can meet some demand after setup costs have been paid. Apart from the many obvious dissimilarities between the models, ours also includes setup costs (or fixed costs as we call them), which tend to have similar effects in the model, i.e., the cleanup or restoration date tends to be postponed due to such costs.

Finally, our paper makes a more technical contribution as well. We present a version of the envelope theorem, which differs from standard theorems that impose stringent assumptions regarding regularity and uniqueness. Additionally, these classical envelope theorems presume a convex and topological structure for the choice set, while disregarding optimal points located on the boundary. Moreover, for example, in contrast to Milgrom and Segal (2002), where a choice set may vary depending on the parameter as in our theorem, we do not assume the concavity of either the objective function or constraint functions. Our envelope theorem also deals with optimal points on the boundary, where the rate of change of the value function might not be given only by the Lagrange multiplier.

2. Model

2.1. **Assumptions.** The decision maker (e.g. an agency) is faced with the problem of minimizing the total discounted damages or losses from the stocks under the budget constraint by choosing restoration dates and investments levels. The planning interval is $[0, T]$, where T is either finite or infinite. In what follows, we take $T = \infty$, and discuss the finite planning interval separately, when needed. There are s sites with predetermined damaged stocks and the size of each stock at time t is given by $N_i(t)$, where $i = 1, \dots, s$.[7] The size of the damaged stock i at time zero is denoted by $n_{i,0}$.

The restoration operation and its effect on the damaged stock is modelled as follows. If no restoration investment is made at the site, the site's stock decreases through natural regeneration or attenuation processes. This is called here passive restoration, and, depending on the application, it can be, for example, regeneration of a habitat or decay of a pollution stock. However, the decision maker can speed up the process by investing once in restoration by exerting effort $k_i > 0$ at a restoration date $\tau_i \in [0, \infty)$. We call this active restoration.

[7]The index i refers here both to the ith site and the ith stock, and we use the words site and stock interchangeably.

Importantly, note that active restoration is modeled here as a one-time investment and not as a gradual investment process.[8]

Specifically, before the restoration investment at time τ_i the damaged stock follows the differential equation $\dot{N}_i = -h_i(N_i)$ with an initial value $n_{i,0}$, where h_i is a given site-specific function that describes the natural attenuation or regeneration of the stock i (i.e., passive restoration). After the investment, the stock decreases according to the equation $\dot{N}_i = -f_i(N_i, k_i)$, where $f_i > 0$ is a different site-specific function that describes the restoration process induced by the investment (i.e., active restoration). We will use natural regeneration (or attenuation) to mean passive restoration that occurs without any costly investment, and restoration to mean active restoration that requires investment, and thus spending of the limited budget.

The stock dynamics at each site i is then described with the following equation:

$$\dot{N}_i(t) = \begin{cases} -h_i(N_i(t)), & t \leq \tau_i, \\ -f_i(N_i(t), k_i), & t > \tau_i, \end{cases} \tag{2.1}$$

together with the initial value $N_i(0) = n_{i,0}$. The following assumption describes what happens to the stock i before and after the restoration investment:

Assumption 1. Natural regeneration and restoration processes satisfy the following conditions for all $n_i > 0$, $k_i > 0$ and for each $i = 1, \ldots, s$: functions h_i and f_i are continuously differentiable in damaged stocks and restoration investments, and

(A1) $f_i(n_i, 0) = h_i(n_i)$,
(A2) $f_i(n_i, k_i) > 0$,
(A3) $\partial f_i / \partial k_i > 0$,
(A4) $f_i(0, k_i) = 0$.

In addition, both differential equations (2.1) have global and unique solutions (e.g. h_i and f_i are globally Lipschitz continuous or satisfy Osgood's criterion).

The first assumption, (A1), implies that, without investment, the natural regeneration process described by function h_i matches the restoration process f_i. Assumption (A2) says that the restoration process decreases the stock. Assumption (A3) means that a larger investment level implies faster restoration. Assumption (A4) implies that the stock is always positive, i.e., $N_i > 0$ for all t, since solutions to the equation (2.1) cannot intersect. Assumption 1 allows for the special case of no natural regeneration (i.e., $h_i = 0$), which may hold, e.g., for heavily degraded sites, and also the case in which the stock degrades even further

[8] Many active restoration processes are approximated by one-time restoration investments, e.g. peatland restoration and contaminated site cleanup.

(i.e., $h_i < 0$).[9] In addition, it is assumed that both differential equations (with the relevant initial values) in (2.1) have global and unique solutions.[10]

The solution to the equation $\dot{N}_i(t) = -h_i(N_i(t))$ with the initial condition $n_{i,0}$ is denoted by $N_i(t; n_{i,0})$ for all $t \in [0, \tau_i)$, but the dependence on the initial value $n_{i,0}$ is often omitted to simplify the notation. After the restoration date, the stock follows equation $\dot{N}_i(t) = -f_i(N_i(t), k_i)$ with $N_i(\tau_i; n_{i,0})$ as the "initial" value, and its solution is denoted by $N_i(t; \tau_i, k_i, n_{i,0})$ for all $t \in [\tau_i, \infty)$. Here too, the dependence on $n_{i,0}$ is often suppressed from the notation. This solution is continuously differentiable in τ_i and k_i, see, e.g., Theorem 3.3 in (Hale, 1980, Chapter I). A possible evolution of the damaged stocks for a two-site case is illustrated in Figure 1.

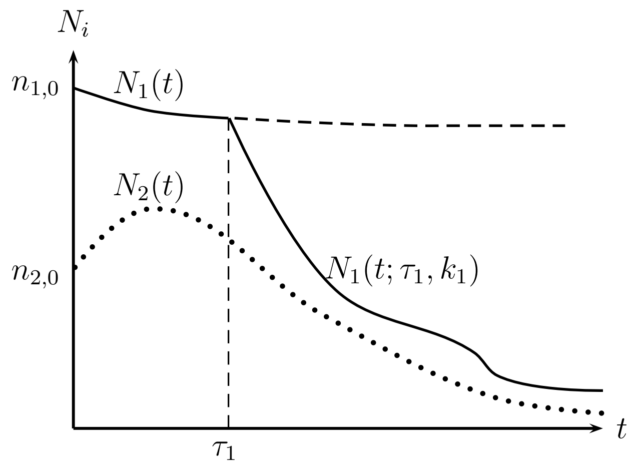

Figure 1. An illustration of the evolution of the stocks on two sites, 1 and 2. Natural regeneration rate is relatively low on site 1. Site 2 degrades initially, but benefits from the relatively high natural regeneration rate later on. The restoration investment on site 1 is made at date τ_1, and the solid curves describe the size of the stock at site 1 given this investment. The dashed curve is the stock size at site 1, if the restoration operation is postponed. The dotted curve is the stock of site 2 and this stock is not restored in this illustration.

Next we define the cost structure, the budget constraint, and the total payoff from restoration. Restoration investment at any site i is costly, and this cost is captured by the restoration cost function C_i whose value depends on the investment level k_i and on the size of the damaged stock. The value is thus given by $C_i(k_i, n_i)$. In the optimization problem we will evaluate this at the size of the damaged stock at the restoration date, i.e. at $N_i(\tau_i)$.

Assumption 2. Restoration cost function C_i satisfies the following conditions for all $n_i > 0$, $k_i > 0$ and for each $i = 1, \ldots, s$: C_i is continuously differentiable in restoration investments and damaged stocks, and

[9]This is not only relevant for natural assets, but also for assets related for example to World Heritage sites such as Venice, where the stocks (i.e., buildings and infrastructure) continue to degrade.

[10]There are many sufficient conditions for the functions h_i and f_i to guarantee global uniqueness, see, e.g., Chapter III.6 in Hartman (1982) or Constantin (2023).

(A5) $\partial C_i/\partial k_i > 0$,
(A6) $\partial^2 C_i/\partial k_i^2 \geq 0$ for all $k_i > \hat{k}_i$ for some $\hat{k}_i > 0$,
(A7) $\lim_{\tau_i \to \infty} C_i(0, N_i(\tau_i))e^{-r\tau_i} = 0$.

The inequality in assumption (A5) means that the marginal cost of investment is positive, and the inequality in assumption (A6) means that the marginal cost is increasing for sufficiently large investment levels. Hence the cost function can be, for example, first concave and then convex. These assumptions allow for fixed costs. By fixed costs we mean any costs that are realized when $k_i = 0$, i.e., the fixed costs are $C_i(0, n_i)$. Assumption (A7) guarantees that the fixed costs do not grow at too high a pace.

The restoration cost of site i is deducted from the available budget at the restoration date τ_i, and the amount of money left is the amount available for saving and for the later restoration of other sites. At the beginning of the planning period, the size of the restoration budget is b_0, and its evolution is described by the following equations:

$$\dot{B} = rB \geq 0, \quad B(0^-) = b_0 > 0, \tag{2.2}$$

$$B(\tau_i^-) - B(\tau_i^+) = C_i(k_i, N_i(\tau_i)) \quad \text{for all} \quad i, \tag{2.3}$$

where $B(\tau_i^+) := \lim_{t\to\tau_i^+} B(t)$ is the right-sided limit and $B(\tau_i^-) := \lim_{t\to\tau_i^-} B(t)$ is the left-sided limit. Equations (2.2) and (2.3) mean that the available budget drops by the restoration investment cost at a restoration date, but increases at the rate of interest between each consecutive restoration. Figure 2 illustrates a possible evolution of the restoration budget.

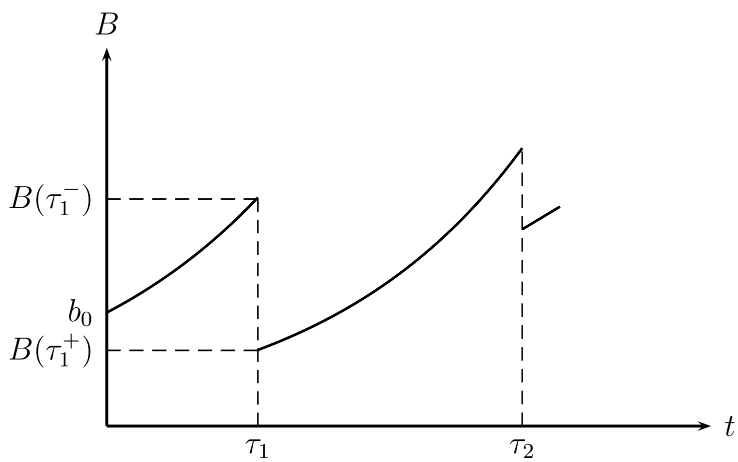

Figure 2. An illustration of the evolution of the available funds. Only two restoration dates, τ_1 and τ_2, are shown. The sizes of the jumps are $C_1(k_1, N(\tau_1))$ and $C_2(k_2, N(\tau_2))$, respectively.

To define the payoff from restoration, we assume that the decision maker has preferences over the different sizes of the damaged stocks, and that these preferences are captured by a loss function $\mathcal{W}$. As the losses can be controlled by choosing the restoration dates and investments, we write losses as a function of these variables directly. That is, the losses are $\mathcal{W}((\tau_i, k_i)_{i=1}^s)$. The next assumption includes its properties.

Assumption 3. The loss function $\mathcal{W}$ is continuously differentiable in restoration dates τ_i and investments k_i, and

(A8) $\partial\mathcal{W}/\partial\tau_i > 0$ and $\partial\mathcal{W}/\partial k_i < 0$ for all $\tau_i > 0$ and $k_i > 0$, and for each $i = 1, \dots, s$.

Hence, by (A8), the losses are strictly increasing in the date of restoration, i.e., as the restoration operation is postponed, the losses grow. The second partial derivative means that the losses decrease as the restoration investment is increased. We make no assumptions about the curvature of the loss function. The above assumption leaves open the exact dependence of the total losses on the damaged stocks. This is application specific and in the next section we give examples which are explicit about the dependence.

2.2. **Applications.** The model has multiple potential applications. To develop some of these, we first give the following result, which shows how the restoration date and the investment affect the post-restoration stock.

Lemma 2.1. *The post-restoration stock of the site i satisfies inequalities*

$$\frac{\partial N_i(t;\tau_i,k_i)}{\partial \tau_i} > 0 \quad \textit{and} \quad \frac{\partial N_i(t;\tau_i,k_i)}{\partial k_i} < 0$$

for all $t > \tau_i$.

Proof. The proof is in Appendix A.1. □

The first partial derivative means that postponing the restoration operation a little results in an increased post-restoration stock for each time instant after the restoration date. The reason for this is that the restoration investment decreases the stock and this decrease does not occur when the restoration operation is postponed. The second partial derivative says that increasing the restoration investment level translates into decreased post-restoration stock, because the restoration production process becomes more effective.

Contaminated sites: Our first application is the cleanup of contaminated sites, where the damaged stocks are contaminant stocks. In this case, a reasonable target for minimization is the total discounted damages from the contaminant stocks, i.e.,

$$\mathcal{W}((\tau_i,k_i)_{i=1}^s) := \sum_{i=1}^s \left(\int_0^{\tau_i} D_i(N_i(t))e^{-rt}\, \mathrm{d}t + \int_{\tau_i}^{\infty} D_i(N_i(t;\tau_i,k_i))e^{-rt}\, \mathrm{d}t \right), \tag{2.4}$$

where D_i is the damage function related to the stock i, and $N_i(t)$ denotes the solution to the stock differential equation on interval $[0,\tau_i)$ and $N_i(t;\tau_i,k_i)$ on interval $[\tau_i,\infty)$. Here the first integral is the discounted damages from the contaminant stock before the cleanup and the second integral consists of the discounted damages after the cleanup investment. Clearly,

$N_i(\tau_i) = N_i(\tau_i; \tau_i, k_i)$. Applying this, Lemma 2.1 and the typical assumption about stock damages, i.e., $D_i'(n_i) > 0$ for all i, shows that this objective satisfies the assumption (A8). Indeed, differentiation gives

$$\begin{aligned}\frac{\partial \mathcal{W}((\tau_i, k_i)_{i=1}^s)}{\partial \tau_i} &= \int_{\tau_i}^{\infty} D_i'(N_i(t;\tau_i,k_i)) \frac{\partial N_i(t;\tau_i,k_i)}{\partial \tau_i} e^{-rt} \,\mathrm{d}t > 0,\\ \frac{\partial \mathcal{W}((\tau_i, k_i)_{i=1}^s)}{\partial k_i} &= \int_{\tau_i}^{\infty} D_i'(N_i(t;\tau_i,k_i)) \frac{\partial N_i(t;\tau_i,k_i)}{\partial k_i} e^{-rt} \,\mathrm{d}t < 0,\end{aligned} \tag{2.5}$$

for each $i = 1, \ldots, s$. This loss function is additive in the sense that cleanup investment at the site i only affects the payoff from that site.[11] No assumption is made about the second derivative of the damage function, but $D_i''(n_i) \geq 0$ is allowed, and therefore the damage function assumption includes the typical strictly increasing and convex damages.

The main differences to the existing models, including Caputo and Wilen (1995), Keohane *et al.* (2007) and Lappi (2018), are that here the decision maker is in charge of cleaning multiple sites using a given budget with one-time cleanup investments, and has the option to distribute the initial budget optimally between multiple contaminated sites. In Caputo and Wilen (1995) the decision maker aims to minimize the discounted sum of cleanup costs and damages from a single contaminated site using a continuous control. Similarly, Keohane *et al.* (2007) model a (stochastically) deteriorating quality of a resource using a continuous control. However, their model also includes another control missing from Caputo and Wilen, i.e. restoration, which pushes the quality of the resource up instantaneously. The model in Lappi (2018) has also an instantaneous cleanup. In contrast, our model focuses on the multi-site case, on the budget allocation problem and on one-time cleanup investments (instead of continuous controls). The framework nevertheless allows both for gradual cleanup processes through passive restoration that are more in line with actual practice (cf. Superfund) and for almost instantaneous cleanup achieved by applying large investment levels.[12]

Habitats and carbon stocks: Second, we can interpret the stocks as state of the habitats at each site and reinterpret the above contaminated sites model as a lost habitat model. Namely, let $N_i(t)$ be the area of lost habitat at the site i at time t and let function D_i be the utility loss related to the lost habitat area. For example, let $D_i(N_i) := U_i(a_i) - U_i(a_i - N_i)$, where U_i is an increasing and concave utility function and a_i is the "pristine" habitat state at the site i.

Also, the model can be interpreted as a model, where the decision maker is interested in increasing the carbon stocks at the sites, which have been diminished by prior economic activity. For this, put $D_i(N_i) := pN_i$, where p is the (possibly time-dependent) carbon price.

[11] Additively separable objective function will be analyzed in Section 3.2.

[12] Of course such a cleanup should be very costly. Another important difference to Lappi (2018) is that in our model cleanup costs depend on investment and pollution stock levels.

In this case the decision maker aims to minimize the total discounted value of the lost carbon stock.

Biodiversity and species conservation: While multi-site cleanup problems are relatively scarce in the literature, it has been usual to model biodiversity and species conservation using such settings (e.g. Costello and Polasky (2004); Luby *et al.* (2022) and the references therein). The loss function $\mathcal{W}$ can be used to model the minimization of the expected number of lost species or the minimization of the lost utility (or value) from extinct species.

For the former interpretation, we suppose following Costello and Polasky (2004) and Luby *et al.* (2022) that the planning interval is finite (i.e., $\tau_i \in [0, T]$ and $T > 0$), and that the decision maker cares about the expected number of lost species at the date T. Hence, each species has the same value for the decision maker, and contrary to the assumptions in Weitzman (1998), even pandas and mosquitoes are equal in the eyes of this particular decision maker. For this, we let the total number of species over all sites be $\mathcal{S}$, and take the stock at the site i, n_i, to be the deforested area or the lost habitat. We define the probability that species $j \in \mathcal{S}$ survives in the site i with $P_{ij}(n_i)$, where $P'_{ij} < 0$, i.e., a larger lost habitat implies the lower probability of survival for the species j. This assumption is similar to Luby *et al.* (2022), where this probability depends on the forest area. The probability that the species j does not survive in any of the sites is given by $\prod_{i=1}^{s}(1 - P_{ij}(n_i))$, and therefore the objective for minimization becomes

$$\mathcal{W}((\tau_i, k_i)_{i=1}^s) := \sum_{j \in \mathcal{S}} \prod_{i=1}^{s} \Big(1 - P_{ij}(N_i(T; \tau_i, k_i))\Big). \tag{2.6}$$

This is the expected number of lost species as a function of restoration dates and investments. Our Lemma 2.1 and a calculation shows, that this function also satisfies the assumption (A8):

$$\begin{aligned}
\frac{\partial \mathcal{W}((\tau_i, k_i)_{i=1}^s)}{\partial \tau_i} &= \sum_{j \in \mathcal{S}} -P'_{ij}(N_i(\cdot)) \frac{\partial N_i(\cdot)}{\partial \tau_i} \prod_{l \neq i} \Big(1 - P_{lj}(N_l(\cdot))\Big) > 0, \\
\frac{\partial \mathcal{W}((\tau_i, k_i)_{i=1}^s)}{\partial k_i} &= \sum_{j \in \mathcal{S}} -P'_{ij}(N_i(\cdot)) \frac{\partial N_i(\cdot)}{\partial k_i} \prod_{l \neq i} \Big(1 - P_{lj}(N_l(\cdot))\Big) < 0,
\end{aligned} \tag{2.7}$$

for all $i = 1, \dots, s$. Costello and Polasky (2004), Luby *et al.* (2022) and the current model share some similarities (e.g. the decision maker is faced with a budget constraint), but diverge in some key dimensions. First, the current model features both active and passive restoration, which means that the lost habitat decreases even without restoration investment. Second, Costello and Polasky (2004) and Luby *et al.* (2022) include in their frameworks periodical budget additions, while in our model the given initial budget is replenished only through saving. Also, in our model restoration is a one-time investment chosen at the optimal date, while in Luby *et al.* (2022) restoration can be performed at multiple time instants.

