JKYLÄ
30.3.2016

(1)

STRUCTURE OF HOMEOMORPHIC SOLUTIONS TO THE BELTRAMI EQUATION

joint works with Astala, Clop, Faraco, Koski, Székelyhidi Jr

The (nonlinear) Beltrami equation

$$\boxed{f_{\bar z} = \mathcal{H}(z, f_z) \quad \text{a.e.}} \qquad (*)$$

- $\mathcal{H}: \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ s.t.
  - $|\mathcal{H}(z,\xi_1) - \mathcal{H}(z,\xi_2)| \leq k(z)|\xi_1 - \xi_2|$, for a.e. z

    $k(z) \leq k < 1$ (uniformly elliptic)
  - $\mathcal{H}(z,0) \equiv 0$ (homogeneity)
  - $z \mapsto \mathcal{H}(z,\xi)$ is measurable for every $\xi \in \mathbb{C}$.
- complexification $\partial_{\bar z} f = f_{\bar z} = \frac{1}{2}(f_x + i f_y)$ $\partial f = f_z = \frac{1}{2}(f_x - i f_y)$

Examples
- $f_{\bar z} = 0$, Cauchy-Riemann equations f analytic
- $\mathbb{C}$-linear

  $f_{\bar z} = \mu(z) f_z$, $\|\mu\|_{L^\infty} \leq k < 1$
- $\mathbb{R}$-linear

  $f_{\bar z} = \mu(z) f_z + \nu(z)\overline{f_z}$, $\|\mu\|_{L^\infty} + \|\nu\|_{L^\infty} \leq k < 1$

  (governs linear uniformly elliptic systems in 2D)
- Nonlinear examples
  - $f_{\bar z} = \mathcal{H}(f_z)$ autonomous
  - $f_{\bar z} = \mu(z)|f_z|$
  - $f_{\bar z} = \mu(z)\,\mathrm{dist}(\gamma, f_z)$ (Key in the solution of Tartar's conjecture) Faraco-Székelyhidi Jr 08

History remarks of (*)

- introduced by Bojarski and Iwaniec in '74
- governs planar elliptic systems
  (homotopic to Cauchy-Riemann operators)
- Good existence theory for $W^{1,2}_{loc}$-solutions
  $$f_{\bar z} = \mathcal{H}(z, f, f_z)$$
  (Lusin measurability), Astala-Iwaniec-Martin 2009
- L^p-theory, Astala-Iwaniec-Saksman 2001

Quasiconformal maps

Solutions to $f_{\bar z} = \mathcal{H}(z, f_z)$ satisfy

$$|f_{\bar z}| \le |\mathcal{H}(z, f_z) - \mathcal{H}(z, 0)| \le k|f_z|$$

Thus homeomorphic $W^{1,2}_{loc}$-solutions are quasiconformal

Non-homeomorphic solutions are quasiregular;
one can factorize

$$f = \underset{\text{analytic}}{H} \circ \underset{\text{qc}}{F} \qquad \text{(Stoilow factorisation)}$$

Structure of homeomorphic solutions to $\mathcal{H}$-eq. 3

is there (even) uniqueness of homeomorphisms

mapping $0 \mapsto 0$, $1 \mapsto 1$, $\infty \mapsto \infty$?

- Conformal map ($f_{\bar z} = 0$)
  $z \mapsto z$ is the unique normalised

- $\mathbb{C}$-linear; $f_{\bar z} = \mu f_z$
  YES by Stoïlow factorisation ($g = H \circ f$)

- $\mathbb{R}$-linear; $f_{\bar z} = \mu f_z + \nu \overline{f_z}$
  YES, Astala-Iwaniec-Martin 2009 (reduction to $f_{\bar z} = \mu \, \mathrm{Im}(f_z)$)

- $\mathcal{H}$-equation?

<u>Theorem</u> (Astala, Clop, Faraco, J, Székelyhidi Jr, 2011)

Let $f_{\bar z} = \mathcal{H}(z, f_z)$. Then homeomorphic solution $f: \mathbb{C} \longrightarrow \mathbb{C} \in W^{1,2}_{loc}$ with $f(0) = 0$, $f(1) = 1$

- is unique if

$$\limsup_{|z| \to \infty} K(z) = \frac{1 + k(z)}{1 - k(z)} < \sqrt{2}$$

- if $f(z) = z$, uniqueness holds for

$$\limsup_{|z| \to \infty} K(z) < 2.$$

Is there always uniqueness for normalised homeomorphic solutions to $\mathcal{H}$-equation?

No! The bounds are sharp!

