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UNIQUENESS IN NON-LINEAR BELTRAMI EQUATIONS

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$\mathbb{C} \to \mathbb{C}$

f is a homeomorphism, $f \in W^{1,2}_{loc}(\mathbb{C})$, and it solves non-linear Beltrami eq.

Question: If we have homeomorphic $f \in W^{1,2}_{loc}(\mathbb{C}, \mathbb{C})$ that solves non-linear Beltrami equations, when does (knowing where it maps) two points define it uniquely?

- representative to ∂c-eq
- some regularity between solutions

Elliptic PDEs in the complex plane

Complexification / notation: $z = x+iy$, $x,y \in \mathbb{R}$

$$\bar\partial f(z) = \frac{1}{2}(\partial_x f(z) + i\,\partial_y f(z)) \quad \text{Cauchy-Riemann operator}$$
$$\partial f(z) = \frac{1}{2}(\partial_x f(z) - i\,\partial_y f(z)) \quad \text{formal adjoint}$$

Classical Beltrami equation

$$(B) \quad \bar\partial f(z) = \mu(z)\,\partial f(z), \quad \mu \text{ measurable}, \ \|\mu\|_\infty \le k < 1$$

- $\mathbb{C}$-linear, two-dimensional $\}$ K-quasi~~conformal~~ regular mappings, $K = \frac{1+k}{1-k}$
- If $f \in W^{1,2}_{loc}(\mathbb{C}, \mathbb{C})$ homeomorphic solution to (B), then any other solution g takes the form

natural domain of definition

$$g = H \circ f$$

where H holomorphic (Stoilow factorization)

- if we normalize $f(0) = 0$, $f(1) = 1$, f is uniquely defined
  know at two points

G-convergence problems, properties of G-families, elasticity plates and films

Uniformly elliptic non-linear Beltrami equations

(*) $\quad \bar{\partial} f(z) = \mathcal{H}(z, f, \partial f)$

(homotopic to Cauchy-Riemann component; i.e., can be continuously deformed to $\bar{\partial} f$) Petrovsky, linearisation

— $f : \mathbb{C} \to \mathbb{C} \in W^{1,2}_{loc}(\mathbb{C}, \mathbb{C})$ ~~[illegible]~~

EXISTENCE OF SOLUTIONS can be established in great generality

Thm (*) admits normalized homeomorphic solution

$\mathcal{H} : \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$

1° $\quad |\mathcal{H}(z, f, w_1) - \mathcal{H}(z, f, w_2)| \leq k\,|w_1 - w_2| \quad 0 \leq k < 1$

contraction in ∂f-variable

Note 1° & 2° $\Rightarrow$ f is qr

uniform ellipticity

2° $\quad \mathcal{H}(z, f, 0) \equiv 0$

$2^\circ \Rightarrow$ constants

homogeneity

3° Lusin-measurable:

$H : Z_j \times F_j \times W_j \to \mathbb{C}$ is continuous

and increasing sequence of compact sets

$A_1 \subset A_2 \subset \ldots \subset \mathbb{C}$, $\bigcup_j A_j$ has full measure

Astala, Iwaniec, and Martin 2009

Bojarski & Iwaniec 1974, Iwaniec 1976 measurable in z

continuous in f

~~[illegible]~~

Remark.

Lusin-measurability $\Rightarrow$ $z \mapsto \mathcal{H}(z, f(z), \partial f(z))$ is measurable

Question of uniqueness is more subtle even for a system

(**) $\qquad \bar\partial f(z) = \mathcal{H}(z, \partial f(z))$

1° $|\mathcal{H}(z,w_1) - \mathcal{H}(z,w_2)| \le k(z)\,|w_1 - w_2|$, $\quad 0 \le k(z) \le k < 1$

2° $\mathcal{H}(z,0) \equiv 0$

3° $z \mapsto \mathcal{H}(z,w)$ measurable

Remark.

More tools (solution f is $K(z)$-qr, $K = \frac{1+k}{1-k}$)

the difference $f-g$ is $K(z)$-qr

(does not (necessary) solve (**)) — all solutions do not have the same μ in (B)

Ex. $\quad |\bar\partial f(z)| = |\mathcal{H}(z, \partial f)| \le k \operatorname{dist}(\partial f(z), \Gamma)$,

where Γ is closed and $\mathcal{H}(z,\Gamma) \equiv 0$.

Thm. If

$$\limsup_{|z|\to\infty} k(z) < 3 - 2\sqrt{2} = 0.17157\ldots,$$

then (**) admits a unique homeomorphic solution $f \in W^{1,2}_{loc}(\mathbb{C},\mathbb{C})$ normalized by $f(0)=0$ and $f(1)=1$.