In addition, our model has a general loss function. This is important, because it allows us to model also the latter interpretation related to biodiversity made above, i.e., the minimization of the lost utility or value from extinct species without any reference to a fixed terminal date T. It will be developed in Section 4, where we use it to model habitat restoration in Europe.

2.3. **Optimization problem.** The decision maker aims to minimize total discounted losses under the budget constraint, that is, to solve

$$\inf_{\{(\tau_i,k_i)_{i=1}^s\}} \mathcal{W}\Big((\tau_i,k_i)_{i=1}^s\Big) \tag{2.8}$$

$$\text{s.t.} \quad \dot{B} = rB \geq 0, \quad B(0^-) = b_0 > 0, \tag{2.9}$$

$$B(\tau_i^-) - B(\tau_i^+) = C_i(k_i, N_i(\tau_i)), \tag{2.10}$$

$$\tau_i \geq 0, \quad k_i \geq 0, \quad \text{for all} \quad i. \tag{2.11}$$

We will show in Theorem 3.2 (see Section 3) that this problem is equivalent to another optimization problem that is more easily analyzed.

Namely, the structure of constraints on the use of restoration funds, i.e., equations (2.9) and (2.10), will be used to show that any optimal solution to this problem is also an optimal solution to a simplified problem (and vice versa). We call this problem the discounted cost problem, because it takes the form

$$\inf_{\{(\tau_i,k_i)_{i=1}^s\}} \mathcal{W}\Big((\tau_i,k_i)_{i=1}^s\Big) \tag{2.12}$$

$$\text{s.t.} \quad b_0 = \sum_{i=1}^s C_i(k_i, N_i(\tau_i))e^{-r\tau_i}, \tag{2.13}$$

$$\tau_i \geq 0, \quad k_i \geq 0, \quad \text{for all} \quad i. \tag{2.14}$$

In this problem, the use of funds is governed by Equation (2.13), which states that the initial budget must equal the total discounted restoration costs. This equivalence is important, because it allows us to use Lagrange multipliers in the analysis of the model, particularly when we derive the optimal waiting and investment rules. Another reason why this equivalence is important is related to the optimal allocation of funds between assets in need of restoration and other public goods. When the decision maker is a government, the optimization problem can be used to find a budget that results in a Lagrange multiplier that equals the marginal cost of public funds. Such a budget represents, in principle, the amount of funds that should be allocated to restoration.

In our optimization problems, we cannot ask for the minimum instead of the infimum (which always exists), as in some cases the optimal choice is not to restore a site or sites. In our model, not to restore means that optimal τ_i equals ∞ – and that is why the minimum might not exist in every situation. We say that $(\tau_i^*, k_i^*)_{i=1}^s$ is an optimal solution to the optimization problem, if $\tau_i^* \in [0, \infty]$ and $k_i^* \in [0, \infty]$ satisfy the budget constraints (feasibility) and such that the infimum is found at that point. Here $\tau_i^* = \infty$ means that the infimum is attained by the sequence of feasible points $(\tau_i^n, k_i^n)_{i=1}^s$ such that $\tau_i^n \to \tau_i^*$ and $k_i^n \to k_i^*$ as $n \to \infty$.

3. Results

3.1. **General results.** It is possible, at least in principle, for some of the funds to be left unused after restoration operations. One may think that a sufficiently large budget would always leave funds unused even if all sites are restored. In the current model, however, the last restoration uses all remaining funds, as is shown next.

Lemma 3.1. *All the remaining budget is used to restore the last site in any optimal solution to problem (2.8)–(2.11). Also, at least one site is restored, i.e., $\tau_i < \infty$ and $k_i > 0$ for at least one site i.*

Proof. The proof is in Appendix A.2. □

The reason for using all of the funds is found in the description of the restoration process. Increasing the restoration investment will increase the rate at which the stock and hence the damages decrease after the restoration date. Such an increase will make the decision maker better off, but it will also increase restoration costs. In effect, the decision maker can be made better off by each feasible increase in the restoration investment until everything is spent.

Next we turn to the equivalence of the optimization problems. We first show that the original problem and the discounted cost problem are equivalent in the sense that the optimal values of the problems are the same, and that any optimal solution to one of those problems can be used to construct an optimal solution to the other.

Theorem 3.2. *A point $(\tau_i^*, k_i^*)_{i=1}^s$ is an optimal solution to the original problem (2.8)–(2.11) if and only if it is an optimal solution to the discounted cost problem (2.12)–(2.14). Optimal values (i.e., infima) of these problems are the same at any solution.*

Proof. The proof is in Appendix A.3. □

This result is useful because it allows us to obtain information about an optimal solution to the original problem through the analysis of a simpler problem, namely the discounted problem, which can be approached with Lagrange multipliers. In fact, we will use the discounted cost problem (2.12)–(2.14) to characterize the optimal postponement of restoration and the related investment.

The next proposition states the optimal waiting and investment rules for restoration operations for an interior solution. To obtain that result, we use the following lemma, where the Lagrangian related to the problem (2.12)–(2.14) is

$$L((\tau_i, k_i)_{i=1}^s) = \lambda_0 \mathcal{W}((\tau_i, k_i)_{i=1}^s) + \lambda \left(\sum_{i=1}^s C_i(k_i, N_i(\tau_i)) e^{-r\tau_i} - b_0 \right) - \sum_{i=1}^s \omega_i \tau_i - \sum_{i=1}^s \gamma_i k_i,$$

and the constants λ_0, λ, and ω_i and γ_i, $i = 1, \ldots, s$, are Lagrange multipliers.

Lemma 3.3. *Let a point $(\tau_i, k_i)_{i=1}^s$ be an optimal solution to the discounted cost problem (2.12)–(2.14). Then we can choose Lagrange multipliers so that $\lambda_0 = 1$. Furthermore, with the choice $\lambda_0 = 1$, we have $\lambda > 0$ and, if $\tau_i \in (0, \infty)$,*

$$\frac{\partial C_i(k_i, N_i(\tau_i)) e^{-r\tau_i}}{\partial \tau_i} < 0. \tag{3.1}$$

Proof. The proof is in Appendix A.4. □

Inequality (3.1) means that the discounted restoration cost is strictly decreasing in the restoration date, and it can be written as

$$\left(rC_i(k_i, N_i(\tau_i)) - \frac{\partial}{\partial t}\bigg|_{t=\tau_i} C(k_i, N_i(t)) \right) e^{-r\tau_i} > 0.$$

The first term in the parenthesis is the interest on the saved restoration cost and the second term is the effect of the postponement on the restoration cost as the damaged stock passively restores. Thus, if the optimal restoration date is interior, the first term trumps the second. These effects are missing from Caputo and Wilen (1995) and Luby *et al.* (2022), because of our focus on restoration timing, and they show how saving the budget and passive restoration affect the optimal date to restore.

Proposition 3.4. *Suppose that it is optimal to postpone the restoration of the site i. Then the optimal restoration date and the corresponding restoration investment in site i solve the pair of equations*

$$\frac{\partial \mathcal{W}((\tau_i, k_i)_{i=1}^s)}{\partial \tau_i} = \lambda \left(rC_i(k_i, N_i(\tau_i)) - \frac{\partial}{\partial t}\bigg|_{t=\tau_i} C(k_i, N_i(t)) \right) e^{-r\tau_i}, \tag{3.2}$$

and

$$-\frac{\partial \mathcal{W}((\tau_i, k_i)_{i=1}^s)}{\partial k_i} = \lambda \frac{\partial C_i(k_i, N_i(\tau_i))}{\partial k_i} e^{-r\tau_i}, \tag{3.3}$$

where λ is the Lagrange multiplier related to constraint (2.13).

Proof. The waiting and investment rules for a site i, whose restoration is not postponed indefinitely (i.e. $\tau_i \in (0, \infty)$ and $k_i > 0$) follow from (A.7) and (A.9) by direct calculation. □

To interpret the optimal waiting rule (3.2), we suppose now that the restoration cost is strictly increasing in the stock size at the optimal restoration date, which implies that the term $-\partial/\partial t\big|_{t=\tau_i} C(k_i, N_i(t))$ is non-negative. This rule then says that (for a given k_i) the restoration of the site i is postponed until the marginal cost of the postponement or the waiting on the left side equals the marginal benefit of the postponement on the right side. The marginal cost is the additional loss incurred by a small postponement of restoration. The (discounted) marginal benefit of waiting contains the sum of the interest on funds not used for the restoration and the decrease in restoration costs due to the natural regeneration process.[13] This sum is valued at the Lagrange multiplier λ, which is the marginal value of funds, when all sites are restored. The optimal investment rule (3.3) equalizes the marginal benefit of investment with the marginal cost of investment (the marginal cost of restoration valued at the marginal value of funds).

Contaminated sites: In the contaminated sites context, i.e., when the objective is given by (2.4), the waiting and investment rules are obtained by combining Proposition 3.4 with equations in (2.5). This gives the following result:

Corollary 3.5. *When the cleanup of a contaminated site i is postponed, the optimal cleanup date and investment solve the pair of equations*

$$\int_{\tau_i}^{\infty} D_i'(N_i(t;\tau_i,k_i)))\frac{\partial}{\partial \tau_i}N_i(t;\tau_i,k_i)e^{-r(t-\tau_i)}\,\mathrm{d}t = \lambda\left(rC_i(k_i,N_i(\tau_i)) - \frac{\partial}{\partial t}\bigg|_{t=\tau_i} C(k_i,N_i(t))\right), \tag{3.4}$$

and

$$-\int_{\tau_i}^{\infty} D_i'(N_i(t;\tau_i,k_i))\frac{\partial}{\partial k_i}N_i(t;\tau_i,k_i)e^{-r(t-\tau_i)}\,\mathrm{d}t = \lambda\frac{\partial C_i(k_i,N_i(\tau_i))}{\partial k_i}. \tag{3.5}$$

The right-side of of the waiting rule (3.4) is the marginal benefit of the waiting found in Lappi (2018) using a jump-investment model without a budget, except that here it is multiplied by the marginal value of funds. The left-side of (3.4) is the discounted value of all future marginal damages from a delay in the restoration of the site i. Similarly, the left-side of the investment rule (3.5) is the marginal decrease in the discounted value of all future marginal damages from an increase in the investment level.[14]

With a single contaminated stock, Caputo and Wilen (1995) show that the continuous cleanup control is chosen such that at each time instant the current marginal cleanup cost equals the future marginal benefits of cleanup, which consists of the difference between reductions in future damages and increases in future cleanup costs. Future cleanup costs increase in their model as current cleanup is increased, because of stock effects in the cost

[13]If $-\partial/\partial t\big|_{t=\tau_i} C(k_i, N_i(t)) < 0$, then the absolute value of this term should be counted as a marginal cost of waiting, because waiting increases the restoration cost.

[14]This investment rule is not found in a jump-investment model of Lappi (2018) as there the investment is maximal in the sense that the stock simply jumps downwards (to zero).

formulation. In our model, the marginal clean-up investment cost multiplied by the shadow value of funds is equalized with the marginal decrease in all future damages, and the one-time cleanup does not influence, by definition, the cost of future cleanups. However, our cleanup cost function depends on the contaminated stock as in Caputo and Wilen and this dependence influences the marginal benefit of waiting in (3.4), although we do not restrict this dependence by any assumption.

Habitats and carbons stocks: Corollary 3.5 can be applied also for lost habitats and carbon stocks. With lost habitats, where $D_i(N_i) := U_i(a_i) - U_i(a_i - N_i)$, the left-side of (3.4) reads

$$\int_{\tau_i}^{\infty} U_i'(a_i - N_i(t;\tau_i,k_i)))\frac{\partial}{\partial \tau_i}N_i(t;\tau_i,k_i)e^{-r(t-\tau_i)}\,\mathrm{d}t. \tag{3.6}$$

Thus the waiting rule equalizes the discounted value of all future lost marginal utility from the delay with the marginal benefit. Similarly, the investment rule for lost habitats equalizes the marginal decrease in the discounted value of all future lost marginal utility with the marginal cost of investment. For the carbon stocks that are valued using the carbon price, setting U' to equal the carbon price p gives the waiting and investment rules.

Biodiversity and species conservation: For a decision maker interested in the minimization of the expected number of lost species, that is, for a decision maker with an objective given by (2.6), the waiting and investment rules are as follows:

Corollary 3.6. *When the restoration of site i is postponed, the optimal restoration date and investment solve the pair of equations*

$$\begin{aligned}\sum_{j\in\mathcal{S}} -P_{ij}'(N_i(\cdot))\frac{\partial N_i(\cdot)}{\partial \tau_i}\prod_{l\neq i}\Big(1-P_{lj}(N_l(\cdot))\Big) &= \\ \lambda\left(rC_i(k_i,N_i(\tau_i)) - \frac{\partial}{\partial t}\bigg|_{t=\tau_i} C(k_i,N_i(t))\right)&,\end{aligned} \tag{3.7}$$

and

$$\sum_{j\in\mathcal{S}} P_{ij}'(N_i(\cdot))\frac{\partial N_i(\cdot)}{\partial k_i}\prod_{l\neq i}\Big(1-P_{lj}(N_l(\cdot))\Big) = \lambda\frac{\partial C_i(k_i,N_i(\tau_i))}{\partial k_i}. \tag{3.8}$$

The left-side of (3.7) is the marginal cost of postponement. The term $\prod_{l\neq i}\Big(1-P_{lj}(N_l(T;\tau_l,k_l))\Big)$ is the probability that the species j does not survive on sites other than i, and the term $-P_{ij}'(N_i(T;\tau_i,k_i))\frac{\partial N_i(T;\tau_i,k_i)}{\partial \tau_i}$ is the marginal increase in the probability of the extinction of the species j when the restoration of the site i is postponed. Summing the product of these terms over all species gives the marginal cost of the postponement of the restoration of the site i, and this is equalized with the marginal benefit of waiting. The marginal benefit of investment is on the left-side of (3.8). The term $P_{ij}'(N_i(T;\tau_i,k_i))\frac{\partial N_i(T;\tau_i,k_i)}{\partial k_i}$ is the marginal increase in the probability of the survival of the species j when the restoration investment at the site i is increased. Multiplying this by the probability that the species does not survive anywhere else, and summing the product over all species, gives the marginal benefit of investment, or the marginal increase in the expected number of saved species from increased restoration investment at the site i.

The objective function in this application is the same as in Luby *et al.* (2022) except for the minor difference that the probability that species survives on some site depends here on the lost habitat instead of the forest area. More importantly, we allow both active and passive restoration of habitats and saving of the budget for future use, while in Luby *et al.* (2022) passive restoration and saving are not allowed. The terms in the parenthesis on the right-side of the waiting rule (3.7) include the interest on the saved restoration cost and the effect of the restoration postponement on the cost through passive restoration.

Examples 1 and 2 given later include cases in which a site is restored immediately on date zero, on an interior date, or not at all, and Proposition 3.4 gives the rules that characterize an interior optimum. But our next result gives sufficient conditions for a site to be restored at a finite time.

Proposition 3.7. *Suppose that we restore sites i, $i = 1, \ldots, s$. Let p be any other site that is not restored. Suppose that there exists a date $\tau_p < \infty$ such that there are no fixed costs for the restoration of the site p and assume that the right derivative $\frac{\partial}{\partial k_p} C_p(0, N_p(\tau_p))$ exists. Further, let λ^* be the Lagrange multiplier related to the discounted cost problem (2.12)–(2.14) with losses $\mathcal{W}((\tau_i, k_i)_{i=1}^s, (\tau_p, 0))$, where the timing τ_p and the effort $k_p = 0$ at the site p have been fixed.*

If the inequality

$$-\frac{\partial \mathcal{W}((\tau_i^*, k_i^*)_{i=1}^s, (\tau_p, 0))}{\partial k_p} > \lambda^* \frac{\partial C_p(0, N_p(\tau_p))}{\partial k_p} e^{-r\tau_p} \tag{3.9}$$

holds, then reallocating money from other sites to the site p decreases the total discounted damages, i.e., it is optimal to allocate some money to the site p.

Proof. The proof is in Appendix A.5 □

Hence, if there are no fixed costs related to the restoration of the site p, and if inequality (3.9) holds at some date, then the site p is restored at a finite time. Of the two conditions, the latter, i.e., inequality (3.9), is the more interesting one. It says that the benefit from a small restoration effort is greater than its corresponding cost. The cost is equal to the (discounted) marginal restoration cost of the site p at the zero restoration effort multiplied by the Lagrange multiplier related to the budget constraint of the discounted cost problem with sites $i, \ldots, s$. This product gives the value of the funds that are used for a small restoration operation at the site p, and if this value is less than the marginal benefit (and there are no fixed costs), then the restoration at the site p is not postponed indefinitely. Importantly, this result means that under these conditions at least some money is used to restore the site p, but it does not mean that all the sites (i.e. sites $1, \ldots, s$ and p) are restored. It is possible that there is a site or sites among the first s sites that are not restored at the optimum when also the site p is included.

3.2. **Special case: additive losses.** Some of the model's potential applications have losses that are separable in the sense that investment at any of the sites influences only the loss of that particular site. This is the case, for example, for the loss function (2.4) related to the contaminated stocks. Such losses are defined as follows:

Definition 3.8. The loss function $\mathcal{W}$ is additively separable if it can be written as

$$\mathcal{W}((\tau_i, k_i)_{i=1}^s) = \sum_{i=1}^{s} W_i(\tau_i, k_i),$$

where functions W_i satisfy the same assumption as $\mathcal{W}$, i.e., assumption (A8).

