On Linear and Nonlinear Beltrami Systems

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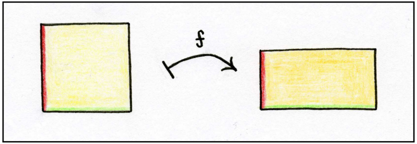
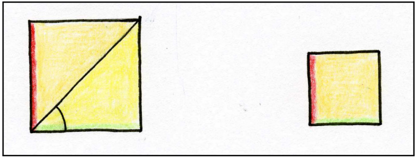
Lectio praecursoria

Grötzch Problem

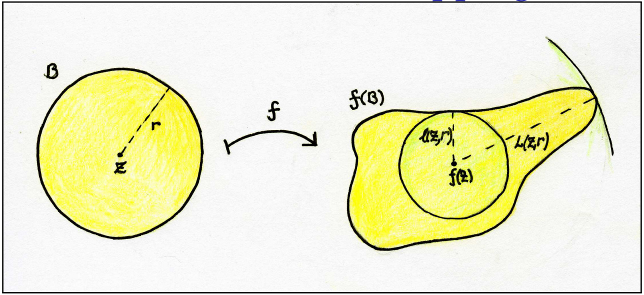
Herbert Grötzsch asked in 1928 *the most nearly conformal mapping* between a square and a rectangle.



Conformal mapping preserves angles locally.



Quasiconformal Mappings



$$\frac{\text{radius of the smallest disk outside}}{\text{radius of the biggest disk inside}} = \frac{L(z,r)}{l(z,r)}$$

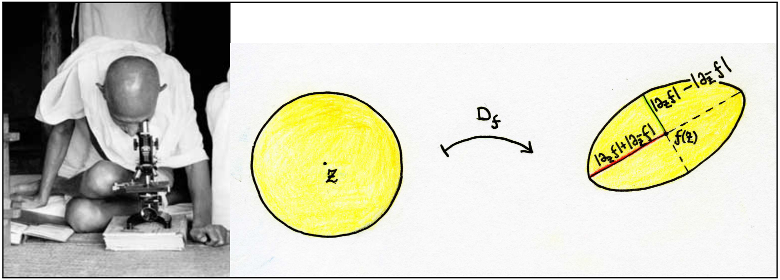
A homeomorphism $f : \mathbb{C} \to \mathbb{C}$ is quasiconformal if for every point z

$$\limsup_{r \to \infty} \frac{L(z,r)}{l(z,r)} \leqslant K$$

one point to one point, mapping and its inverse are continuous (if points are close then the image of the points are close)

Quasiconformal Mappings

Infinitesimal behaviour can be characterized by the derivative.



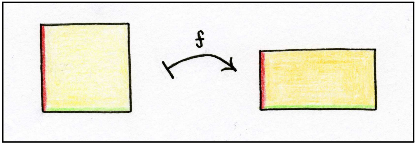
$$\text{disks} \mapsto \text{ellipsoids}$$

$$\frac{\text{major axis}}{\text{minor axis}} = \frac{|\partial_z f| + |\partial_{\bar z} f|}{|\partial_z f| - |\partial_{\bar z} f|} \leqslant K(z) \leqslant K$$

Conformal functions (preserve angles locally) map infinitesimally disks to disks.

Back to Grötzch

Herbert Grötzsch asked in 1928 *the most nearly conformal mapping* between a square and a rectangle.



$$\frac{\text{major axis}}{\text{minor axis}} = \frac{|\partial_z f| + |\partial_{\bar{z}} f|}{|\partial_z f| - |\partial_{\bar{z}} f|} \leqslant K(z) \leqslant K \qquad \text{most nearly = smallest } K$$

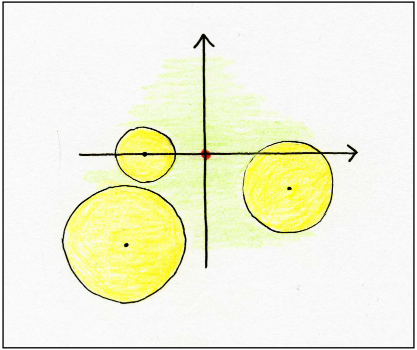
$$f(z) = \frac{1}{2}\left(\frac{a_R}{a_S} + \frac{b_R}{b_S}\right) z + \frac{1}{2}\left(\frac{a_R}{a_S} - \frac{b_R}{b_S}\right) \bar{z}$$

This natural generalization of conformal maps is a superficial reason to study quasiconformal functions. Quasiconformal mappings arise in many questions of geometry and analysis (holomorphic dynamics, elliptic PDEs, etc.)