Moreover, k is sharp, i.e, if $k > 3-2\sqrt{2}$, the is $\mathcal{H}$ with 1°–3° which admits at least two normalized solutions

Remarks. i) 2° is for some $k<1$ in $\mathcal{H}$ (bound only at ∞)

ii) in terms of qc distortion bound

$$\limsup_{|z|\to\infty} K(z) < \sqrt{2}.$$

Proof Assume there exist two normalized homeomorphic solutions $f, g \in W^{1,2}_{loc}(\mathbb{C})$ to (**).

Let $K_\infty := \limsup_{|z| \to \infty} K(z) < \sqrt{2}$. Then for any $K > K_\infty$,

$\boxed{\text{Claim}}$ $|f(z)| \leq C_K(1+|z|)^K$ and $|g(z)| \leq C_K(1+|z|)^K$

Indeed, f, g are $K(z)$-qc and we can decompose

$$f = H \circ F,$$

where H, F are normalized quasiconformal homeomorphisms ~~and~~

$$\mu_F = \chi_{\mathbb{C} \setminus \mathbb{D}(0,R)} \mu_f .$$

Further, we may choose R so large that F is K-qc in $\mathbb{C}$. Then (by quasisymmetry)

$$\frac{1}{C_K} |z|^{1/K} \leq |F(z)| \leq C_K |z|^K, \quad |z| \geq 1.$$

Since H is conformal near ∞, $H(z) = cz + \mathcal{O}(\frac{1}{z})$ and claim follows.

The difference is quasiregular, but not necessary injective. By Stoïlow factorization

$$f - g = P \circ h,$$

where P is holomorphic and h is normalized $K(z)$-qc.

$$|P(h(z))| = |f(z) - g(z)| \overset{\boxed{\text{claim}}}{\leq} C_K |z|^K$$

$$= C_K |(h^{-1}(h(z))|^K \leq C_K |h(z)|^{K \|K\|_\infty}, \quad |z| \geq 1.$$

h^{-1} is $\|K(z)\|_\infty$ - qc.

Hence P is a polynomial.

Two zeros, 0 and 1, implies $\deg(P) \geq 2$.

As above we can decompose $h = H_1 \circ F_1$.

Thus

$$\frac{1}{C_K} |z|^{1/K} \leq |h(z)|.$$

Combining estimates, for $|z|$ large,

$$\frac{1}{C_K} |z|^{2/K} \leq |P(h(z))| = |f(z) - g(z)| \leq C_K |z|^K.$$

This implies $K \geq \sqrt{2}$ ↯

Counterexample

Goal Find for ~~...~~ every $3-2\sqrt{2} < k < 1$ a function $\mathcal{H}: \mathbb{C}\times\mathbb{C}\times\mathbb{C} \to \mathbb{C}$ such that

1° ~~...~~ k-Lipschitz in the second variable

2° measurable in the first variable

3° $\mathcal{H}(z,0) \equiv 0$

and $\bar\partial f = \mathcal{H}(z, \partial f)$ has at least two different homeomorphic solutions $f \in W^{1,2}_{loc}(\mathbb{C}, \mathbb{C})$ normalized by $f(0)=0$, $f(1)=1$.

For any $0<t<1$, set

$$F_t(z) = \begin{cases} (1+t)z|z| - tz^2, & |z|>1 \\ (1+t)z - tz^2, & |z|\le 1 \end{cases}$$

$$G_t(z) = \begin{cases} (1+t)z|z| - tz, & |z|>1 \\ z, & |z|\le 1 \end{cases}$$

Remark.- Both are normalized at 0 and 1.

- modifications of radial stretching $\psi(z) = z|z|^{K-1}$
- difference is a polynomial $(t(z-z^2))$ vanishing at 0 and 1

Next, reduce distortion constants by composing with extra qc-factor φ

$$\varphi(z) = \begin{cases} z|z|^{\sqrt{2}-1}, & |z|>1 \\ z, & |z|\le 1. \end{cases}$$

Consider

$$f_t(z) = F_t \circ \varphi^{-1}$$
$$g_t(z) = G_t \circ \varphi^{-1}$$

Remark. — f_t, g_t are injective and normalized (direct)
— $f_t - g_t$ is K-qr, $0 < k = 3 - 2\sqrt{2}$, $K = \frac{1+k}{1-k}$.
— f_t is k_f-qc and g_t is k_g-qc, where

$$0 < k_f = \frac{\sqrt{2}-1+t}{\sqrt{2}+1-t} < 1, \quad 0 < k_g = \frac{2-\sqrt{2}+t}{2+\sqrt{2}+t} < 1$$