For this class of loss functions we first define a budget allocation problem, which we later show to be equivalent with the previous discounted cost formulation, and then simplify few of our results. Namely, we argue that in this case the sufficient condition for restoration is only related to Lagrange multipliers of individual restoration budgets instead of depending on the specifics of the losses stemming from these sites as in the discounted cost formulation (recall Proposition 3.7). In addition, we show that the value functions are differentiable, connect their derivatives to Lagrange multipliers, and argue that one unit of additional money is equally valuable whether it is allocated to individual budgets or to the total initial budget.

The equivalence result presented below states that when the loss function $\mathcal{W}$ is additively separable, the original problem (2.8)–(2.11) is equivalent to a budget allocation problem of the form

$$\inf_{(b_i)_{i=1}^s} \sum_{i=1}^{s} V_i(b_i) \quad \text{s.t.} \quad b_i \in [0, b_0], \quad \sum_{i=1}^{s} b_i = b_0, \tag{3.10}$$

where the function V_i is the optimal value of losses for the site i that is allocated a budget b_i for the restoration, i.e.,

$$V_i(b_i) = \inf_{\{\tau_i, k_i\}} \left\{ W_i(\tau_i, k_i) : b_i = C_i(k_i, N_i(\tau_i))e^{-r\tau_i} \right\}, \tag{3.11}$$

when $b_i > 0$, and

$$V_i(0) = \lim_{\tau_i \to \infty} W_i(\tau_i, 0) = W_i(\infty, 0). \tag{3.12}$$

Hence, when the loss function is additively separable, the original problem can be represented as a problem in which each individual site is first optimally restored for any given budget, and then the initial budget is optimally allocated for each individual site. We denote an optimal solution to the budget allocation problem in (3.10) by $(b_i^*)_{i=1}^s$.[15]

[15]The relationship between the budget b_i^* of the site i and the optimal restoration time τ_i^* and the optimal investment level k_i^* is described by $W_i(\tau_i^*, k_i^*) = V_i(b_i^*)$. Here we set $\tau_i^* = \infty$ and $k_i^* = 0$, if $b_i^* = 0$. Naturally,

We begin by deriving a corollary to Proposition 3.7 in the additively separable case.

Corollary 3.9. *Suppose that the loss function $\mathcal{W}$ is additively separable and that we have optimally distributed budgets $b_j^* > 0$ to some sites to restore them. Assume that all sites j are restored at an optimal date τ_j^*. Let λ_j^* be the Lagrange multiplier related to the constraint in* (3.11)*. Let p be any other site that has not been allocated a restoration budget. Suppose that there exists a date $\tau_p < \infty$ such that there are no fixed costs and that the right derivative $\frac{\partial}{\partial k_p} C_p(0, N_p(\tau_p))$ exists.*

If the inequality

$$-\frac{\partial W_p(\tau_p, 0)}{\partial k_p} > \lambda_j^* \frac{\partial C_p(0, N_p(\tau_p))}{\partial k_p} e^{-r\tau_p}, \tag{3.13}$$

holds for τ_p, then it is optimal to restore site p at some instant $\tau_p < \infty$.

Proof. Note first that by a similar argument as in Proposition 3.7 with $s = 1$ money moved away from the site j is more valuable in the site p.

As we have assumed that the original sites have optimally distributed budgets, the money is not more valuable at any site that has already been restored. □

Inequality (3.13) states that the marginal benefit of restoration is greater than the marginal cost of restoration for the first unit of restoration at some date τ_p. Compared to the condition (3.9) of Proposition 3.7, this sufficient condition to reallocate money is only related to the value of the budget constraint's Lagrange multiplier of a restored site and not to the specific properties of the losses from the restored sites. The reason for this simplification is that our loss function is additively separable, i.e., the investment at any of the sites influences only the loss of that particular site.

Furthermore, suppose we have allocated budgets $b_i > 0$ to some sites to restore them. How can one test whether this allocation is optimal? Should one reallocate money to some other site p that has not received funds? To answer these questions, let site j be restored at an optimal date τ_j^* given this allocation and let λ_j^* be the corresponding Lagrange multiplier related to the constraint in (3.11). Suppose that the stated properties for site p given in Corollary 3.9 hold. Then, given inequality (3.13), reallocating money from site j to site p decreases the total discounted losses. This corollary therefore gives the following test for the optimal budget allocation: a budget is not optimally allocated if there is a site p (that has not been allocated any money) that satisfies the above marginal condition, which states that the ratio of the marginal benefit and the marginal cost of restoration at zero investment level is greater than the shadow value of funds. In such a case there is a reallocation of funds that decreases losses. Similarly, if one discovers, for example, a new habitat or a contaminated site,

when the budget is 0 one does not restore the site – thus we set the optimal restoration time in this case to be ∞.

and if the site's properties are such that the inequality does not hold, then no reoptimization of the money allocation is needed and the new site should be left for passive restoration.

For the equivalence of the original problem and the budget allocation problem, we assume that the objective function is additively separable and that the loss function $\mathcal{W}$ satisfies the following technical assumptions

$$W_i(\infty, 0) = \lim_{\tau_i \to \infty} W_i(\tau_i, k_i) = \lim_{\tau_i \to \infty, k_i \to \infty} W_i(\tau_i, k_i) \tag{3.14}$$

for each $i = 1, \ldots, s$, where the limit $\tau_i \to \infty, k_i \to \infty$ is taken for points in the choice set. Further, we assume also that

$$W_i(\infty, 0) = W_i(\tau_i, 0) = \lim_{k_i \to 0} W_i(\tau_i, k_i), \qquad \text{if there are no fixed costs.} \tag{3.15}$$

These assumptions guarantee that the loss function achieves the same value, when we do not restore. These equations hold, for instance, for our example, where damaged stocks are contaminant stocks (2.4), see Appendix A.6 for a proof.

Theorem 3.10. *Suppose that the loss function $\mathcal{W}$ is additively separable, equations (3.14) and (3.15) hold, and that $\lim_{\tau_i \to \infty} C_i(k_i, N(\tau_i))e^{-r\tau_i} = 0$.*

(*a.*) *If $(b_i^*)_{i=1}^s$ is an optimal solution to the budget allocation problem (3.10), then $(\tau_i^*, k_i^*)_{i=1}^s$ is an optimal solution to the discounted cost problem (2.12)–(2.14), where $(\tau_i^*, k_i^*)_{i=1}^s$ is such that*

$$W_i(\tau_i^*, k_i^*) = V_i(b_i^*) \quad \text{for each} \;\; i. \tag{3.16}$$

If $(\tau_i^*, k_i^*)_{i=1}^s$ is an optimal solution to the discounted cost problem (2.12)–(2.14), then $(\beta_i(\tau_i^*, k_i^*))_{i=1}^s$ is an optimal solution to the budget allocation problem (3.10), where the function β_i is the discounted restoration cost of site i, i.e.,

$$\beta_i(\tau_i, k_i) = C_i(k_i, N_i(\tau_i))e^{-r\tau_i} \quad \text{for each} \;\; i.$$

(*b.*) *Optimal values (i.e., infima) of the problems are the same at any solution.*

Proof. The proof is in Appendix A.7. □

Hence under mild assumptions the discounted cost problem and the budget allocation problem are equivalent for separable losses, which means that the above corollaries also apply to the discounted cost problem and the original problem. The additional assumption, $\lim_{\tau_i \to \infty} C_i(k_i, N(\tau_i))e^{-r\tau_i} = 0$, rules out cases where the restoration cost grows at a rate greater than the interest, given the natural regeneration process (described by function h_i) and the related solution to the stock equation.

Our last results before examples and the empirical application illustrating the model are related to Lagrange multipliers. Namely, we will show that the multipliers are equalized for sites where funds are used and that one additional unit of money is equally valuable whether it is allocated to an individual budget or to the total initial budget. We show these results by connecting the value functions of single site problems with Lagrange multipliers whether the optimal restoration date of the site i is at $\tau_i^* = 0$ or at $\tau_i^* > 0$. First, we show that the value functions are differentiable (see Appendix A.8, where we also present and prove the applied envelope theorem). For the Lagrange multipliers result, we use the following notation: let λ^* be the Lagrange multiplier of the discounted cost problem (2.12)–(2.14), μ^* be the Lagrange multiplier of the budget allocation problem (3.10), and λ_i^* be the Lagrange multiplier of the single site optimization problem (3.11).

In the next proposition, we only study the subset of sites where the money is used, which we index from 1 to s.

Proposition 3.11. *Assume that the loss function $\mathcal{W}$ is additively separable, equations (3.14) and (3.15) hold and that $\lim_{\tau_i\to\infty} C_i(k_i, N(\tau_i))e^{-r\tau_i} = 0$. Suppose that there is a unique optimal solution (τ_i^*, k_i^*) for a given budget $b_i > 0$ to the single site problem (3.11) and that $b_i > 0$ for all $i = 1, \ldots, s$. Suppose also that the unique solution is differentiable with respect to the budget.*

(a.) *Then*

$$-\mu^* = V_i'(b_i^*) = -\lambda_i^*.$$

(b.) *If there is a unique budget allocation $(b_1^*, \ldots, b_s^*)$ to (3.10) with $b_0 = \sum_{i=1}^s b_i^*$, then*

$$\frac{\mathrm{d}}{\mathrm{d}b_0}\sum_{i=1}^s V_i(b_i^*) = -\mu^* = -\lambda_i^* = -\lambda^* = \frac{\mathrm{d}}{\mathrm{d}b_0}\sum_{i=1}^s W_i(\tau_i^*, k_i^*).$$

Proof. The proof is in Appendix A.9. □

Note first that parts (a.) and (b.) imply that

$$\frac{\mathrm{d}}{\mathrm{d}b_0}\sum_{i=1}^s V_i(b_i^*) = V_i'(b_i^*) = \frac{\mathrm{d}}{\mathrm{d}b_0}\sum_{i=1}^s W_i(\tau_i^*, k_i^*),$$

which means that one additional unit of money (for the sites where money is used) is equally valuable whether it is added to the total initial budget or added to the individual site's budget. Moreover, for a unique optimal solution (τ_i^*, k_i^*) to the single site optimization problem, all the problems have the same Lagrange multipliers that are related to the respective budget constraints, i.e., the shadow values of the budget constraints are equalized. For the general loss function $\mathcal{W}$ the last equality of Proposition 3.11 (b.) holds, i.e., $-\lambda^* = \frac{\mathrm{d}}{\mathrm{d}b_0}\mathcal{W}((\tau_i^*, k_i^*)_{i=1}^s)$ (see Proposition A.8). This result is connected to Corollary 3.9 and also to the test result related to it. Because Lagrange multipliers are the same it suffices to take any of the restored sites and its multiplier when testing for optimal allocation of funds.

We conclude the analysis of the model by presenting two examples with specific function forms, which highlight different possibilities for where the optimal points may lie. Example 1 is a two-site case, in which it is optimal to restore one site on the initial date, i.e., $\tau^* = 0$, and to postpone the restoration of the other site indefinitely. Example 2 contains a single site case where it is optimal to postpone the restoration.

Example 1. Suppose there are two sites and that the stock dynamics are given by

$$\dot{N}_i(t) = \begin{cases} 0, & t \le \tau_i, \\ -k_i N_i(t), & t > \tau_i, \end{cases}$$

for $i = 1, 2$. These equations mean that there are no natural regeneration processes, and that the stocks remain at their initial values $n_{i,0}$ until the restoration investment, after which the stock evolution is described by $N_i(t; \tau_i, k_i) = n_{i,0}e^{-k_i(t-\tau_i)}$. The stocks in this example could be related to some badly degraded sites, because there is no natural regeneration. The damage function is given by $D_i(n_i) := d_i n_i$ and the restoration cost function by $C_i(k_i, n_i) := c_i k_i n_i$. As there are no natural regeneration processes, the restoration cost is $C_i(k_i, N_i(\tau_i)) = c_i k_i n_{i,0}$ for each possible restoration date.

To see how a fixed budget is used to restore a single site, we consider the single site problem in (3.11) with the above specifications. Namely, the problem is to

$$\inf_{\{\tau_i, k_i\}} \left(\int_0^{\tau_i} d_i n_{i,0} e^{-rt} \, \mathrm{d}t + \int_{\tau_i}^{\infty} d_i n_{i,0} e^{-k_i(t-\tau_i)} e^{-rt} \, \mathrm{d}t \right) \tag{3.17}$$

$$\text{s.t.} \quad b_i = c_i k_i n_{i,0} e^{-r\tau_i}, \tag{3.18}$$

$$\tau_i \ge 0, \quad k_i \ge 0.$$

Here $b_i > 0$. We solve the constraint (3.18) for k_i and plug it into the objective function W_i and then take the derivative with respect to τ_i. This shows that the objective is a strictly increasing function of the restoration date τ_i:

$$\begin{aligned} \frac{\mathrm{d}W_i}{\mathrm{d}\tau_i} &= \int_{\tau_i}^{\infty} d_i n_{i,0} \left((\tau_i - t) \frac{b_i r e^{r\tau_i}}{c_i n_{i,0}} + \frac{b_i e^{r\tau_i}}{c_i n_{i,0}} \right) e^{-k_i(t-\tau_i)} e^{-rt} \, \mathrm{d}t \\ &= \frac{b_i d_i e^{(r+k_i)\tau_i}}{c_i} \int_{\tau_i}^{\infty} \Big((\tau_i - t) r + 1 \Big) e^{-(r+k_i)t} \, \mathrm{d}t = \frac{b_i d_i}{c_i} \frac{k_i}{(r+k_i)^2} > 0. \end{aligned}$$

Hence, if a strictly positive budget is allocated to the restoration of site i, then the optimal restoration date is $\tau_i = 0$ and the optimal investment level is the ratio between the site's budget and its marginal restoration cost, that is, the optimal investment is $k_i = b_i/(c_i n_{i,0})$.

Next we analyze the budget allocation problem (3.10) in this two-site setup to ask whether it always pays to allocate funds to both sites. We denote the value function of site i by V_i. Then

$$V_i(b_i) = \int_0^{\infty} d_i n_{i,0} e^{-(b_i/(c_i n_{i,0}) + r)t} \, \mathrm{d}t = \frac{d_i n_{i,0}^2 c_i}{b_i + r c_i n_{i,0}}. \tag{3.19}$$

Furthermore, $V_i'(b_i) = -d_i n_{i,0}^2 c_i/(b_i + r c_i n_{i,0})^2$. The value function's derivative is strictly increasing, because $V_i''(b_i) > 0$ for all $b_i > 0$. For $b_i = 0$, the value function gives the total discounted damages from the constant stock, and $V_i'(0) = -d_i/(r^2 c_i)$. The main message of this example is that it may not be optimal to allocate money for the restoration of both sites, which is not unlike the "extreme policy" of Weitzman (1998) in conservation planning. For example, if the parameters are such that

$$\arg\max_{b_1 \in [0,b_0]} V_1'(b_1) = -\frac{d_1 n_{1,0}^2 c_1}{(b_0 + r c_1 n_{1,0})^2} < -\frac{d_2}{r^2 c_2} = \arg\min_{b_2 \in [0,b_0]} V_2'(b_2),$$

then $V_1'(b_1) < V_2'(b_2)$ for any pair (b_1, b_2) such that $b_1 + b_2 = b_0$, $b_1 \geq 0$ and $b_2 \geq 0$. If it were the case that both sites receive funds for restoration, then equation $V_1'(b_1) = V_2'(b_2)$ would hold. This implies that, given the above requirement for the parameters, it is not optimal to allocate money to restore both sites, and because $V_1'(b_1) < V_2'(b_2)$, it is site 2 that is never restored. The reason is that each unit of money is more valuable when allocated to site 1 than to site 2.[16]

Example 2. To continue and extend the previous example, we let the cost function of site 1 be $C_1(k_1, n_1) := c_1 k_1 n_1 + F_1$, where $F_1 > 0$ is a fixed cost. The purpose of this example is to show that it can be optimal to postpone the restoration. We suppose that there are no other sites and that $F_1 > b_1$. The budget constraint gives $k_1 = (b_1 e^{r\tau_1} - F_1)/(c_1 n_{1,0})$, which shows that there are no feasible restoration investments at date zero (as k_1 would be negative at $\tau_1 = 0$). Hence the site is either not restored at all or the restoration is postponed, and, as there is only one site, the optimal restoration is at $\tau_1 < \infty$, because otherwise money would be left unspent (recall Lemma 3.1).

4. Habitat restoration in Europe

In this section, we illustrate the model by focusing on habitat restoration and species protection in European terrestrial ecoregions. Ecoregions are defined by Olson *et al.* (2001) "*as relatively large units of land containing a distinct assemblage of natural communities and species, with boundaries that approximate the original extent of natural communities prior to major land-use change*". The habitats in these regions have suffered from, e.g., land-use change, which affects the number of species able to survive. We will relate the lost habitat area to the probability that a species inhabiting the ecoregion goes extinct. The aim of the decision maker is to minimize the expected loss from species extinctions.

As we are unable to obtain ecological models for the passive restoration processes in each ecoregion, we focus only on the decision to place a fraction of the lost habitat area under restoration at some date, and, in particular, on the allocation of the budget between the regions. The restoration process means here that once the restoration investment is made at

[16]Similar results as above can be obtained with other cost functions, for example with function $C_i(k_i, n_i) := c_i k_i / n_i$. This follows in this example from the assumption that there are no natural regeneration processes, which implies that the cost is in both cases the investment level multiplied by a positive constant.

the restoration date, the habitat in the area designated for restoration will begin to improve. This implies that species will survive longer with restoration than they would without it, and thus continue to provide benefits.

Data. We study the allocation of restoration funds for 46 terrestrial ecoregions located in Europe.[17] Our data on habitat loss, as a fraction of the total ecoregion area, is from The Nature Conservancy (2009). The extent of the habitat loss varies considerably from near zero to over 80 percent.[18] The same is true for the ecoregion sizes.[19]

The species data for each ecoregion is based on Kier *et al.* (2005) plant data, which contains estimates on the total number of vascular plant species for each ecoregion. These values range from about 330 to 5000 species per ecoregion.[20] The total number of species is approximately 80 000. We also use data mainly from Eurostat on arable land prices, when we construct the cost function below.