Beltrami Inequality

$$\frac{\text{major axis}}{\text{minor axis}} = \frac{|\partial_z f| + |\partial_{\bar{z}} f|}{|\partial_z f| - |\partial_{\bar{z}} f|} \leqslant K(z) \leqslant K$$

$$|\partial_{\bar{z}} f| \leqslant k |\partial_z f|, \qquad k = \frac{K-1}{K+1} < 1$$



Beltrami Equation

Beltrami inequality

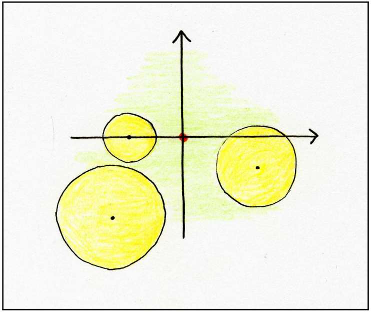
$$|\partial_{\bar z} f| \leqslant k |\partial_z f|, k = \frac{K-1}{K+1} < 1.$$

Beltrami equation

$$\partial_{\bar z} f(z) = \mu(z) \partial_z f(z), \qquad |\mu(z)| \leqslant k,$$

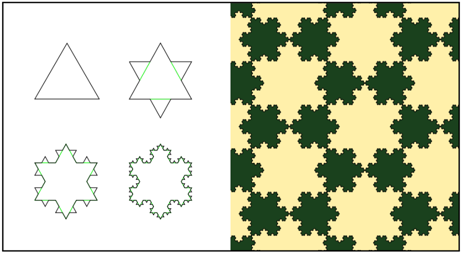
named after *Eugenio Beltrami* (1835–1899); studied differential geometry and mathematical physics.

Already *Carl Friedrich Gauss* (1777–1855) used the equation when he studied isothermal coordinates.

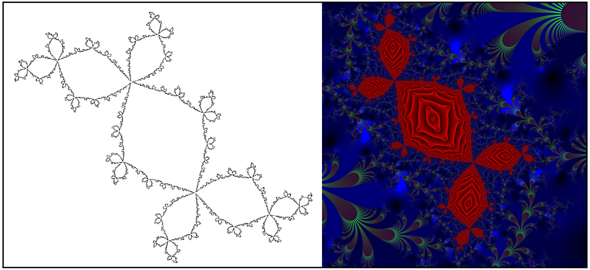


Examples (Quasicircles)

Koch snowflake

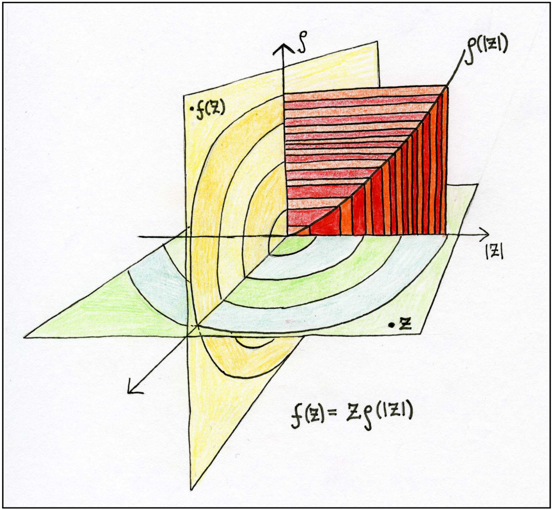


Douady Rabbit



the rabbit on right: RBerenguel

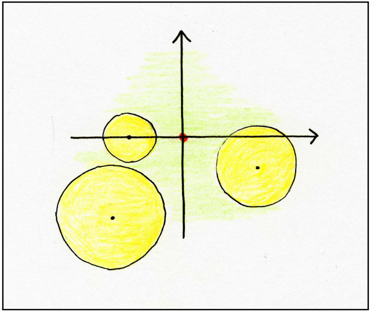
Examples (Radial Stretching)



Reduced Beltrami Equation

Beltrami equation

$$\partial_{\bar{z}} f(z) = \mu(z) \partial_z f(z), \qquad |\mu(z)| \leqslant k.$$

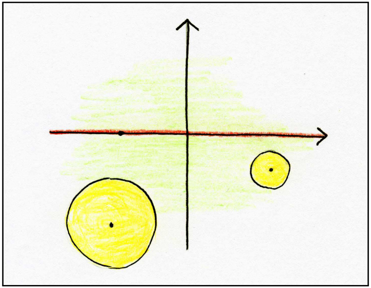


Reduced Beltrami Equation

$$|\partial_{\bar{z}} f| = \lambda(z) \, \mathrm{Im} \, \partial_z f(z), \quad |\lambda(z)| \leqslant k < 1$$

Reduced Beltrami inequality

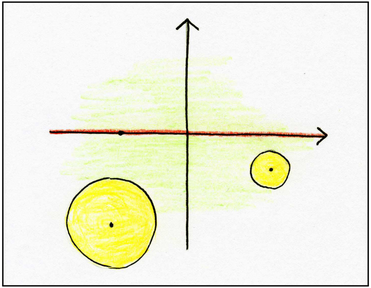
$$|\partial_{\bar{z}} f| \leqslant k |\, \mathrm{Im} \, \partial_z f|.$$