<u>Sketch of proof</u> $K_\infty = \limsup_{|z|\to\infty} K(z) < \sqrt{2}$

Let f, g solve the same $\mathcal{H}$-equation. Then

$$|f_{\bar z} - g_{\bar z}| = |\mathcal{H}(z, f_z) - \mathcal{H}(z, g_z)| \le k|f_z - g_z|,$$

i.e. $f-g$ is qr

Suppose that f and g are homeomorphic, $0 \mapsto 0$, $1 \mapsto 1$

Then

$$\deg(f-g) \le K^2 \quad , \quad K = \frac{1+k}{1-k}$$

Indeed $f - g = P(\Phi)$, P analytic, $P(0) = 0 = P(1)$

Φ qc $\Phi(0) = 0$, $\Phi(1) = 1$

$$|P(\Phi(z))| = |f-g| \le C|z|^K \le C|\Phi(z)|^{K^2}$$

behave like K-qc near ∞

$\Longrightarrow$ P polynomial of degree $\le K^2$

If $K^2 < 2$, degree ≤ 1. Thus $f = g$

If $f \equiv z$, a topological argument gives $|f-g| \leq C|z|$ and thus degree $< K$

Counterexamples (sharpness)

Let

$$F_t(z) = \begin{cases} (1+t)z|z| - tz^2, & |z| > 1 \\ (1+t)z - tz^2, & |z| \leq 1 \end{cases}$$

$$G_t(z) = \begin{cases} (1+t)z|z| - tz, & |z| > 1 \\ z, & |z| \leq 1 \end{cases}$$

Then $F_t - G_t = t(z - z^2)$, $K_{F_t}, K_{G_t} \longrightarrow 2$ as $t \to 0$

Both F_t, G_t $0 \mapsto 0$, $1 \mapsto 1$

$\mathcal{H}(z, 0) = 0$, $\mathcal{H}(z, (F_t)_z) = (F_t)_z$, $\mathcal{H}(z, (G_t)_z) = (G_t)_z$

Extend by Kirzbraun

- the case $K \to \sqrt{2}$

$F_t \circ \varphi^{-1}$

$G_t \circ \varphi^{-1}$,

where $\varphi(z) = \begin{cases} z|z|^{\sqrt{2}-1}, & |z| > 1 \\ z, & |z| \leq 1 \end{cases}$

Structure of homeomorphic solutions $f \in W^{1,2}_{loc}(\mathbb{C}, \mathbb{C})$ makes sense only under suitable uniqueness properties.

Equation

$$f_{\bar{z}} = \mathcal{H}(z, f_z)$$

has uniqueness property if

for every $z_0, z_1, w_0, w_1 \in \mathbb{C}$ there is unique solution f with $f(z_0) = w_0$, $f(z_1) = w_1$

Note: • $\mathcal{H}$ has uniqueness property $\Longleftrightarrow$ f, g homeomorphic solutions, $f - g$ is qc or constant

• under the bound

$$\limsup_{|z| \to \infty} K(z) < \sqrt{2}$$

$\mathcal{H}$ has uniqueness property

Consider the field / family of solutions

$$\mathcal{F}_{\mathcal{H}} = \{ \varphi_a \in W^{1,2}_{loc}(\mathbb{C}, \mathbb{C}) \; : \; a \in \mathbb{C} \}$$

1) $\varphi_a(0) = 0$, $\varphi_a(1) = a$, φ_a qc for $a \neq 0$

2) $\varphi_0 \equiv 0$

3) $\varphi_a - \varphi_b$ is qc for $a \neq b$

Q: Structure of $\mathcal{F}_{\mathcal{H}}$?

$\mathcal{F} \longleftrightarrow \mathcal{H}$?

- $\mathbb{C}$- linear $\quad \mathcal{H}(z, \xi) = \mu(z)\xi$

$\mathcal{F}_\mu = \{a\varphi_1 : a \in \mathbb{C}\}$ a complex line in $W^{1,2}_{loc}(\mathbb{C}, \mathbb{C})$

- $\mathbb{R}$- linear $\quad \mathcal{H}(z, \xi) = \mu(z)\xi + \nu(z)\bar{\xi}$

$\mathcal{F}_{\mu,\nu} = \{s\varphi_1 + t\varphi_i : s, t \in \mathbb{R}\}$

a two dimensional subspace of $W^{1,2}_{loc}(\mathbb{C}, \mathbb{C})$

Theorem (Astala, Clop, Faraco, J 15) For any field $\mathcal{F} = \{\varphi_a\}$

$a \mapsto \varphi_a$ is bi-Lipschitz $\mathbb{C}\setminus\{0\} \to W^{1,2}_{loc}(\mathbb{C}, \mathbb{C})$

that is, topological embedding (homeomorphism onto its image)

Theorem (Astala, Clop, Faraco, J 15) If $\xi \mapsto \mathcal{H}(z, \xi)$ is C^1

$\mathcal{F}_{\mathcal{H}}$ is C^1-embedded submanifold of $W^{1,2}_{loc}(\mathbb{C}, \mathbb{C})$

topological embedding + immersion such that $D_a\varphi_a$ splits

$D_a\varphi_a : \mathbb{C} \to W^{1,2}_{loc}(\mathbb{C}, \mathbb{C})$ $\quad D_a\varphi_a : T_a\mathbb{C} \to T_{\varphi_a} W^{1,2}_{loc}(\mathbb{C}, \mathbb{C})$ injective

isomorphism onto its image, range is two-dimensional and complemented (thus $D_a\varphi_a$ splits)

What is the tangent plane of $\mathcal{F}_{\mathcal{H}}$?