Model specification. The total area of ecoregion i in hectares is A_i and the fraction of lost habitat at the initial date is $n_{i,0} \in [0,1]$. Hence the lost habitat area in region i is $n_{i,0}A_i$, and this area forms the target area for restoration. As this area is based on the habitat loss data from The Nature Conservancy, it must be noted here that the lost habitat only means the land area that has been transformed by humans into something else that may or may not support biodiversity (The Nature Conservancy, 2009). We therefore assume that the habitat loss fractions adequately approximate the relative differences in the actual habitat losses between ecoregions.

Some fraction $k_i \in [0,1]$ of this area may be brought under restoration at date $\tau_i \geq 0$. The total area of the region i under restoration is $k_i n_{i,0} A_i$ and the habitats in this land area will improve over time. This state is described with the function N_i, which is assumed to take the following form:

$$N_i(t;\tau_i,k_i) = (1-k_i)n_{i,0} + \frac{2k_i n_{i,0}}{1+e^{t-\tau_i}}. \tag{4.1}$$

[17]Specifically, the ecoregions are located in the following countries: the UK, and the EU, EFTA, and CEFTA countries, excluding Liechtenstein and Kosovo. Some ecoregions overlap with other countries, and for these ecoregions we focus only on the area included in the above countries. For example, Aegean and Western Turkey Sclerophyllous and Mixed Forests overlaps Albania, Bulgaria, Greece, and Turkey, but the Turkish area is not included in the study. In particular, areas in Belarus, Monaco, Russian Federation, San Marino, Turkey, and Ukraine are excluded. We also exclude the ecoregions Mediterranean Woodlands and Forests and Kola Peninsula Tundra, because they overlap the countries of interest in insignificant amounts.

[18]In the Scandinavian Montane Birch and Grassland, and in the Iceland Boreal Birch Forests and Alpine Tundra ecoregions, the habitat loss percent is virtually zero. On the other extreme, the habitat loss percent is 88.5 in Po Basin Mixed Forests.

[19]The smallest ecoregions are located on islands, e.g., the Madeira Evergreen Forests ecoregion is only about 800 square kilometers. The largest ecoregion is Scandinavian and Russian Taiga, with an area of over 2 million square kilometers (of which about 684 thousand square kilometers are included in the study).

[20]In this regard the most species rich ecoregion is Alps Conifer and Mixed Forests. The lowest estimate is for Faroe Islands Boreal Grasslands. If an ecoregion overlaps with excluded countries, the species data is adjusted by multiplying the species number by a fraction of the area of the ecoregion in the included countries and the total area of the ecoregion.

This captures the idea that restoration is initially fast, but will slow down later on and that the post-restoration lost habitat area decreases towards $(1-k_i)n_{i,0}$ as time approaches infinity. Hence, if all of the area is placed under restoration, the habitat loss will approach zero. Also, the habitat loss fraction equals the initial habitat loss $n_{i,0}$ at the time τ_i when the area is moved to restoration. This means that before this date the habitat loss neither increases nor decreases.

Bringing one hectare of land under restoration is costly. We assume that the restoration cost is increasing and convex in the area brought under restoration. In addition, as noted for example by Harstad (2023) in a different context, it is reasonable to suppose that this cost is decreasing in the lost habitat area that is not restored. Hence, the cost function for ecoregion i is

$$C_i(k_i, n_i) := \frac{c_i(k_i n_{i,0} A_i)^2}{(1-k_i)n_{i,0}A_i} = c_i n_{i,0} A_i \frac{k_i^2}{1-k_i}, \tag{4.2}$$

where $c_i > 0$. Importantly, restoring the first hectares is very cheap, but as more land is allocated for restoration, the cost rises at an increasing rate. We use the average price per hectare of agricultural land as the values for cost parameters c_i.[21] Note that, in the case of otherwise identical regions, these values order the restoration costs from the lowest to the highest.

To define the objective, we note, first, that the vascular plant species are only a relatively small subset of all species, and the same species can be located in multiple ecoregions. We assume that the total number of vascular plant species on any ecoregion i, α_i, is a good approximation of the number of endemic species of that region. The reason for this assumption is that we do not have data on the total number of plant, animal, and other species, let alone for endemic species, in each ecoregion.

We assume that each species produces an equal flow benefit and we normalize this value to unity. The value of a single species at any of the sites is discounted and is given by $\int_0^\dagger e^{-rt}\,\mathrm{d}t$, where † is the date of extinction. Hence each species has value as long as it is alive.[22] This means, for example, that the extinct bird species dodo (†*Raphus cucullatus*) has no value anymore, and more generally, that the memory of an extinct species has no value (either positive or negative).

[21]Specifically, we take the average price of each country in ecoregion i, and calculate parameter c_i as the weighted average of these prices, where the weights are the fraction of country's land on that ecoregion. We apply 120 477 euros average price for the ecoregions in the Canary Islands, the island ecoregions in the Atlantic and in Madeira, because in those relatively small areas the Spanish and Portuguese average prices are a poor indicator of land value. The same price is used for Malta.

[22]The value of a single species could be modelled with a more general utility function: the above expression could be replaced with $u(\dagger)$, where u is a non-decreasing function.

The extinction of a species located in ecoregion i is uncertain and depends on the available habitat. The expected value of this species is given by

$$\mathbb{E}\int_0^{\dagger} e^{-rt}\ \mathrm{d}t = \int_0^{\infty}\int_0^{\dagger} e^{-rt}\ \mathrm{d}t F_i'(\dagger;\tau_i,k_i)\ \mathrm{d}\dagger, \tag{4.3}$$

where F_i is the distribution function of $\dagger$, which depends on the restoration date τ_i and investment k_i of ecoregion i. We assume that the intensity function obtains value $\rho n_{i,0}$ for all $t \in [0,\tau_i)$ and $\rho N_i(t)$ for all $t > \tau_i$, which implies that the distribution F_i depends on the restoration date τ_i and the investment k_i.[23]

The formula for the distribution F is first developed for all $t < \tau$. For this, we let the probability that the species goes extinct on a small time interval $(t, t+\mathrm{d}t)$ given that it did not go extinct before t be given by

$$P(\dagger \in (t, t+\mathrm{d}t) \mid \dagger > t) = \rho n_0 \mathrm{d}t + \mathrm{d}t\ E(\mathrm{d}t),$$

where $\lim_{\mathrm{d}t\to 0}\ E(\mathrm{d}t) = 0$ uniformly in τ and k. Using the properties of the conditional probability, multiplying by $P(\dagger > t)$, dividing by $\mathrm{d}t$, and letting $\mathrm{d}t \to 0$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}P(\dagger < t) = \rho n_0(1 - P(\dagger < t)), \quad t \in [0,\tau).$$

We define $F(t) := P(\dagger < t)$, when $t < \tau$, and note that $F(0) = 0$. The solution to the above initial value problem is

$$F(t) = 1 - e^{-\rho n_0 t}. \tag{4.4}$$

The density is $F'(t) = \rho n_0 e^{-\rho n_0 t}$ and $1 - F(\tau) = P(\dagger > \tau) = e^{-\rho n_0 \tau}$.

When $t > \tau$,

$$P(\dagger \in (t, t+\mathrm{d}t) \mid \dagger > t) = \rho N(t;\tau,k)\mathrm{d}t + \mathrm{d}t\ E(\mathrm{d}t).$$

We obtain from this and from $t > \tau$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}P(\dagger < t) = \rho N(t)(1 - P(\dagger < t)), \quad t > \tau.$$

Letting $F(t) := P(\dagger < t)$ and noticing from (4.4) that $F(\tau) = 1 - e^{-\rho n_0 \tau}$, gives an initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}F(t) = \rho N(t;\tau)(1 - F(t)), \quad F(\tau) = 1 - e^{-\rho n_0 \tau}.$$

The solution is

$$F(t) = 1 - e^{-\rho n_0 \tau - \int_\tau^t \rho N(s;\tau,k)\ \mathrm{d}s}, \tag{4.5}$$

which is the distribution F, when $t > \tau$.

The value of this species, if it never goes extinct, is $\int_0^\infty e^{-rt}\ \mathrm{d}t$. Subtracting the expected value of the species in (4.3) from it gives the expected loss from losing this species. Multiplying it by the number of species at the site, and summing over all the sites, gives the

[23] For now, we suppress the subscript i to avoid clutter.

expected total loss from extinctions,

$$\mathcal{W}((\tau_i, k_i)_{i=1}^s) := \sum_{i=1}^{s} \alpha_i \left(\int_0^\infty e^{-rt} \, \mathrm{d}t - \int_0^{\tau_i} \int_0^\dagger e^{-rt} \, \mathrm{d}t \rho n_{i,0} e^{-\rho n_{i,0} \dagger} \, \mathrm{d}\dagger \right.$$
$$\left. - \int_{\tau_i}^\infty \int_0^\dagger e^{-rt} \, \mathrm{d}t \rho N_i(\dagger; \tau_i, k_i) e^{-\rho n_{i,0}\tau_i - \int_{\tau_i}^\dagger \rho N_i(s;\tau_i,k_i) \, \mathrm{d}s} \, \mathrm{d}\dagger \right), \tag{4.6}$$

where N_i is given in Equation (4.1). We assume that the intensity rate $\rho = 0.03$ and the interest rate $r = 0.03$. Finally, we show in Appendix A.10 that a first order stochastic dominance argument shows that $\mathcal{W}$ satisfies Assumption 3, and in particular, that $\partial\mathcal{W}/\partial\tau_i > 0$ and $\partial\mathcal{W}/\partial k_i < 0$.

Results. We present the allocation of the budget of 10 billion euros between ecoregions in Figure 3. This figure shows that most of the restoration funds should be allocated to the Mediterranean area and to the Balkan Peninsula. The largest budget shares are allocated to Dinaric Mountains Mixed Forests (approximately 15 percent), Appenine Deciduous Montane Forests (11 percent), and Crete Mediterranean Forests (10 percent). Multiple ecoregions obtain virtually zero restoration funds. These regions include North Atlantic Moist Mixed Forests, Caledon Conifer Forests, and most ecoregions in Fennoscandia. In addition, the funds are allocated immediately (i.e. there is no waiting), which is partly explained by the cost specification and by the relatively fast post-investment restoration. The cost formulation only includes the initially lost habitat, which means that the marginal benefit of waiting consists only of the interest on the saved restoration cost. These results are similar to Luby *et al.* (2022), in which a relatively small fraction of areas is allocated funds.

The allocation of funds is driven by a combination of four factors: the number of species, the cost parameter, the fraction of initially lost habitat, and the total area of the ecoregion. As expected, if the initially lost habitat of the ecoregion is near zero (here, less than 0.1), there is little need for restoration and therefore the budget allocation for such an ecoregion is zero. However, when a region is allocated a positive budget (which happens when habitat loss is more than 10 percent), a combination of the above four factors is typically required to explain a high budget allocation. In many cases a high number of species tends to imply a large budget allocation, but in general, a high species count is not sufficient for high budget allocation. Instead, the unit cost and the initially lost habitat area must also be relatively low.

At the country level, the list of countries that receive most of the restoration funds are located in the Mediterranean. Greece is allocated around 1.7 billion, Spain 1.4 billion and Italy 1.4 billion euros. On the other extreme, Iceland, for example, receives virtually zero euros, and Finland, Norway, and Estonia less than 100 000 euros each. These allocations are a reflection of the fact that the high-budget-share ecoregions are located in the Mediterranean.

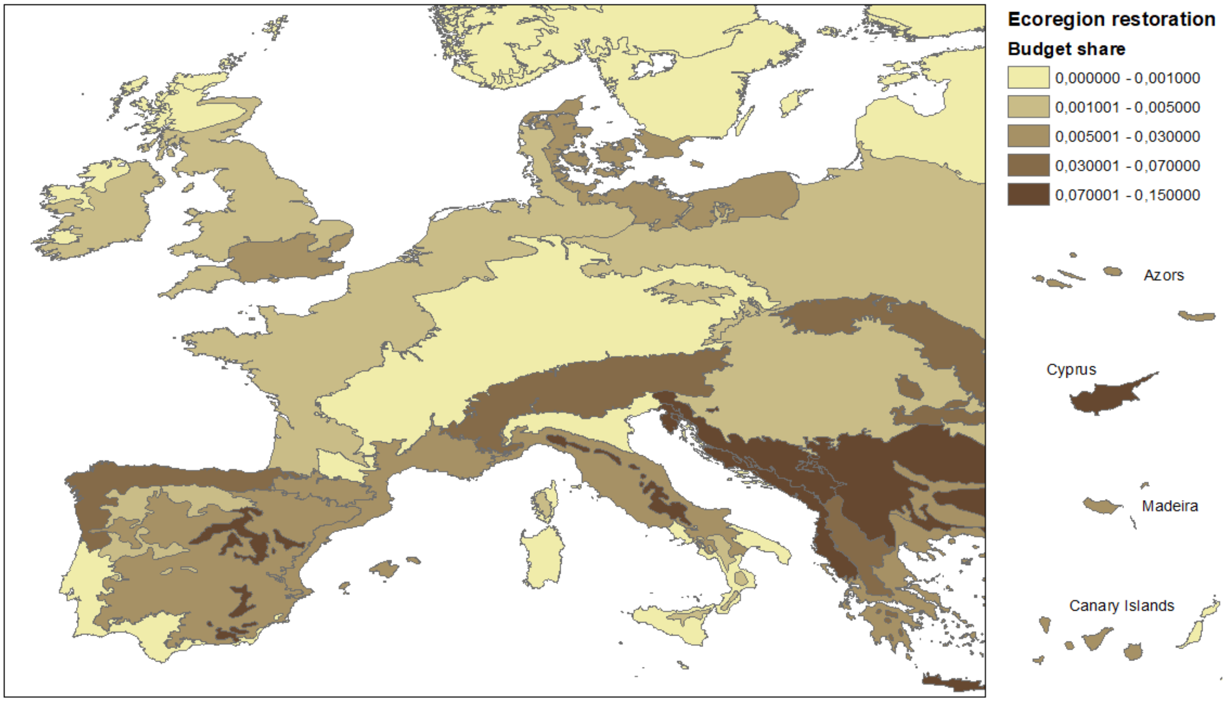

Figure 3. Optimal allocation of a restoration budget in Europe. This figure shows the optimal budget share for each terrestrial ecoregion when the restoration budget is 10 billion euros.

5. Concluding remarks

The paper formulates a dynamic model for the optimal restoration of damaged stocks, such as biodiversity, ecosystems, contaminated sites and carbon stocks. The model includes a fixed budget that can be used to finance restoration investments either at the beginning of the planning interval or after a delay, if such a delay results in a greater total benefit. The analysis yields optimal waiting and investment rules, along with a sufficient condition for restoring any damaged stock.

Our model is well-suited for planning biodiversity and ecosystem restoration and conservation, especially when there are multiple stocks or sites in need of restoration. Whether restoration should rely on natural processes or more active restoration actions is currently debated and studied in relation to restoration planning. In our dynamic model, we analyze the allocation of funds for restoration and conservation, considering both investments and natural regeneration. By integrating factors like delays, costly restoration actions, and their associated benefits, our model provides insights into determining the optimal approach between active restoration and restoration through natural processes. Furthermore, restoration or regeneration of natural forests is important for carbon storage and the model can address carbon stocks as well. Additionally, our model can be applied to the remediation of contaminated sites.

Future research should extend the theoretical model to allow for multiple restoration operations per site and budget additions. In addition, the budget in the current model is exogenous, which excludes trading off restoration benefits with benefits from other assets to which the funds could be invested. Furthermore, the stock dynamics should also be extended to cover further damages from economic activity as in many economic growth models on environment (see e.g. Menuet *et al.*, 2024). Moreover, research should also focus on quantifying the model and exploring its potential applications to policy issues related to ecosystem restoration, carbon stock growth, and the reclamation of contaminated sites at both national and international levels. Particular attention in the quantification should be paid to uncertainty regarding payoffs from restoration.

Appendix A.

A.1. **Proof of Lemma 2.1.** Recall that $N_i(t;\tau_i,k_i)$ is the solution to the initial value problem $\dot{N}_i(t) = -f_i(N_i(t),k_i)$ with $N_i(\tau_i;n_{i,0})$ as the initial value. Function $\partial N_i(t;\tau_i,k_i)/\partial k_i$ solves the linear differential equation

$$\dot{Z}(t) = -\frac{\partial f_i(N_i(t;\tau_i,k_i),k_i)}{\partial n_i}Z(t) - \frac{\partial f_i(N_i(t;\tau_i,k_i),k_i)}{\partial k_i}, \tag{A.1}$$

with the initial value

$$\frac{\partial N_i(\tau_i;\tau_i,k_i)}{\partial k_i} = 0, \tag{A.2}$$

see, e.g., Theorem 3.3 on the page 21 in Hale (1980). The time derivative of $\partial N_i(t;\tau_i,k_i)/\partial k_i$ at $t=\tau_i$ is

$$\frac{\partial}{\partial t}\frac{\partial N_i(t;\tau_i,k_i)}{\partial k_i}\bigg|_{t=\tau_i} = -\frac{\partial f_i(N_i(\tau_i;\tau_i,k_i),k_i)}{\partial k_i} < 0,$$

by (A3) of Assumption 1. If $\partial N_i(t;\tau_i,k_i)/\partial k_i = 0$ at some $t>\tau_i$, then Equation (A.1) implies that

$$\frac{\partial}{\partial t}\frac{\partial N_i(t;\tau_i,k_i)}{\partial k_i} < 0.$$

But this means that $\partial N_i(t;\tau_i,k_i)/\partial k_i$ cannot be zero at any $t>\tau_i$, since the initial value is 0 by (A.2). Hence $\partial N_i(t;\tau_i,k_i)/\partial k_i < 0$ for all $t>\tau_i$.

Partial derivative $\partial N_i(t;\tau_i,k_i)/\partial \tau_i$ (see, e.g., Theorem 3.3 on the page 21 in Hale (1980)) solves the linear differential equation

$$\dot{Z}(t) = -\frac{\partial f_i(N_i(t;\tau_i,k_i),k_i)}{\partial n_i}Z(t), \tag{A.3}$$

with the initial value

$$\frac{\partial N_i(\tau_i;\tau_i,k_i)}{\partial \tau_i} = f_i(N_i(\tau_i),k_i),$$

where $f_i(N_i(\tau_i),k_i) > 0$ by (A2) of Assumption 1. As two distinct solutions cannot intersect and $Z(t)\equiv 0$ is a solution to (A.3), we have $\partial N_i(t;\tau_i,k_i)/\partial \tau_i > 0$ for all $t>\tau_i$.