Reduced Beltrami Equation

Reduced Beltrami Equation

$$|\partial_{\bar z} f| = \lambda(z)\,\mathrm{Im}\,\partial_z f(z), \quad |\lambda(z)| \leqslant k < 1$$

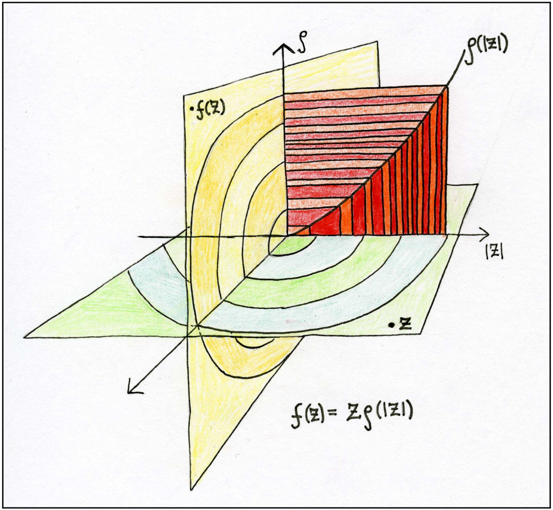


Identity is always a solution, $f(z) = z$.

The identity is the unique homeomorphic solution fixing points $z = 0$ and $z = 1$, i.e., $f(0) = 0$ and $f(1) = 1$.

$\mathrm{Im}\,\partial_z f$ is non-vanishing almost everywhere.

Radial Stretching



When rotated by 90°, the mapping is reduced quasiregular.

Linear Families

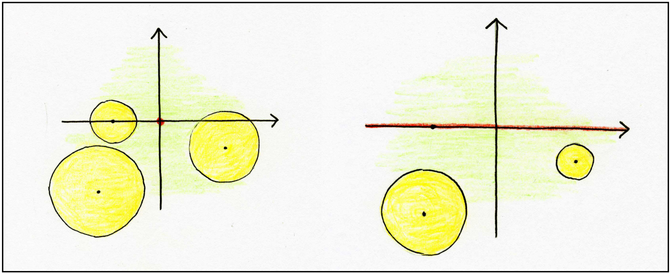
Linear families of quasiconformal mappings can be studied with the help of a reduced Beltrami equation. That is families of quasiconformal mappings $af + bg$, where a and b are real numbers, e.g., $f+g$, $f-g$, $0.1f-6g$, etc.

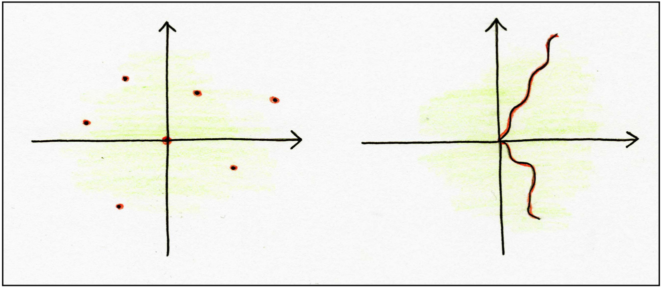
A linear family of quasiconformal mappings has a unique associated linear Beltrami equation,

$$\partial_{\bar{z}} f(z) = \mu(z)\partial_z f(z) + \nu(z)\overline{\partial_z f(z)}.$$

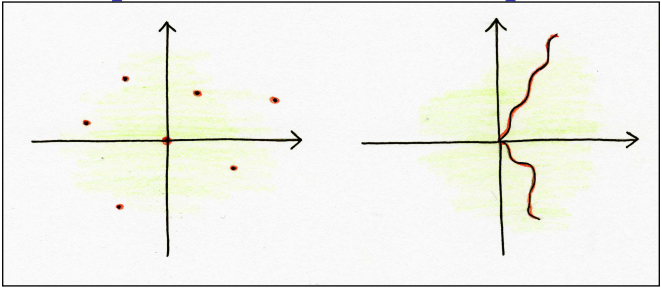
$\partial_{\bar{z}} f(z) = \mu(z)\partial_z f(z)$ and $\partial_{\bar{z}} f(z) = \lambda(z)\,\mathrm{Im}\,\partial_z f(z)$ are linear equations, i.e., if f, g are solutions, then $af + bg$ is a solution.

Nonlinear Situation





Uniqueness in Nonlinear Equation



Beltrami Systems = we have uniform ellipticity bound $k < 1$.

The identity was the unique homeomorphic solution to the reduced Beltrami equation such that $0 \mapsto 0$ and $1 \mapsto 1$. There is a unique solution in the case of the Beltrami equation.

The same is true in the nonlinear case under an explicit bound of the ellipticity. Namely, if we fix two points, say $f(0) = 0$ and $f(1) = 1$, then there is a unique solution to nonlinear Beltrami equation if $k(z) < 3 - 2\sqrt{2} = 0.17157\ldots$ when z is near ∞.