Theorem (ACFJ15) The tangent at φ_a is given by the solutions to an $\mathbb{R}$-linear equation

$$T_{\varphi_a}\mathcal{F}_{\mathcal{H}} = \mathcal{F}_{\mu_a, \nu_a},$$

where $\mu_a = \mathcal{H}_{\xi}(z, (\varphi_a)_{\bar{z}})$, $\nu_a = \mathcal{H}_{\bar{\xi}}(z, (\varphi_a)_{\bar{z}})$.

Sketch

$$\eta_t^e = \frac{\varphi_{a+te} - \varphi_a}{t} \quad qc; \quad \eta_t^e(0) = 0$$

$$\eta_t^e(1) = e$$

$$\partial_{\bar{z}}\eta_{t_j}^e = \mu_a(z)\,\partial_z\eta_{t_j}^e + \nu_a(z)\,\overline{\partial_z\eta_{t_j}^e} + h_{t_j}(z)$$

Choosing t_j sparse enough

$$\frac{|h_{t_j}|}{|\partial_z\eta_{t_j}^e|} \longrightarrow 0, \quad j \to \infty \qquad \text{a.e. } \mathbb{D}_R$$

$$\Rightarrow \int_{\mathbb{C}} \xi(z)\,h_{t_j}(z)\,dA \longrightarrow 0 \qquad \text{for any } \xi \in C_0^\infty(\mathbb{C})$$

Hence

$$\eta^e \in \mathcal{F}_{\mu_a,\nu_a}$$ (qc solution with $0 \mapsto 0$, $1 \mapsto e$ is unique, the whole sequence converges in L^∞_{loc})

$\{\eta_t^{\varepsilon}\}$ is an approximative solution to the $\mathbb{R}$-linear equation

Approximate versions of Caccioppoli's inequality yield convergence in $W^{1,2}_{loc}$-norm.

Theorem (Giannetti, Iwaniec, Kovalev, Moscariello, Sbordone 2003; Bojarski, -"-, D'Onofrio, -"- 2005; Alessandrini-Nesi, Astala-J 2009)

There is one-to-one correspondence between $\mathbb{R}$-linear families of quasiconformal maps and linear Beltrami equations

$$\{s\varphi_1 + t\varphi_2 : s,t \in \mathbb{R}\} \Leftrightarrow F_{\mu,\nu}$$

Nonlinear case?

$$\mathcal{H} \longrightarrow F_{\mathcal{H}} \qquad \checkmark$$

$$F = \{\varphi_a\} \longrightarrow \mathcal{H}_F$$

$$\mathcal{H}_F(z, \xi) = \partial_{\bar z}\varphi_a(z), \quad \xi = \partial_z \varphi_a(z).$$

(well-defined by $\varphi_a - \varphi_b$ qr)

How much this commutes? $\mathcal{H}_{F_{\mathcal{H}}} = \mathcal{H}$?

$F_{\mathcal{H}_F} = F$?

Key

$a \mapsto \partial_z \varphi_a(z)$ is a global homeomorphism $\mathbb{C} \to \mathbb{C}$, $\forall z \in \mathbb{C}$ under regularity assumptions on $\mathcal{H}$!

- $\det [D_a \partial_z \varphi_a(z)] \neq 0$

  (gives $a \mapsto \partial_z \varphi_a(z)$ local homeomorphism in $\mathbb{C}$)

Needed:
Schauder estimates + Null Lagrangians

- $\partial_z \varphi_\infty(z) = \infty$, i.e.

$$\frac{1}{C(R)} \leq \frac{|\partial_z \varphi_a(z)|}{|a|} \leq C(R) \quad \begin{array}{l} z \in \mathbb{D}_R \\ a \neq 0 \end{array}$$

Needed:
Non-vanishing of Jacobians

Theorem (Astala, Clop, Faraco, J, Koski, 2015)
Let $z \mapsto \mathcal{H}(z, \xi) \in C^\alpha$ and $f_{\bar z} = \mathcal{H}(z, f_z)$ a.e.

- $f \in C^{1,\gamma}_{loc}$, $\gamma = \alpha$ if $\alpha < \frac{1}{K}$
  otherwise one can choose
  $\gamma < \frac{1}{K}$

- if f is a homeomorphism

  $J(z, f) \geq C > 0$

There is a genuine dependence on the ellipticity bound K. ($f = z^2 |z|^{\frac{3}{2K+1} - 1}$ solves autonomous equation with K-ellipticity)
Sharpness not known
(if $\xi \mapsto \mathcal{H}(z, \xi) \in C^1$, $\gamma = \alpha$)

Theorem ACFJK15
$z \mapsto \mathcal{H}(z, \xi) \in C^\alpha$
$(z, w) \mapsto D_\xi \mathcal{H}(z, \xi) \in C^\alpha$
$\mathcal{H}_{F_{\mathcal{H}}} = \mathcal{H}$

Theorem ACFJK15
Smoothness on $\mathcal{F}$ + ellipticity bound near ∞
$\mathcal{F}_{\mathcal{H}_{\mathcal{F}}} = \mathcal{F}$