A.2. **Proof of Lemma 3.1.** Let $(\tau_i^*, k_i^*)_{i=1}^s$ be an optimal solution to (2.8)–(2.11). Suppose, on the contrary, that not all the remaining budget is used. That is, suppose $B(\tau_i^{*,-}) > C_i(k_i^*, N_i(\tau_i^*))$, where i is the last restored site. Then the value of the objective can be decreased by increasing k_i a little (without changing τ_i from τ_i^*), because $\partial\mathcal{W}/\partial k_i < 0$. Because a higher value for the investment is feasible due to inequality $B(\tau_i^{*,-}) > C_i(k_i^*, N_i(\tau_i^*))$, the original choice for the investment is not optimal.

Note that, as $b_0 > 0$, it cannot be optimal to not restore any of the sites. In such a case, by assumption (A7), the budget grows in finite time to cover a possible fixed cost of one of the sites, say of the site j, after which it is feasible to decrease the value of the objective function (2.8) from the case where no restoration is done by a small investment in the site j.

A.3. **Proof of Theorem 3.2.** It is assumed without a loss of generality that all sites are restored, that is, $\tau_i^* < \infty$ and $0 < k_i^*$, $i = 1, \dots, s$ (if not, then re-index sites from the first restored site to the last restored site, and apply the reasoning below for that list of sites).

Lemma A.1. *If a point $(\tau_i^*, k_i^*)_{i=1}^s$ is an optimal solution to the problem (2.8)–(2.11), then it is an optimal solution to the problem (2.12)–(2.14).*

Proof. Let $(\tau_i^*, k_i^*)_{i=1}^s$ be an optimal solution to the problem (2.8)–(2.11). Because the objectives of the problems are the same, it is enough to show that this point is a feasible point of the problem (2.12)–(2.14). Define a short-hand notation $C_i^* := C_i(k_i^*, N_i(\tau_i^*))$ and suppose first that all sites are restored at distinct times. Lemma 3.1 implies $B(\tau_s^{*,-}) = C_s^*$. Constraint $B(t) \geq 0$ for all t implies that $B(\tau_i^{*,+}) > 0$ for all $1 \leq i \leq s-1$, and, because all the money is spent, $B(\tau_s^{*,+}) = 0$. Equations (2.9) and (2.10) imply that for each site $1 \leq i \leq s-1$,

$$B(\tau_i^{*,-}) = C_i^* + B(\tau_i^{*,+}) \quad \text{and} \quad B(\tau_i^{*,+}) = B(\tau_{i+1}^{*,-})e^{-r(\tau_{i+1}^* - \tau_i^*)}. \tag{A.4}$$

In addition, $b_0 = B(\tau_1^{*,-})e^{-r\tau_1^*}$. Using these equations repeatedly gives

$$\begin{aligned} b_0 &= \left(C_1^* + B(\tau_1^{*,+})\right)e^{-r\tau_1^*} = \left(C_1^* + B(\tau_2^{*,-})e^{-r(\tau_2^* - \tau_1^*)}\right)e^{-r\tau_1^*} \\ &= C_1^* e^{-r\tau_1^*} + (C_2^* + B(\tau_2^{*,+}))e^{-r\tau_2^*} \\ &= C_1^* e^{-r\tau_1^*} + C_2^* e^{-r\tau_2^*} + B(\tau_3^{*,-})e^{-r(\tau_3^* - \tau_2^*)}e^{-r\tau_2^*} = \dots \\ &= \sum_{i=1}^s C_i^* e^{-r\tau_i^*}, \end{aligned}$$

as required.

Suppose then that some sites are restored on the same dates. Let $j \in \{1, \dots, s\}$ be any site that is restored on the same date as some other site, and let J_j be the set of sites that are restored on the same date τ_j as the site j. Then $B(\tau_j^{*,-}) = \sum_{i \in J_j} C_i^* + B(\tau_j^{*,+})$. Using this equation in a similar way as above for any set of sites that are restored simultaneously gives $b_0 = \sum_{i=1}^s C_i^* e^{-r\tau_i^*}$.

Hence $(\tau_i^*, k_i^*)_{i=1}^s$ solves also the problem (2.12)–(2.14). □

Lemma A.2. *If a point $(\tau_i^*, k_i^*)_{i=1}^s$ is an optimal solution to the problem (2.12)–(2.14), then it is an optimal solution to the problem (2.8)–(2.11).*

Proof. Suppose that the point $(\tau_i^*, k_i^*)_{i=1}^s$ solves the problem (2.12)–(2.14) and that all τ_i^*s are distinct. Any point $(\tau_i, k_i)_{i=1}^s$ that solves equations

$$B(\tau_i^-) = C_i(k_i, N_i(\tau_i)) + B(\tau_i^+), \quad \text{and} \quad B(\tau_i^+) = B(\tau_{i+1}^-)e^{-r(\tau_{i+1}-\tau_i)},$$

for $1 \le i \le s-1$, and equations $B(\tau_s^-) = C_s(k_s, N_s(\tau_s))$ and $b_0 = B(\tau_1^-)e^{-r\tau_1}$, is a feasible point of the problem (2.8)–(2.11) as the above equations define B recursively at possible points of the discontinuity, i.e., at τ_is. It is shown that $(\tau_i^*, k_i^*)_{i=1}^s$ is such a point using the following recursive argument: define $B(\tau_s^{*,-}) := C_s^*$, and

$$B(\tau_i^{*,+}) := B(\tau_{i+1}^{*,-})e^{-r(\tau_{i+1}^*-\tau_i^*)} \quad \text{and} \quad B(\tau_i^{*,-}) := C_i^* + B(\tau_i^{*,+}), \tag{A.5}$$

for each $1 \le i \le s-1$. If $b_0 = B(\tau_1^{*,-})e^{-r\tau_1^*}$, then $(\tau_i^*, k_i^*)_{i=1}^s$ is a feasible point of problem (2.8)–(2.11). Note that

$$\begin{aligned} B(\tau_1^{*,+}) &= B(\tau_2^{*,-})e^{-r(\tau_2^*-\tau_1)^*} = \left(C_2^* + B(\tau_2^{*,+})\right)e^{-r(\tau_2^*-\tau_1)^*} \\ &= \left(C_2^* + B(\tau_3^{*,-})e^{-r(\tau_3^*-\tau_2)^*}\right)e^{-r(\tau_2^*-\tau_1)^*} = \dots \\ &= \sum_{i=2}^s C_i^* e^{-r(\tau_i^*-\tau_1^*)} = \left(\sum_{i=2}^s C_i^* e^{-r\tau_i^*}\right) e^{r\tau_1^*}. \end{aligned}$$

This and (A.5) imply then that $B(\tau_1^{*,-}) = C_1^* + \left(\sum_{i=2}^s C_i^* e^{-r\tau_i^*}\right) e^{r\tau_1^*}$. Thus $B(\tau_1^{*,-})e^{-r\tau_1^*} = b_0$ as (2.13) holds for $(\tau_i^*, k_i^*)_{i=1}^s$. Hence the point $(\tau_i^*, k_i^*)_{i=1}^s$ is a feasible point, and an optimal solution of the problem (2.8)–(2.11), since the objectives are the same.

If some sites are restored on the same date under the optimal solution of (2.12)–(2.14), then the definition (A.5) is redefined to include the sum of restoration costs that occur on the date with multiple restorations. □

Lemmas A.1 and A.2 prove the first part of the proposition. The second part (i.e., that the infima are the same) holds, since problems (2.8)–(2.11) and (2.12)–(2.14) have the same objective functions.

A.4. **Proof of Lemma 3.3.** We suppose that all the sites $i = 1, \dots, s$ are restored (if not, then we exclude the sites that are not restored from the the set of analyzed sites).

By the Fritz John conditions, there exists a vector

$$(\lambda_0, \lambda, \omega_1, \dots, \omega_s, \gamma_1, \dots, \gamma_s) \ne 0, \tag{A.6}$$

with $\lambda_0 \in \{0,1\}$, such that the following conditions hold for each $i = 1, \dots, s$:

$$\lambda_0 \frac{\partial \mathcal{W}}{\partial \tau_i} + \lambda \frac{\partial C_i(k_i, N_i(\tau_i))e^{-r\tau_i}}{\partial \tau_i} - \omega_i = 0, \tag{A.7}$$

$$\omega_i \geq 0, \quad -\tau_i \leq 0, \quad \omega_i \tau_i = 0, \tag{A.8}$$

$$\lambda_0 \frac{\partial \mathcal{W}}{\partial k_i} + \lambda \frac{\partial C_i(k_i, N_i(\tau_i))e^{-r\tau_i}}{\partial k_i} - \gamma_i = 0, \tag{A.9}$$

$$\gamma_i \geq 0, \quad -k_i \leq 0, \quad \gamma_i k_i = 0, \tag{A.10}$$

$$\sum_{i=1}^{s} C_i(k_i, N_i(\tau_i))e^{-r\tau_i} - b_0 = 0.$$

Suppose on the contrary to the claim that $\lambda_0 = 0$. If also $\lambda = 0$, then conditions (A.7) and (A.9) imply $\omega_i = 0$ and $\gamma_i = 0$ for all i, which contradicts (A.6). Hence, if $\lambda_0 = 0$, then $\lambda \neq 0$. Now, if $k_i > 0$ for any i, $\gamma_i = 0$ by (A.10). This, together with (A.9) and assumption (A5), gives a contradiction, i.e., $0 \neq \lambda \partial C_i(k_i, N_i(\tau_i))e^{-r\tau_i}/\partial k_i = 0$. Hence, if $\lambda_0 = 0$, then $k_i = 0$ for each i, which is not optimal by Lemma 3.1. Thus we can choose $\lambda_0 = 1$.

Furthermore, with $\lambda_0 = 1$, we get from (A.9) that $\lambda > 0$, since $\partial \mathcal{W}/\partial k_i < 0$ (assumption (A8)) and $\partial C_i(k_i, N_i(\tau_i))e^{-r\tau_i}/\partial k_i > 0$ (assumption (A5)).

If $\tau_i > 0$, (3.1) follows from (A.7) as $\lambda > 0$, $\omega_i = 0$ (by (A.8)) and $\partial \mathcal{W}/\partial \tau_i > 0$ (by assumption (A8)).

A.5. **Proof of Proposition 3.7.** We move a part of the budget from sites i to the site p. Let us denote this amount by $\epsilon \in [0, b_0)$. We will use this money at the finite time τ_p. The amount of effort $k_p(\epsilon) \geq 0$ we use on-site solves the equation

$$\epsilon = C_p(k_p, N_p(\tau_p))e^{-r\tau_p} \tag{A.11}$$

with respect to k_p, as the budget (2.13) must be balanced, i.e.,

$$b_0 = C_p(k_p(\epsilon), N_p(\tau_p))e^{-r\tau_p} + \sum_{i=1}^{s} C_i(k_i^*, N_i(\tau_i^*))e^{-r\tau_i^*} - \epsilon.$$

For a given ϵ there is an optimal solution to (A.11), since the cost function C_p is strictly increasing in the k_p variable, by (A5) in Assumption 2, and we assume that fixed costs are zero.

At the same time the loss at original sites i increases as we take away an ϵ amount from the initial budget b_0. Altogether the loss changes at least by

$$\Delta(\epsilon) := \mathcal{W}((\tau_i^*, k_i^*)_{i \neq j}, (\tau_j^*, k_j(\epsilon)), (\tau_p, k_p(\epsilon))) - \mathcal{W}((\tau_i^*, k_i^*)_{i=1}^{s}, (\tau_p, 0)),$$

where we take a small amount of effort away at the site j with $\tau_j^* < \infty$ to compensate for the loss of ϵ amount of money. Similarly, as in site p, the amount of effort $0 < k_j(\epsilon) \leq k_j^*$ we use on-site solves the equation

$$b_j - \epsilon = C_j(k_j, N_j(\tau_j^*))e^{-r\tau_j^*} \tag{A.12}$$

with respect to k_j, where b_j is the part of the budget initially used in the site j.

Our goal is to show that $\Delta(\epsilon) < 0$ for some ϵ, which means that the full loss decreases when we move money from the site j to the site p. In other words, money taken away from the site j is more valuable in the site p. We will prove our claim using a linear approximation.

We take the right derivative of Δ at 0. To this end, note that $\lim_{\epsilon\to 0+} k_p(\epsilon) = 0$ follows by (A.11) as C_p is strictly increasing in the k_p variable, by (A5), and we assume that fixed costs are zero. Thus

$$\Delta'(0) = \frac{\partial \mathcal{W}((\tau_i^*, k_i^*)_{i=1}^s, (\tau_p, 0))}{\partial k_p} k_p'(0) + \frac{\partial \mathcal{W}((\tau_i^*, k_i^*)_{i=1}^s, (\tau_p, 0))}{\partial k_j} k_j'(0), \tag{A.13}$$

where $k_p'(0)$ and $k_j'(0)$ are the right derivatives of $k_p = k_p(\epsilon)$ and $k_j = k_j(\epsilon)$ (if they exist).

Now, deriving the right derivative at 0 in (A.11) gives

$$\frac{\partial}{\partial k_p} C_p(0, N_p(\tau_p))e^{-r\tau_p}\ k_p'(0) = 1. \tag{A.14}$$

Note that the right derivative $\partial C_p(0, N_p(\tau_p))/\partial k_p$ exists by assumption. Thus, if

$$\frac{\partial}{\partial k_p} C_p(0, N_p(\tau_p)) > 0,$$

the right derivative of $k_p = k_p(\epsilon)$ exists and

$$k_p'(0) = \frac{1}{\frac{\partial}{\partial k_p} C_p(0, N_p(\tau_p))e^{-r\tau_p}}. \tag{A.15}$$

Note that similarly as in the case of the site p, in (A.12) we take the right derivative for $k_j(\epsilon)$ at $\epsilon = 0$, where the derivative $\partial C_j(k_j^*, N_j(\tau_j^*))e^{-r\tau_j^*}/\partial k_j$ exists and is positive, since $k_j^* > 0$ as an optimal effort.

As (τ_j^*, k_j^*) is the optimal date and effort, with $k_j^* > 0$, we have by Fritz John conditions, see Lemma 3.3 and Equation (A.9),

$$\frac{\partial \mathcal{W}((\tau_i^*, k_i^*)_{i=1}^s, (\tau_p, 0))}{\partial k_j} = -\lambda^* \frac{\partial C_j(k_j^*, N_j(\tau_j^*))e^{-r\tau_j^*}}{\partial k_j}, \tag{A.16}$$

where $\lambda^* > 0$.

We can deduce our claim using a linear approximation. Note first that $\Delta(0) = 0$ and that, for $\epsilon > 0$,

$$\frac{\Delta(\epsilon)}{\epsilon} = \frac{\Delta(\epsilon) - \Delta(0)}{\epsilon}.$$

We have, by (A.13), (A.15) and (A.16),

$$\Delta'(0) = \frac{\mathcal{W}((\tau_i^*, k_i^*)_{i=1}^s, (\tau_p, 0))}{\partial k_p} \left(\frac{1}{\frac{\partial}{\partial k_p} C_p(0, N_p(\tau_p)) e^{-r\tau_p}} \right) + \lambda^* < 0$$

The inequality follows by our assumption (3.9). Thus

$$\frac{\Delta(\epsilon)}{\epsilon} \to \Delta'(0) < 0, \qquad \text{when } \epsilon \to 0+$$

and we have shown that $\Delta(\epsilon) < 0$ at least when ϵ is close to 0.

We are left to study the case when $\partial C_p(0, N_p(\tau_p))/\partial k_p = 0$. Since the right derivative of $C_p(\cdot, N_p(\tau_p))$ at 0 exists and $C_p(k_p, N_p(\tau_p)) \geq 0$ for every $k_p > 0$ with $C_p(0, N_p(\tau_p)) = 0$, by our no fixed costs assumption and Assumption 2, if

$$\frac{\partial}{\partial k_p} C_p(0, N_p(\tau_p)) = 0,$$

then we see from (A.14) that the right derivative of $k_p = k_p(\epsilon)$ diverges to the positive infinity

$$k_p'(0) = \infty, \tag{A.17}$$

i.e., $k_p'(\epsilon) \geq M$ for any $M \geq 0$, whenever ϵ is close enough to 0. As we have already noticed in (A.16),

$$\frac{\partial \mathcal{W}((\tau_i^*, k_i^*)_{i=1}^s, (\tau_p, 0))}{\partial k_j} k_j'(0) = \lambda^*$$

and it is a real number. Now, since $\partial \mathcal{W}/\partial k_p < 0$ by (A8), we deduce from (A.13) and (A.17) that

$$\frac{\Delta(\epsilon) - \Delta(0)}{\epsilon} = \frac{\Delta(\epsilon)}{\epsilon} \to -\infty + \frac{\partial \mathcal{W}((\tau_i^*, k_i^*)_{i=1}^s, (\tau_p, 0))}{\partial k_j} k_j'(0) < 0$$

when $\epsilon \to 0+$.

Thus we have shown also in this case that $\Delta(\epsilon) < 0$ at least when ϵ is close to 0.

A.6. **Claim on page 20.** Here we prove the claim on page 20 that equations (3.14) and (3.15) hold for our example, where damaged stocks are contaminant stocks (2.4). Indeed, $W_i(\tau_i, 0) = W_i(\infty, 0) = \lim_{\tau_i \to \infty} W_i(\tau_i, 0)$ as

$$\begin{aligned} W_i(\tau_i, 0) &= \int_0^{\tau_i} D_i(N_i(t)) e^{-rt} \, \mathrm{d}t + \int_{\tau_i}^{\infty} \underbrace{D_i(N_i(t; \tau_i, 0))}_{D_i(N_i(t))} e^{-rt} \, \mathrm{d}t \\ &= \int_0^{\infty} D_i(N_i(t)) e^{-rt} \, \mathrm{d}t = W_i(\infty, 0). \end{aligned} \tag{A.18}$$

Here $D_i(N_i(t;\tau_i,0)) = D_i(N_i(t))$, because $f_i(n_i,0) = h_i(n_i)$ in the stock dynamics (2.1), by our assumption (A1) in Assumption 1. Moreover,

$$\begin{aligned}\lim_{\tau_i\to\infty} W_i(\tau_i,k_i) &= \lim_{\tau_i\to\infty}\left(\int_0^{\tau_i} D_i(N_i(t))e^{-rt}\,\mathrm{d}t + \int_{\tau_i}^{\infty} D_i(N_i(t;\tau_i,k_i))e^{-rt}\,\mathrm{d}t\right)\\ &= W_i(\infty,0) + \lim_{\tau_i\to\infty}\left(\int_{\tau_i}^{\infty} D_i(N_i(t;\tau_i,k_i))e^{-rt}\,\mathrm{d}t\right) = W_i(\infty,0)\end{aligned}$$

and

$$\begin{aligned}\lim_{k_i\to 0} W_i(\tau_i,k_i) &= \lim_{k_i\to 0}\left(\int_0^{\tau_i} D_i(N_i(t))e^{-rt}\,\mathrm{d}t + \int_{\tau_i}^{\infty} D_i(N_i(t;\tau_i,k_i))e^{-rt}\,\mathrm{d}t\right)\\ &= \int_0^{\tau_i} D_i(N_i(t))e^{-rt}\,\mathrm{d}t + \int_{\tau_i}^{\infty} D_i(N_i(t;\tau_i,0))e^{-rt}\,\mathrm{d}t = W_i(\infty,0),\end{aligned}$$

where the limit can be taken inside the integral by the dominated convergence theorem and the last equality follows as in (A.18). Above all the integrals converge and we can use the dominated convergence theorem, since $W_i(\tau_i,k_i)$ is bounded

$$\begin{aligned}0 &\le \int_{\tau_i}^{\infty} D_i(N_i(t;\tau_i,k_i))e^{-rt}\,\mathrm{d}t\\ &\le \underbrace{\int_0^{\tau_i} D_i(N_i(t))e^{-rt}\,\mathrm{d}t + \int_{\tau_i}^{\infty} D_i(N_i(t;\tau_i,k_i))e^{-rt}\,\mathrm{d}t}_{W_i(\tau_i,k_i)}\\ &\le \int_0^{\infty} D_i(N_i(t))e^{-rt}\,\mathrm{d}t\end{aligned} \tag{A.19}$$

and the indefinite integral

$$\lim_{\tau_i\to\infty} W_i(\tau_i,0) = \int_0^{\infty} D_i(N_i(t))e^{-rt}\,\mathrm{d}t \tag{A.20}$$

converges, as $D_i(N_i(t))$ is bounded.