(estimating)

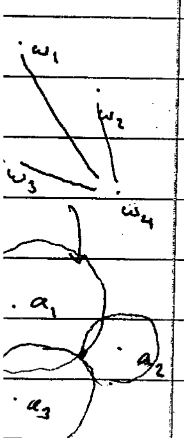
Define for fixed $z \notin \partial\mathbb{D}$ the mapping $w \mapsto \mathcal{H}(z,w)$:

$$\mathcal{H}(z,0) = 0, \quad \mathcal{H}(z, \partial f_t(z)) = \overline{\partial f_t(z)}, \quad \mathcal{H}(z, \partial g_t(z)) = \overline{\partial g_t(z)}$$

Note $\mathcal{H}(z,\cdot) : \{0, \partial f_t(z), \partial g_t(z)\} \to \mathbb{C}$ is k_0-Lipschitz,
$k_0 = \max\{k, k_f, k_g\} \left(\xrightarrow[t\to 0]{} 3 - 2\sqrt{2}\right)$

Question of extending! Kirszbraun extension theorem implies there is an extension $\mathcal{H}(z,\cdot) : \mathbb{C} \to \mathbb{C}$ that is k_0-Lipschitz.

It is not clear from an abstract use of Kirszbraun that the obtained extension is measurable in z!



We can use a constructive proof:
— Fix countable dense set $\mathcal{D} \subset \mathbb{C}$ enumerated by
$\mathcal{D} = \{w_4, w_5, \dots\}$, $w_1 := 0$, $w_2 := \partial f(z)$, $w_3 := \partial g(z)$
— Define $\mathcal{H}(z, w_k)$ recursively:
If $\mathcal{H}(z, w_k)$ defined for $k \le N$, $N \ge 3$, set

$$Y_s(z) = \bigcap_{j=1}^{N} \overline{\mathbb{D}}(a_j(z), s r_j),$$

$a_j(z) = \mathcal{H}(z, w_j)$ and $r_j = k_0 |w_j - w_{N+1}|$

Let $s_0(z) = \inf\{s > 0 : Y_s(z) \neq \emptyset\}$.

"Federer: Geometric Measure Theory"

$\Rightarrow$ Y_{s_0} consist of a single point, i.e., $Y_{s_0}(z) = b(z)$ and $s_0(z) \le 1$.

Elementary argument shows

$$(a_1, \dots, a_N) \mapsto b \quad \text{is continuous.}$$

Set $\mathcal{H}(z, w_{N+1}) = b(z)$.

Since $a_1(z), a_2(z), a_3(z)$ are measurable, recursively $a_i(z)$ is measurable.

We obtained k_0-Lipschitz map $\mathcal{H}(z, \cdot): \mathcal{D} \to \mathbb{C}$ such that, for $w \in \mathcal{D}$, $z \mapsto \mathcal{H}(z, w)$ is measurable.

Since $\mathcal{D}$ is dense, for fixed z, we can extend to $\mathbb{C} \to \mathbb{C}$ k_0-Lipschitz map that is measurable in z-variable.

Letting $t \to 0$, $k_0 \to 3 - 2\sqrt{2}$.

More symmetries

— $\mathcal{H}(z,1)=0 \Rightarrow id$ is a solution
(unique if $k(z)<\frac{1}{3}$ near ∞)
(unique if there exists a path γ between 0 and 1 s.t. $\mathcal{H}(z,\gamma(t)) \in L^{p_0}(\mathbb{C})$, $p_0<2$, uniformly in t)
— $\mathbb{R}$-homogeneous in the second variable
— $\bar\partial f = \mathcal{H}(\partial f)$, homeomorphic solutions are affine

No uniqueness for general systems $\bar\partial f(z) = \mathcal{H}(z, f(z), \partial f(z))$

Ex Fix $0<k<1$

Let

$$\mu(z,\zeta) = \begin{cases} \dfrac{|\zeta - z|}{|\zeta - \bar z|}, & 0 \le |\zeta - z| \le k|\zeta - \bar z| \\ k, & \text{otherwise} \end{cases}$$

and $\mathcal{H}(z,\zeta,w) = \mu(z,\zeta)\,w$.

μ is continuous outside $z \in \mathbb{R}$

Then $z \mapsto t\bar z + (1-t)z$, $0<t<\frac{k}{1+k}$, is normalized solution.