A.7. **Proof of Theorem 3.10.** In this section, we will prove that the budget allocation problem (3.10) is equivalent with optimization problems (2.12)–(2.14) and (2.8)–(2.11) when $\mathcal{W}$ satisfies

$$\mathcal{W}((\tau_i,k_i)_{i=1}^s) = \sum_{i=1}^s W_i(\tau_i,k_i). \tag{A.21}$$

We have already shown in the above Lemmas A.1 and A.2 that the discounted cost problem (2.12)–(2.14) and the original problem (2.8)–(2.11) are equivalent. Thus we only need to look at the equivalence of the budget allocation problem and the discounted cost problem (2.12)–(2.14), i.e., the problem

$$\begin{aligned}&\inf_{\left\{(\tau_i,k_i)_{i=1}^s\right\}} \sum_{i=1}^s W_i(\tau_i,k_i)\\ &\text{s.t.}\quad b_0 = \sum_{i=1}^s C_i(k_i,N_i(\tau_i))e^{-r\tau_i},\end{aligned}$$

$$\tau_i \geq 0, \quad k_i \geq 0, \quad \text{for all } i.$$

We first show that both problems have the same optimal value, that is, the infima are the same. After this we show that points where these infima are attained are connected by (3.16).

Denote

$$\beta_i(\tau_i, k_i) := C_i(k_i, N_i(\tau_i))e^{-r\tau_i}$$

and note that the optimal value (given a budget b_i) for the site i is

$$V_i(b_i) = \inf_{\{\tau_i, k_i\}} \{W_i(\tau_i, k_i) : \beta_i(\tau_i, k_i) = b_i\}. \tag{A.22}$$

Now, our budget constraint (2.13), i.e., $b_0 = \sum_{i=1}^s C_i(k_i, N_i(\tau_i))e^{-r\tau_i}$, can be written as $b_0 = \sum_{i=1}^s \beta_i(\tau_i, k_i)$ and the optimization problem (2.12)–(2.14) as

$$\inf_{\substack{b_i \in \mathcal{R}_i \\ b_0 = \sum_{i=1}^s b_i}} \inf_{\left\{(\tau_i, k_i)_{i=1}^s\right\}} \sum_{i=1}^s \{W_i(\tau_i, k_i) : \beta_i(\tau_i, k_i) = b_i\}, \tag{A.23}$$

where $\mathcal{R}_i$ is the range of the function $(\tau_i, k_i) \mapsto \beta_i(\tau_i, k_i)$ defined on $[0, \infty) \times [0, \infty)$.

The fact that the infimum of a sum is the sum of infima and the equation (A.22) give

$$\inf_{\left\{(\tau_i, k_i)_{i=1}^s\right\}} \sum_{i=1}^s \{W_i(\tau_i, k_i) : \beta_i(\tau_i, k_i) = b_i\} = \sum_{i=1}^s V_i(b_i). \tag{A.24}$$

Thus the infimum of the discounted cost problem in (A.23) equals

$$\inf_{\substack{b_i \in \mathcal{R}_i \cap [0, b_0] \\ b_0 = \sum_{i=1}^s b_i}} \sum_{i=1}^s V_i(b_i),$$

because $b_i \in [0, b_0]$ for each i.

Actually, we can take the infimum without the above restriction on the range.

Lemma A.3. *The discounted cost problem* (2.12)–(2.14) *has the same value with the budget allocation problem*

$$\inf_{\substack{b_i \in [0, b_0] \\ b_0 = \sum_{i=1}^s b_i}} \sum_{i=1}^s V_i(b_i), \tag{A.25}$$

where $V_i \colon [0, b_0] \to \mathbb{R}$*,*

$$V_i(b_i) = \begin{cases} \inf_{\{\tau_i, k_i\}} \{W_i(\tau_i, k_i) : \beta_i(\tau_i, k_i) = b_i\}, & b_i > 0, \\ \lim_{\tau_i \to \infty} W_i(\tau_i, 0) = W_i(\infty, 0), & b_i = 0. \end{cases} \tag{A.26}$$

Proof. Recall that $\mathcal{R}_i$ is the range of the function $(\tau_i, k_i) \mapsto \beta_i(\tau_i, k_i)$ defined on $[0, \infty) \times [0, \infty)$.

Claim 1. The following hold for the function β_i:

a) The range $\mathcal{R}_i \cap [0, b_0]$ is $(0, b_0]$ or $[0, b_0]$.

b) If $\beta_i(\tau_i, k_i) \to 0$, then $k_i \to 0$ or $\tau_i \to \infty$.

Proof of Claim 1. Note that, $C_i(k_i, N_i(\tau_i))$ is nonnegative and strictly increasing and unbounded in the k_i variable, by assumptions (A5) and (A6) in Assumption 2. Moreover, $\lim_{\tau_i \to \infty} C_i(0, N(\tau_i))e^{-r\tau_i} = 0$, by (A7) in Assumption 2. Thus the continuous function $(\tau_i, k_i) \mapsto \beta_i(\tau_i, k_i) = C_i(k_i, N_i(\tau_i))e^{-r\tau_i}$ defined on $[0, \infty) \times [0, \infty)$ has the range $\mathcal{R}_i$ is $[0, \infty)$ or $(0, \infty)$ and a) follows.

Since the stock $N_i(\tau_i) > 0$ for every τ_i, the cost function C_i is strictly positive for $k_i > 0$. Hence $\beta_i(\tau_i, k_i) = 0$, if and only if $k_i = 0$ and fixed costs are zero. Thus the point 0 belongs to the range, if there are no fixed costs, that is, $C_i(0, N_i(\tau_i)) = 0$. Moreover, $\beta_i(\tau_i, k_i) \to 0$ implies that $k_i \to 0$ (if fixed costs are zero) or that $\tau_i \to \infty$. ☐

We will show that we do not need to restrict the range $\mathcal{R}_i$ and actually the infimum can be taken over $[0, b_0]$.

We (re)define our value function $V_i\colon [0, b_0] \to \mathbb{R}$ using (A.26). Here the definition for $b_i > 0$ is the same as before in (A.22). We have to show that the case of $b_i = 0$ gives the same value as the definition (A.22) and that our mapping has only real values.

Recall that $\beta_i(\tau_i, 0) = 0$, if fixed costs are zero. Moreover, $\beta_i(\tau_i, k_i) \to 0$ implies that $k_i \to 0$ (if fixed costs are zero) or that $\tau_i \to \infty$ as noted in Claim 1. We show that the function V_i is well-defined by (A.26) at $b_i = 0$. We have by (3.14)

$$W_i(\infty, 0) = \lim_{\tau_i \to \infty} W_i(\tau_i, 0) = \lim_{\tau_i \to \infty} W_i(\tau_i, k_i) = \lim_{\tau_i \to \infty, k_i \to \infty} W_i(\tau_i, k_i),$$

where the last limit is taken with feasible points, i.e., (τ_i, k_i) such that $\beta_i(\tau_i, k_i) \leq b_0$. If there are no fixed costs, by (3.15), also

$$W_i(\tau_i, 0) = W_i(\infty, 0) = \lim_{k_i \to 0} W_i(\tau_i, k_i)$$

holds. Thus we have shown above that V_i can be defined at 0 so that the infimum can be taken over the whole interval $[0, b_0]$. ☐

We are left to show that for optimal points the equality (3.16) holds. Lemma A.3 shows the claim ($b.$) of Theorem 3.10, i.e., infima are the same for all three problems, since problems (2.8)–(2.11) and (2.12)–(2.14) have the same objectives and we have already shown that their optimal solutions are equivalent (Theorem 3.2). Next we prove the equivalence of optimization problems.

Lemma A.4. *Assume that* $\lim_{\tau_i \to \infty} C_i(k_i, N(\tau_i))e^{-r\tau_i} = 0$. *The budget allocation problem* (A.25) *is equivalent to the discounted cost problem* (2.12)–(2.14). *Namely,*

- *If* $(b_i^*)_{i=1}^s \in [0, b_0]^s$ *is an optimal solution to the budget allocation problem* (A.25), *then*

$(\tau_i^*, k_i^*)_{i=1}^s \in ([0,\infty] \times [0,\infty))^s$ is an optimal solution to the discounted cost problem (2.12)–(2.14), where $(\tau_i^*, k_i^*)_{i=1}^s$ is such that

$$W_i(\tau_i^*, k_i^*) = V_i(b_i^*) \quad \textit{for each} \quad i. \tag{A.27}$$

Here we set $\tau_i^* = \infty$ and $k_i^* = 0$, if $b_i^* = 0$.

- *If $(\tau_i^*, k_i^*)_{i=1}^s \in ([0,\infty] \times [0,\infty])^s$ is an optimal solution to the discounted cost problem (2.12)–(2.14), then $(\beta_i(\tau_i^*, k_i^*))_{i=1}^s \in [0, b_0]^s$ is an optimal solution to the budget allocation problem (A.25).*

Note that $(\tau_j^*, k_j^*) = (\infty, \infty)$ for some site j can be a point in the solution set to the optimization problem (2.12)–(2.14). In this case, though, the budget b_j must be zero as otherwise one would use money and not to restore at all as an optimal solution, which cannot hold (Lemma 3.1). This is the only case where $k^* = \infty$ as the cost function is strictly increasing and unbounded by assumptions (A5) and (A6) in Assumption 2.

Proof. Let $(b_i^*)_{i=1}^s \in [0, b_0]^s$ be an optimal solution to the budget allocation problem (A.25).

Claim 2. There is $(\tau_i^*, k_i^*)_{i=1}^s \in ([0,\infty] \times [0,\infty))^s$ such that (A.27) holds, i.e.,

$$W_i(\tau_i^*, k_i^*) = V_i(b_i^*), \quad \text{for each} \quad i.$$

Proof of Claim 2. If $b_i^* > 0$, then the infimum in (A.26) is attained at a point with $\tau_i < \infty$. Indeed, this follows as W_i is a continuous function, and the boundary values are the same, i.e., $\lim_{\tau_i \to \infty} W_i(\tau_i, k_i) = W_i(\infty, 0)$ by (3.14). But a point with $\tau_i = \infty$ cannot be the infimum as we assume $b_i^* > 0$, in which case it is optimal to restore, see Lemma 3.1. If $k_i \to \infty$, then $\tau_i \to \infty$ as $\beta_i(\tau_i, k_i) = C_i(k_i, N_i(\tau_i))e^{-r\tau_i} \le b_0$ and C_i is strictly increasing and unbounded in the k_i variable, as we already noted. Hence the infimum cannot be attained when variables increase without boundary/go to the infinity.

If $b_i^* = 0$, the claim follows by our definition (A.27). □

The choice of $(\tau_i^*, k_i^*)_{i=1}^s \in ([0,\infty] \times [0,\infty))^s$ such that (A.27) holds is not necessary unique, but the following argumentation works for all choices.

If $b_j^* \notin \mathcal{R}_j$, for some j, we need an approximation argument because even though $(b_i^*)_{i=1}^s$ is an optimal solution to the budget allocation problem (A.25) and thus especially $b_0 = \sum_{i=1}^s b_i^*$, it may happen that $b_j^* = 0 \notin \mathcal{R}_j$ for some j, as we noticed in Claim 1 and thus $(b_i^*)_{i=1}^s$ does not belong to the feasible set of the discounted cost problem, see (A.23). If $b_i^* > 0$, b_i^* belongs to the range.

Claim 3. Let $(\tau_i^*, k_i^*)_{i=1}^s \in ([0,\infty] \times [0,\infty])^s$ and $0 < b_0 = \sum_{i=1}^s \beta_i(\tau_i^*, k_i^*)$ (here $\beta_i(\infty, k_i) = 0$). There exist points $(\tau_i^n, k_i^n) \in [0,\infty) \times [0,\infty)$ such that

$$b_0 = \sum_{i=1}^s \beta_i(\tau_i^n, k_i^n), \quad \lim_{n\to\infty} (\tau_i^n, k_i^n) = (\tau_i^*, k_i^*), \quad \lim_{n\to\infty} W_i(\tau_i^n, k_i^n) = W_i(\tau_i^*, k_i^*). \tag{A.28}$$

Proof of Claim 3. If $\beta_i(\tau_i^*, k_i^*) > 0$ for all i, we can take $(\tau_i^n, k_i^n) = (\tau_i^*, k_i^*)$, since $\tau_i^* < \infty$ as an optimal point (see Lemma 3.1).

If $\beta_j(\tau_j^*, k_j^*) = 0$ for $j \in \mathcal{J}$, where $\mathcal{J}$ has at least one element, there is at least one site i_0 where one restores, that is, $\beta_{i_0}(\tau_{i_0}^*, k_{i_0}^*) > 0$, $\tau_{i_0}^* < \infty$ and $k_{i_0}^* > 0$ (see Lemma 3.1).

For $j \in \mathcal{J}$, there are two options: $k_j^* = 0$ or $\tau_j^* = \infty$ (see Claim 1). If $\tau_j^* < \infty$, we can take $(\tau_j^n, k_j^n) = (\tau_j^*, k_j^*)$. If $\tau_j^* = \infty$, for any $\epsilon > 0$, we can choose numbers τ_j^n large enough so that $\beta_j(\tau_j^n, k_j^n) < \epsilon$ for all n and such that $(\tau_j^n, k_j^n) \to (\infty, k_j^*) = (\tau_j^*, k_j^*)$. Let us choose ϵ so small that $\epsilon|\mathcal{J}| < b_0$.

For site i_0, where we restore, as β_{i_0} is strictly increasing and continuous in the k_{i_0} variable, by (A5), we have, for all sequences $(k_{i_0}^n)_{n=1}^\infty$ that converge to $k_{i_0}^*$ and for which $k_{i_0}^n < k_{i_0}^*$ holds, that

$$\beta_{i_0}(k_{i_0}^n, \tau_{i_0}^*) < \beta_{i_0}(k_{i_0}^*, \tau_{i_0}^*) \quad \text{and} \quad \lim_{n\to\infty} \beta_{i_0}(k_{i_0}^n, \tau_{i_0}^*) = \beta_{i_0}(k_{i_0}^*, \tau_{i_0}^*).$$

Since

$$\sum_{j\in\mathcal{J}} \beta_j(\tau_j^n, k_j^n) < \sum_{j\in\mathcal{J}} \epsilon = \epsilon|\mathcal{J}| < b_0,$$

we can choose a sequence $(k_{i_0}^n)_{n=1}^\infty$ such that

$$\beta_{i_0}(k_{i_0}^*, \tau_{i_0}^*) - \beta_{i_0}(k_{i_0}^n, \tau_{i_0}^*) = \sum_{j\in\mathcal{J}} \beta_j(\tau_j^n, k_j^n).$$

For all the other $i \notin \mathcal{J}$, we take $(\tau_i^n, k_i^n) = (\tau_i^*, k_i^*)$.

Now, $b_0 = \sum_{i=1}^s \beta_i(\tau_i^n, k_i^n)$ and $\lim_{n\to\infty}(\tau_i^n, k_i^n) = (\tau_i^*, k_i^*)$. Since function W_i is continuous for every i and it has the limit, when $\tau_i \to \infty$, by (3.14), our claim follows. ☐

Now,

$$\begin{aligned}
\sum_{i=1}^s W_i(\tau_i^n, k_i^n) &\geq \sum_{i=1}^s \inf_{\{\tau_i, k_i\}} \{W_i(\tau_i, k_i) : \beta_i(\tau_i, k_i) = \beta_i(\tau_i^n, k_i^n)\} \\
&= \inf_{\left\{(\tau_i, k_i)_{i=1}^s\right\}} \sum_{i=1}^s \{W_i(\tau_i, k_i) : \beta_i(\tau_i, k_i) = \beta_i(\tau_i^n, k_i^n)\} \\
&\geq \inf_{\substack{b_i \in \mathcal{R}_i \\ b_0 = \sum_{i=1}^s b_i}} \inf_{\left\{(\tau_i, k_i)_{i=1}^s\right\}} \sum_{i=1}^s \{W_i(\tau_i, k_i) : \beta_i(\tau_i, k_i) = b_i\}
\end{aligned} \tag{A.29}$$

where the last inequality follows from Claim 3 as it implies that $\beta_i(\tau_i^n, k_i^n)$ belongs to the feasible set of the discounted cost problem (A.23) (as $b_0 = \sum_{i=1}^s \beta_i(\tau_i^n, k_i^n)$). By Lemma A.3 and Claim 2, the right side in (A.29) is equal to

$$\inf_{\substack{b_i \in [0, b_0] \\ b_0 = \sum_{i=1}^s b_i}} \sum_{i=1}^s V_i(b_i) = \sum_{i=1}^s V_i(b_i^*) = \sum_{i=1}^s W_i(\tau_i^*, k_i^*). \tag{A.30}$$

Hence, we get

$$\sum_{i=1}^{s} W_i(\tau_i^n, k_i^n) \geq \sum_{i=1}^{s} W_i(\tau_i^*, k_i^*)$$

By Claim 3, $\lim_{n\to\infty} W_i(\tau_i^n, k_i^n) = W_i(\tau_i^*, k_i^*)$ and hence we have shown that we attain the infimum at $(\tau_i^*, k_i^*)_{i=1}^s$, that is,

$$\sum_{i=1}^{s} W_i(\tau_i^*, k_i^*) = \inf_{\substack{b_i \in \mathcal{R}_i \\ b_0 = \sum_{i=1}^{s} b_i}} \inf_{\left\{(\tau_i, k_i)_{i=1}^s\right\}} \sum_{i=1}^{s} \{W_i(\tau_i, k_i) : \beta_i(\tau_i, k_i) = b_i\}$$

and $(\tau_i^*, k_i^*)_{i=1}^s \in ([0,\infty] \times [0,\infty))^s$ is an optimal solution to (2.12)–(2.14), see (A.23).

For the other direction, we assume that $(\tau_i^*, k_i^*)_{i=1}^s \in ([0,\infty] \times [0,\infty])^s$ is an optimal solution to the optimization problem (2.12)–(2.14).

Assume that $\beta_j(\tau_j^*, k_j^*) = 0$ for some j (note that this is the only case in which τ_j^* or k_j^* can be ∞). As before, we have a sequence of feasible points $(\tau_i^n, k_i^n)_{i=1}^s$ such that $\tau_j^n \to \infty$ as $n \to \infty$ and $k_j^n = 0$, since the same optimal value is attained also for $\tau_j^* = \infty$ and $k_j^* = 0$, as we noticed in the proof of Lemma A.3. The looked-for sequence of feasible points can be found by Claim 3.

If $\beta_i(\tau_i^*, k_i^*) > 0$ for all i, we take $\tau_i^n = \tau_i^*$ and $k_i^n = k_i^*$ for all n.

Now, $(\beta_i(\tau_i^*, k_i^*))_{i=1}^s \in [0, b_0]^s$ is well-defined (we set $\beta_i(\infty, k_i^*) = 0$). Our goal is to show that $(\beta_i(\tau_i^*, k_i^*))_{i=1}^s$ is an optimal solution to the budget allocation problem (A.25).

Since our sequence is feasible, (A.29) continues to hold, and the right side of (A.29) equals $\sum_{i=1}^s W_i(\tau_i^*, k_i^*)$. In addition,

$$\sum_{i=1}^{s} \inf_{\{\tau_i, k_i\}} \{W_i(\tau_i, k_i) : \beta_i(\tau_i, k_i) = \beta_i(\tau_i^n, k_i^n)\} = \sum_{i=1}^{s} V_i(\beta_i(\tau_i^n, k_i^n)), \tag{A.31}$$

and therefore

$$\sum_{i=1}^{s} W_i(\tau_i^n, k_i^n) \geq \sum_{i=1}^{s} V_i(\beta_i(\tau_i^n, k_i^n)) \geq \sum_{i=1}^{s} W_i(\tau_i^*, k_i^*). \tag{A.32}$$

We want to take the limit when $n \to \infty$. To take the limit inside functions, we need the continuity. By assumptions, W_i and β_i are continuous. We will show the continuity of V_i.

Claim 4. V_i defined in (A.26) is continuous.

Proof of Claim 4. We will modify the proof of a so-called maximum theorem (see, e.g., p. 306 in Ok (2007)). The maximum theorem in question implies the continuity of V_i, but our β_i does not fulfil all assumptions of the theorem. In assumptions of the maximum theorem, one has that W_i is continuous (which holds by our definition) and that β_i^{-1} is a so-called upper and lower hemicontinuous, compact-valued correspondence. Our β_i^{-1} might not be lower hemicontinuous and, luckily, we do not need (in our case) the lower hemicontinuity at all points, if we look at the proof more carefully.

A key element in the continuity proof is that the correspondence

$$\sigma_i(b) := \arg\min\{W_i(\tau_i, k_i) : (\tau_i, k_i) \in \beta_i^{-1}(b)\}$$

is compact-valued and upper hemicontinuous at $b \in [0, b_0]$. We show this technical claim in Lemma A.6 below at the end of the section.

To show the continuity, we can follow Ok (2007). Pick any sequence $(b^m)_{m=1}^\infty$ such that $b^m \to b$ as $m \to \infty$. For continuity it would be enough to show that $\lim_{m\to\infty} V_i(b^m) = V_i(b)$.

$(V_i(b^m))_{m=1}^\infty$ has a subsequence so that $\lim_{j\to\infty} V_i(b^{m_j}) = \limsup_{m\to\infty} V_i(b^m)$. We pick any $(\tau^{m_j}, k^{m_j}) \in \sigma_i(b^{m_j})$ so that $W_i(\tau^{m_j}, k^{m_j}) = V_i(b^{m_j})$. Since σ_i is compact-valued and upper hemicontinuous, there is a subsequence of $(\tau^{m_j}, k^{m_j})_{j=1}^\infty$ that converges to a point (τ, k) in $\sigma_i(b)$, (Ok, 2007, Proposition 2, p. 290). We denote the subsequence in question by the same indexing m_j. Now, as W_i is continuous and (τ, k) in $\sigma_i(b)$

$$V_i(b^{m_j}) = W_i(\tau^{m_j}, k^{m_j}) \to W_i(\tau, k) = V_i(b),$$

as $j \to \infty$. Thus $V_i(b) = \limsup_{m\to\infty} V_i(b^m)$. A similar argumentation shows that $V_i(b) = \liminf_{m\to\infty} V_i(b^m)$ and we have shown the continuity of V_i. □

As W_i and β_i are continuous, their limits at the infinity exist and V_i is continuous (Claim 4), taking the limit in (A.32), when $n \to \infty$, gives

$$\sum_{i=1}^s W_i(\tau_i^*, k_i^*) = \sum_{i=1}^s V_i(\beta_i(\tau_i^*, k_i^*)) \geq \sum_{i=1}^s W_i(\tau_i^*, k_i^*).$$

By Lemma A.3, $\sum_{i=1}^s W_i(\tau_i^*, k_i^*)$ has the same value as the budget allocation problem (A.25) and hence we have shown that $(\beta_i(\tau_i^*, k_i^*))_{i=1}^s \in [0, b_0]^s$ is an optimal solution to the budget allocation problem (A.25). □

Now, Lemma A.4 finishes the proof of the claim $(b.)$ and we have shown Theorem 3.10.

In the above proof we needed the correspondence σ to be compact-valued and upper hemicontinuous to find a converging subsequence. We show this below in Lemma A.6. In that proof, we need certain properties of the correspondence β_i that we will prove next. We need that the costs behave reasonably when stocks go to zero, namely, for $k_i < \infty$,

$$\lim_{\tau_i\to\infty} \beta_i(\tau_i, k_i) = \lim_{\tau_i\to\infty} C_i(k_i, N(\tau_i))e^{-r\tau_i} = 0. \tag{A.33}$$

Note that we always assume that it is reasonable to restore at least one site, since (A7) holds, i.e.,

$$\lim_{\tau_i\to\infty} \beta_i(\tau_i, 0) = \lim_{\tau_i\to\infty} C_i(0, N(\tau_i))e^{-r\tau_i} = 0. \tag{A.34}$$

Lemma A.5. *Taking a preimage by β_i is a compact-valued, upper hemicontinuous correspondence between the metric space $[0, b_0]$ and the Riemann sphere (i.e., the one point compactification of $\mathbb{R}^2$).*

Also the correspondence β^{-1} from $[0, b_0]$ to the product of s Riemann spheres, where $\beta = \sum_{i=1}^{s} \beta_i$, is compact-valued and upper hemicontinuous.

Proof. Indeed, as β_i is continuous, the preimage $\beta_i^{-1}(b)$ is closed for every $b \in [0, b_0]$ and, as the Riemann sphere is a compact metric space, $\beta_i^{-1}(b)$ is a compact.

As $\beta = \sum_{i=1}^{s} \beta_i$, the same holds true for β as a sum of continuous maps.

For the upper hemicontinuity of β_i^{-1} one needs to show that for any sequence $(b^m)_{m=1}^{\infty}$ that converges to b in the interval $[0, b_0]$ and for any sequence $(\tau_i^m, k_i^m) \in \beta_i^{-1}(b^m)$, there is a converging subsequence $(\tau_i^m, k_i^m)_{m=1}^{\infty}$ such that the limit point belongs to the preimage $\beta_i^{-1}(b)$ (Ok, 2007, Proposition 2, p. 290).

Now, in our situation, if $(\tau_i^m, k_i^m)_{m=1}^{\infty}$ has a bounded subsequence, an accumulation point $(\tau_i, k_i) \in [0, \infty) \times [0, \infty)$ exists by the Bolzano-Weierstrass theorem, and the continuity of β_i shows the claim.

If there is not any bounded subsequence, we have an unbounded subsequence and there is an accumulation point (∞, k_i), where $k_i \in [0, \infty]$. Indeed, for the accumulation point, we must have $\tau_i = \infty$, since β_i is strictly increasing and unbounded on the k_i variable, by our assumptions (A5), (A6), and b^m is bounded.

Note that the point (∞, ∞) on the Riemann sphere belongs always to the preimage $\beta_i^{-1}(b)$ when $b \in [0, b_0]$. Indeed, if $(k^m)_{m=m_0}^{\infty}$ is such that $\beta_i(m, k_i^m) = b^m > 0$ and $b^m \to b > 0$ (such sequence exists as (A.34) and β_i is strictly increasing and unbounded on the k_i variable) and the sequence is bounded by L, then $0 \leq \beta_i(m, k_i^m) \leq \beta_i(m, L)$ and $\lim_{m\to\infty} \beta_i(m, L) = 0$, by (A.33), that is a contradiction with $b > 0$. Thus points k_i^m cannot be bounded. If $b = 0$, one sees that (∞, ∞) belongs to the set by studying the sequence $(\tau^m, m)_{m=1}^{\infty}$ such that $\beta_i(\tau^m, m) = b^m > 0$ and τ^ms cannot be bounded because $b^m \to 0$ and β_i is strictly increasing and unbounded on k_i as already noted.

If $k_i < \infty$, we have that $b = 0$, since we assume that our sequence is unbounded and (A.33) holds. When $b = 0$, actually the whole line from $(\infty, 0)$ to (∞, ∞) belongs to $\beta_i^{-1}(0)$: points (∞, k_i) belong to $\beta_i^{-1}(0)$ by (A.33).

Thus our correspondence β_i^{-1} is upper hemicontinuous and hence so is the sum β. □

We will state the following result for the general loss function $\mathcal{W}$ as we will also need properties of σ later in the envelope theorem.

Lemma A.6. *The correspondence*

$$\sigma(b) := \arg\min\{\mathcal{W}((\tau_i, k_i)_{i=1}^{s}) : \beta((\tau_i, k_i)_{i=1}^{s}) = b\}$$

is compact-valued, closed, and upper hemicontinuous at $b \in [0, b_0]$. Here $\beta((\tau_i, k_i)_{i=1}^{s}) = \sum_{i=1}^{s} C_i(k_i, N_i(\tau_i))e^{-r\tau_i} = \sum_{i=1}^{s} \beta_i(\tau_i, k_i)$.

Proof. Note that $\sigma(b) = \sigma(b) \cap \beta^{-1}(b)$ and, as $\beta^{-1}(b)$ is compact-valued and upper hemicontinuous (Lemma A.5), the upper hemicontinuity of σ follows by a standard argument once one has that σ has a closed graph. The closedness is the place where in the original proof in (Ok, 2007, p. 307) one uses the lower hemicontinuity of β^{-1}.

Let $b \in [0, b_0]$. To show the closedness, take any sequence $(b^m)_{m=1}^{\infty}$ such that $b^m \to b$ and points $(\tau_i^m, k_i^m)_{i=1}^s \in \sigma(b^m)$ such that $(\tau_i^m, k_i^m)_{i=1}^s \to (\tau_i, k_i)_{i=1}^s$ as $m \to \infty$. In particular, as $(\tau_i^m, k_i^m)_{i=1}^s \in \sigma(b^m)$, one has a minimal value of W_i at $(\tau_i^m, k_i^m)_{i=1}^s$ given a budget b^m for the site i. Now, σ has a closed graph, if $(\tau_i, k_i)_{i=1}^s \in \sigma(b)$.

If this does not hold, that is, $(\tau_i, k_i)_{i=1}^s \notin \sigma(b)$, there is $(\hat{\tau}_i, \hat{k}_i)_{i=1}^s \in \beta^{-1}(b)$ such that $\mathcal{W}((\hat{\tau}_i, \hat{k}_i)_{i=1}^s) < \mathcal{W}((\tau_i, k_i)_{i=1}^s)$. Now, we look for a sequence $(\hat{\tau}_i^m, \hat{k}_i^m)_{i=1}^s \in \beta^{-1}(b^m)$ such that $\lim_{m\to\infty}(\hat{\tau}_i^m, \hat{k}_i^m)_{i=1}^s = (\hat{\tau}_i, \hat{k}_i)_{i=1}^s$ and $\beta((\hat{\tau}_i, \hat{k}_i)_{i=1}^s) = b$ to get a contradiction. Indeed, as $\mathcal{W}$ is continuous we have that for m large enough $\mathcal{W}((\hat{\tau}_i^m, \hat{k}_i^m)_{i=1}^s) < \mathcal{W}((\tau_i^m, k_i^m)_{i=1}^s)$, too. This is a contradiction with our assumption that $(\tau_i^m, k_i^m)_{i=1}^s \in \sigma(b^m)$.

To find sequences $(\hat{\tau}_i^m, \hat{k}_i^m)_{i=1}^s \in \beta^{-1}(b^m)$ we can argue as follows. If $\hat{k}_i < \infty$, $\hat{\tau}_i < \infty$, for all i, and $0 < \hat{k}_j$ for some j, we have a looked-for sequence of the form $(\hat{\tau}_j, \hat{k}_j^m) \in \beta_j^{-1}(b^m)$ as β_j is strictly increasing and unbounded on the k_j variable, by our assumptions (A5), (A6). For other is one can take $(\hat{\tau}_i^m, \hat{k}_i^m) = (\hat{\tau}_i, \hat{k}_i)$.

Note that, if $\hat{\tau}_j = \infty$ for some j, then $\beta_j(\hat{\tau}_j, \hat{k}_j) = 0$ as it is not optimal to allocate money to a site and not to use it for restoration (Lemma 3.1). One finds sequences similarly as in Claim 3: the first equation in (A.28) holds for b^m in the role of b_0 with the same argument.

The only remaining situation is $\hat{k}_i = 0$ and $\hat{\tau}_i < \infty$ for all i. We use for $b^m > b$, the same choice as in the first case. For $b^m < b$, if there are no fixed costs, $\hat{k}_i = 0$ implies that $b = 0$ and we cannot approach b with $b^m < b$. If there are fixed costs, which implies $\beta_i(\tau, 0) > 0$, for some i, no point $((\hat{\tau}_i, 0))_{i=1}^s$ can belong to $\sigma(b)$ – and we do not need to find the sequence. Indeed, as $\sigma(b)$ consists of optimal choices, $((\hat{\tau}_i, 0))_{i=1}^s$ cannot be one of them, because in our model, if you have money for the site, you restore (if $k = 0$, one does not restore) by Lemma 3.1. The last situation with fixed costs is the one where β_i^{-1} might not be lower hemicontinuous (one might not be able to approximate b from below at point $(\tau, 0)$). □

A.8. **The envelope theorem.**

Proposition A.7. *Assume that the loss function $\mathcal{W}$ is additively separable. Suppose that $\lim_{\tau_i\to\infty} C_i(k_i, N(\tau_i))e^{-r\tau_i} = 0$ and that there is a unique optimal solution (τ_i^*, k_i^*) for a given budget $b_i > 0$ to the single site restoration problem (3.11). Suppose also that the unique solution is differentiable with respect to the budget.*

Then the value function V_i is differentiable.

The following proposition implies Proposition A.7 by choosing $s = 1$.

Proposition A.8. *Suppose that all the sites $i = 1, \ldots, s$ are restored. Assume that there exists a unique optimal solution $(\tau_i^*, k_i^*)_{i=1}^s$ to the discounted cost problem (2.12)–(2.14). Denote $\beta((\tau_i, k_i)_{i=1}^s) = \sum_{i=1}^s C_i(k_i, N_i(\tau_i))e^{-r\tau_i} = \sum_{i=1}^s \beta_i(\tau_i, k_i)$ and suppose that (A.33) holds.*

If $\tau_i^* > 0$ for all i, then we have that the value function V of the discounted cost problem is differentiable and

$$V'(b_0) = -\lambda^*,$$

where λ^* is the Lagrange multiplier of the budget constraint β.

If $\tau_i^* = 0$ and $\sigma(b_0) := \arg\min \mathcal{W}((\tau_i, k_i)_{i=1}^s) = (\tau_i^*(b_0), k_i^*(b_0))_{i=1}^s$ is differentiable, then V is differentiable and

$$V'(b_0) = -\lambda^*. \tag{A.35}$$

We will prove this proposition using the envelope theorem. We will state the theorem next but we will give its proof at the end of the section. We state our own version of the envelope theorem, as standard theorems have quite strict regularity and uniqueness assumptions, and they assume that the choice set has a convex and topological structure. Furthermore, these classical envelope theorems do not consider optimal points that are on the boundary of the choice set. Our envelope theorem also deals with optimal points on the boundary, where the rate of change of the value function might not be given only by the Lagrange multiplier. Moreover, for example, in contrast to Milgrom and Segal (2002), where a choice set may vary depending on the parameter as in our theorem, we do not assume the concavity of either the objective function or constraint functions.

We consider the following nonlinear optimization problem with parameters

$$\min_{x\in X} f(x) \qquad \text{such that}$$
$$h_i(x) = r_i, \quad i = 1, \ldots, M, \qquad \text{and} \qquad g_l(x) \le 0, \quad l = 1, \ldots, J, \tag{A.36}$$

where X is an open subset of $\mathbb{R}^n$, f is differentiable in X, h_i and g_i are continuously differentiable functions in X, and $r_i \in I_i$, where $I_i \subset \mathbb{R}$ is an interval. Let the Lagrangian function be defined as

$$L(x, \lambda, \omega) = \lambda_0 f(x) + \sum_{i=1}^{M} \lambda_i (h_i(x) - r_i) + \sum_{l=1}^{J} \omega_l\, g_l(x),$$

where $x = (x_1, \ldots, x_n)$, $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_M)$ and $\omega = (\omega_1, \ldots, \omega_J)$.

Theorem A.9 (The envelope theorem)**.** *If the following conditions 1–5 are satisfied, then the value function*

$$V(\mathbf{r}) = \min\{f(x) : \text{(A.36) } \textit{holds for } \mathbf{r} = (r_1, r_2, \ldots, r_M)\}$$

has partial derivatives at $\mathbf{r}$.

1. *Given $r_i \in I_i$, $i = 1, 2, \ldots, M$, there is a solution to the optimization problem.*

2. *$\sigma(\mathbf{r}) := \arg\min\{f(x) : $ (A.36) holds for $\mathbf{r} = (r_1, r_2, \ldots, r_M)\}$ is compact-valued and upper hemicontinuous.*
3. *All global minimum points have the same Lagrange multipliers λ^*, ω^* with $\lambda_0^* = 1$ in Fritz John conditions.*
4. *For every $i = 1, \ldots, M$, directional derivatives of h_i do not vanish at a global minimum point.*
5. *If $\omega_l^* > 0$, for some $l = 1, \ldots, J$, then $\sigma\colon \mathbb{R}^M \to \mathbb{R}^n$ is a differentiable map.*

The partial derivatives are given by

$$\partial_{r_i} V = -\lambda_i^* - \sum_{l=1}^{J} \omega_l^* \, Dg_l(x^*) \, \partial_{r_i}\sigma. \tag{A.37}$$

Note that, as x^* is a minimum point, $\omega_l^* \, g_l(x^*) = 0$ and $\omega_l^* \geq 0$ by Fritz John conditions. Thus condition 5 is relevant only when there is an optimal point x^* on the boundary of the inequality constraint, that is, $g_l(x^*) = 0$.

Proof of Proposition A.8. When the budget is positive, i.e., $b_0 > 0$, there is always an optimal solution to the discounted cost problem, where $\tau_i^* < \infty$ and $k_i^* > 0$ at least for one i, using Lemma 3.1. We suppose that all the sites $i = 1, \ldots, s$ are restored. This means that $\tau_i^* < \infty$ and $k_i^* < \infty$ for all is. Thus the minimum point is found.

In the envelope theorem A.9 our functions $f = \mathcal{W}$ and h_1, $h_1((\tau_i, k_i)_{i=1}^s) = \beta((\tau_i, k_i)_{i=1}^s)$, are continuously differentiable. The inequality constraints are given by $g_{2i-1}((\tau_i, k_i)_{i=1}^s) = -\tau_i$ and $g_{2i}((\tau_i, k_i)_{i=1}^s) = -k_i$. Moreover, $r_1 = b_0$ in the theorem.

In our case, conditions 1–5 are satisfied. Indeed, as there is a minimum point and Lemma A.6 holds, conditions 1 and 2 of the envelope theorem are satisfied. Our assumption of the uniqueness of $(\tau_i^*, k_i^*)_{i=1}^s$ with Lemma 3.3 guarantees condition 3. For condition 4, note that our constraint map $\beta((\tau_i, k_i)_{i=1}^s) = \sum_{i=1}^s C_i(k_i, N_i(\tau_i))e^{-r\tau_i}$ has, for $i = 1, \ldots, s$, a nonzero k_i derivative $\partial_{k_i}\beta_i(\tau_i^*, k_i^*) = \partial_{k_i} C_i(k_i^*, N_i(\tau_i^*))e^{-r\tau_i^*}$, since we have that $\partial_{k_i} C_i(k_i^*, N_i(\tau_i^*)) > 0$, by assumption (A5) as $k_i^* > 0$. Further, $\partial_{\tau_i}\beta = \partial_{\tau_i}\beta_i(\tau_i^*, k_i^*) < 0$, by (3.1). Condition 5 is an assumption of our proposition.

Hence the proposition follows by (A.37) once one notices that if $\tau_i^* > 0$, $\omega_{\tau_i}^* = \omega_{2i-1}^* = 0$ as $\omega_{\tau_i}^* \tau_i^* = 0$ by Fritz John conditions, $\nabla g_{2i-1}((\tau_i, k_i)_{i=1}^s)$ is $s \times 1$-matrix, where the row i has an entry -1 and all other rows are 0, and $\omega_{2i}^* = 0$, as $k_i^* > 0$ and $\omega_{2i}^* k_i^* = 0$, by Fritz John conditions. Further, if σ is differentiable,

$$\partial_{b_0}\sigma = \left(\partial_{b_0}\tau_i^*(b_0), \partial_{b_0} k_i^*(b_0)\right)_{i=1}^s .$$

The set on which $\tau_i^*(b_i)$ is zero can consist of isolated b_i values or intervals. The optimal restoration date $\tau_i^*(b_i)$ is a differentiable function by assumption and as such it has zero derivative as zero is always a local minimum. Thus (A.35) holds. □

Proof of the envelope theorem A.9. Let $\mathbf{r} = (r_1, r_2, \ldots, r_M)$. We first study the partial derivative of V with respect to r_{i_0}. We pick any sequence $(\mathbf{r}^m)_{m=1}^{\infty}$ such that $\mathbf{r}^m_{i_0} \in I_{i_0} \setminus \{r_{i_0}\}$ and $\mathbf{r}^m_{i_0} \to r_{i_0}$ when $m \to \infty$, and other coordinates of $\mathbf{r}^m$ always stay the same, that is, $\mathbf{r}^m_i = r_i, i \neq i_0$. For the partial derivative $\partial_{r_i} V$ at $\mathbf{r}$, we need there to be a linear map A (a $1 \times M$-matrix, i.e., $A \in \mathbb{R}^{1\times M}$) and an error function E tending to zero as its argument tends to 0 such that

$$V(\mathbf{r}^m) = V(\mathbf{r}) + A(\mathbf{r}^m - \mathbf{r}) + E(\mathbf{r}^m - \mathbf{r})|\mathbf{r}^m - \mathbf{r}|.$$

Take any $(x^m) \in \sigma(\mathbf{r}^m)$ so that $f(x^m) = V(\mathbf{r}^m)$, these points exist by condition 1. As σ is compact-valued and upper hemicontinuous by assumption 2, there is a subsequence of $(x^{m_j})_{j=1}^{\infty}$ that converges to a minimum point x^* in $\sigma(\mathbf{r})$ (Ok, 2007, Proposition 2, p. 209). Denote $\delta^j := x^{m_j} - x^*$. Now, we have a linear approximation

$$\begin{aligned} V(\mathbf{r}^{m_j}) = f(x^{m_j}) &= f(x^*) + Df(x^*)\delta^j + E^f(\delta^j)|\delta^j| \\ &= V(\mathbf{r}) + Df(x^*)\delta^j + E^f(\delta^j)|\delta^j|, \end{aligned}$$

where the linear map $Df(x^*) = \nabla f(x^*)^\top$ and an error function E^f exist, as f is differentiable. Since x^* satisfies Fritz John conditions with $\lambda_0^* = 1$ (as a minimum point and by condition 3, see, e.g., Girsanov (1972, Theorem 11.4, p. 81)),

$$Df(x^*) = -\sum_{i=1}^{M} \lambda_i^*\, Dh_i(x^*) - \sum_{l=1}^{J} \omega_l^*\, Dg_l(x^*).$$

Hence

$$\begin{aligned} V(\mathbf{r}^{m_j}) &= V(\mathbf{r}) + Df(x^*)\delta^j + E^f(\delta^j)|\delta^j| \\ &= V(\mathbf{r}) + \left(-\sum_{i=1}^{M} \lambda_i^*\, Dh_i(x^*)\delta^j - \sum_{l=1}^{J} \omega_l^*\, Dg_l(x^*)\delta^j\right) + E^f(\delta^j)|\delta^j|. \end{aligned} \tag{A.38}$$

As h_i is also differentiable, $h_i(x^{m_j}) = h_i(x^*) + Dh_i(x^*)\delta^j + E^{h_i}(\delta^j)|\delta^j|$. Thus

$$\mathbf{r}_i^{m_j} - r_i = h_i(x^{m_j}) - h_i(x^*) = Dh_i(x^*)\delta^j + E^{h_i}(\delta^j)|\delta^j| \tag{A.39}$$

and we have

$$Dh_i(x^*)\delta^j = \mathbf{r}_i^{m_j} - r_i - E^{h_i}(\delta^j)|\delta^j|. \tag{A.40}$$

When $\omega_l = 0$ for all l, we have, by (A.38) and (A.40), and our choice that $\mathbf{r}_i^{m_j} = r_i$, when $i \neq i_0$,

$$V(\mathbf{r}^{m_j}) = V(\mathbf{r}) - \lambda_{i_0}^* \left(\mathbf{r}_{i_0}^{m_j} - r_{i_0}\right) + E^V(\delta^j)|\delta^j|,$$

where $E^V(\delta^j) := \sum_{i=1}^{M} \lambda_i^* E^{h_i}(\delta^j) + E^f(\delta^j)$.

Moreover, condition 4 implies that $\min_{|e|=1} |Dh_i(x^*)e| > 0$, since every component of $Dh_i = \nabla h_i^\top$ is nonzero. Hence, for large enough j, $E^{h_{i_0}}(\delta^j) \leq \frac{1}{2}\min_{|e|=1}|Dh_i(x^*)e|$ and, by

(A.39),

$$|\mathbf{r}_i^{m_j} - r_i| = |Dh_i(x^*)\delta^j + E^{h_i}(\delta^j)|\delta^j|| \geq \min_{|e|=1} |Dh_i(x^*)e|\,|\delta^j| - E^{h_i}(\delta^j)|\delta^j| \geq \frac{1}{2}\min_{|e|=1} |Dh_i(x^*)e|\,|\delta^j|. \tag{A.41}$$

Thus

$$|\delta^j| \leq \frac{2}{\min_{|e|=1}|Dh_i(x^*)e|}|\mathbf{r}_{i_0}^{m_j} - r_{i_0}| \tag{A.42}$$

and we have shown that $V(\mathbf{r}^{m_j}) = V(\mathbf{r}) + A(\mathbf{r}^{m_j} - \mathbf{r}) + E(\mathbf{r}^{m_j} - \mathbf{r})|\mathbf{r}^{m_j} - \mathbf{r}|$, where $A_{i_0} = -\lambda_{i_0}^*$.

In the above proof, we take a subsequence $(x^{m_j})_{j=1}^\infty$ that converges to a minimum point x^*. As all minimum points have the same linear approximation A_{i_0}, by our assumptions, we conclude that $\partial_{r_i} V = A_{i_0}$.

If $\omega_l^* > 0$ for some l we have that σ is differentiable, by condition 5, and thus

$$\delta^j = x^{m_j} - x^* = \sigma(\mathbf{r}^{m_j}) - \sigma(\mathbf{r}) = D\sigma(\mathbf{r})\,(\mathbf{r}^{m_j} - \mathbf{r}) + E^\sigma(\mathbf{r}^{m_j} - \mathbf{r})|\mathbf{r}^{m_j} - \mathbf{r}|,$$

where $D\sigma(\mathbf{r}) \in \mathbb{R}^{n\times M}$ and $E^\sigma(\mathbf{r}^{m_j} - \mathbf{r}) \in \mathbb{R}^{n\times 1}$. Now, as above, from (A.38)

$$V(\mathbf{r}^{m_j}) = V(\mathbf{r}) - \lambda_{i_0}^*\left(\mathbf{r}_{i_0}^{m_j} - r_{i_0}\right) - \sum_{l=1}^J \omega_l^*\, Dg_l(x^*)\,\partial_{r_{i_0}}\sigma(\mathbf{r})\left(r_{i_0}^{m_j} - r_{i_0}\right) + E^V(\delta^j)|\delta^j|,$$

where $E^V(\delta^j) := \sum_{i=1}^M \lambda_i^* E^{h_i}(\delta^j) - \sum_{l=1}^J \omega_l^*\, Dg_l(x^*) E^\sigma(\delta^j)\frac{|\mathbf{r}^{m_j} - \mathbf{r}|}{|\delta^j|} + E^f(\delta^j)$ and $\partial_{r_{i_0}}\sigma(\mathbf{r}) \in \mathbb{R}^{n\times 1}$. □

A.9. **Proof of Proposition 3.11.** The Lagrangian function for the budget allocation problem is

$$L(b,\mu,\omega) = \mu_0\sum_{i=1}^s V_i(b_i) + \mu\left(\sum_{i=1}^s b_i - b_0\right) - \sum_{i=1}^s \omega_i b_i.$$

The last term comes from the inequality constraints $b_i \geq 0$. Here $b = (b_1,\dots,b_s)$, $\mu_0 \in \{0,1\}$ and $\omega = (\omega_1,\dots,\omega_s)$.

For an optimal budget allocation, where one uses money at all the sites $i = 1,\dots s$, we have, by Proposition A.7, that $V = \sum_{i=1}^s V_i$ is a differentiable function and thus the budget allocation $(b_1^*,\dots,b_s^*)$ satisfies Fritz John conditions (Girsanov, 1972, Theorem 11.4, p. 81) $\partial_{b_i} V(b^*) = \partial_{b_i}\left(\sum_{i=1}^s V_i(b_i^*)\right) = -\mu^* + \omega_i^*$ and $\omega_i^* b_i^* = 0$. As we assume that $b_i^* > 0$, we have

$$\partial_{b_i} V(b^*) = V_i'(b_i^*) = -\mu^*.$$

Above we can choose $\mu_0 = 1$ in Fritz John conditions, since $b_i^* > 0$. Indeed, if $\mu_0 = 0$, a Fritz John condition $\omega_i^* b_i^* = 0$ implies that $\omega_i^* = 0$ and thus, by another Fritz John condition

$\mu_0 \partial_{b_i} V(b^*) + \mu^* - \omega_i^* = 0$, we get $\mu^* = \omega_i^* = 0$, which is a contradiction as the Lagrange multiplier $(\mu_0, \mu^*, \omega_i^*)$ would be zero.

Note that, by equation (A.35) which follows from the envelope theorem A.9, the Lagrange multiplier related to problem (3.11) satisfies

$$V_i'(b_i^*) = -\lambda_i^*$$

for each i. Thus

$$-\mu^* = V_i'(b_i^*) = -\lambda_i^*.$$

For the second claim, we assume that there is a unique optimal solution $(b_1^*, \dots, b_s^*)$ to (3.10) with $b_i^* > 0$ and $b_0 = \sum_{i=1}^s b_i^*$.

We will first use the envelope theorem A.9 with $f(x) = \sum_{i=1}^s V(b_i)$, $h_1(x) = \sum_{i=1}^s b_i$, and $g_i(x) = -b_i$ with $x = (b_1, \dots, b_s)$. As $b_i^* > 0$ for all i, f is differentiable, by Proposition A.7, and condition 5 is satisfied ($\omega_l^* = 0$ as b_i^*s are inner points for inequality constraints g_i). As the solution is unique, conditions 1 and 3 are satisfied. Condition 4 is immediate from the definition of h_1 (especially all partials are 1).

For condition 2 note that $\sigma(b_0) = (b_1^*, \dots, b_s^*)$ is a point, as we assume that the budget allocation is unique. Thus σ has a closed graph. Since the target domain of σ is compact set $[0, b_0]^s$, σ is compact-valued and upper-hemicontinuous (Ok, 2007, Proposition 3 (a), p. 295). Now, by the envelope theorem A.9,

$$\frac{\mathrm{d}}{\mathrm{d}b_0} \sum_{i=1}^s V_i(b_i^*) = -\mu^*.$$

For the last equality, we use the envelope theorem A.9 for $f(x) = \sum_{i=1}^s W_i(\tau_i, k_i)$, $h_1(x) = \sum_{i=1}^s \beta_i(\tau_i, k_i)$, and the inequality constraints $g_{2i-1}(x) = -\tau_i \le 0$ and $g_{2i}(x) = -k_i \le 0$, where $x = (\tau_1, k_1, \tau_2, k_2, \dots, \tau_s, k_s)$, see Proposition A.8. The assumptions are satisfied, as we assume that the budget allocation $(b_1^*, \dots, b_s^*)$ to (3.10) is unique, and especially, $b_i^* > 0$ for all i (this guarantees, e.g., that the minimum is attained). Moreover, the optimal time and effort (τ_i^*, k_i^*) are also unique, when we are given b_i^*, by assumption. Hence, by (A.35),

$$\frac{\mathrm{d}}{\mathrm{d}b_0} \sum_{i=1}^s W_i(\tau_i^*, k_i^*) = -\lambda(b_0).$$

Since $W_i(\tau_i^*, k_i^*) = V_i(b_i^*)$, by Theorem 3.10,

$$\frac{\mathrm{d}}{\mathrm{d}b_0} \sum_{i=1}^s W_i(\tau_i^*, k_i^*) = \frac{\mathrm{d}}{\mathrm{d}b_0} \sum_{i=1}^s V_i(b_i^*) = -\mu^*$$

and our claim follows.

A.10. **Claim on page 27.** Here we show that Assumption 3 holds for the loss function (4.6). Function $\mathcal{W}$ is continuously differentiable. For the required partial derivatives $\partial \mathcal{W} / \partial \tau_i > 0$

and $\partial \mathcal{W}/\partial k_i < 0$, we note that

$$\frac{\partial F_i}{\partial \tau_i} = \int_{\tau_i}^{t} \rho \partial N_i(s;\tau_i,k_i)/\partial \tau_i \; \mathrm{d}s \; e^{-\rho n_{i,0}\tau_i - \int_{\tau_i}^{t} \rho N_i(s;\tau_i,k_i)\;\mathrm{d}s} > 0$$

by Lemma 2.1. Hence, $\tau_i^1 < \tau_i^2$ implies $F_i(\dagger;\tau_i^1,k_i) < F_i(\dagger;\tau_i^2,k_i)$ for each $\dagger$ (and k_i). Then,

$$\int_0^{\infty}\int_0^{\dagger} e^{-rt}\;\mathrm{d}t F'(\dagger;\tau_i^1,k_i)\;\mathrm{d}\dagger > \int_0^{\infty}\int_0^{\dagger} e^{-rt}\;\mathrm{d}t F'(\dagger;\tau_i^2,k_i)\;\mathrm{d}\dagger,$$

because the function represented by $\int_0^{\dagger} e^{-rt}\;\mathrm{d}t$ is strictly increasing. This inequality implies that the expected loss from losing a single species in ecoregion i is strictly increasing in τ_i, which gives $\partial \mathcal{W}/\partial \tau_i > 0$ after multiplying the loss by α_i.

A similar argument shows that $\partial \mathcal{W}/\partial k_i < 0$, because

$$\frac{\partial F_i}{\partial k_i} = \int_{\tau_i}^{t} \rho \partial N_i(s;\tau_i,k_i)/\partial k_i \; \mathrm{d}s \; e^{-\rho n_{i,0}\tau_i - \int_{\tau_i}^{t} \rho N_i(s;\tau_i,k_i)\;\mathrm{d}s} < 0.$$

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