On Linear and Nonlinear Beltrami Systems

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Quasiregular Mappings

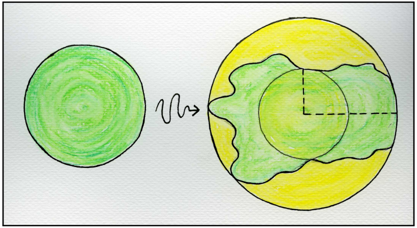
As a reminder, $f : \Omega \to \mathbb{C} \in W^{1,2}_{\text{loc}}(\Omega)$ is K-*quasiregular* if the classical *Beltrami equation* holds for almost every $z \in \Omega$

$$\bar{\partial} f(z) = \mu(z)\, \partial f(z), \qquad |\mu(z)| \leqslant k < 1, \quad K = \frac{1+k}{1-k},$$

where $\quad \bar{\partial} = \frac{1}{2}(\partial_x + i\,\partial_y), \quad \partial = \frac{1}{2}(\partial_x - i\,\partial_y), \quad z = x + i\,y.$

If, in addition, the mapping is also a homeomorphism, then it is called *quasiconformal*.

Infinitesimally a quasiconformal function maps circles into ellipses.



$\mathbb{R}$-linear Beltrami Systems

The classical Beltrami equation $\bar\partial f(z) = \mu(z)\, \partial f(z)$ is $\mathbb{C}$-linear. A general *$\mathbb{R}$-linear Beltrami equation* takes the form

$$\bar\partial f(z) = \mu(z)\, \partial f(z) + \nu(z)\, \overline{\partial f(z)}, \qquad |\mu(z)| + |\nu(z)| \leqslant k < 1, \qquad (*)$$

for almost every $z \in \Omega$. Let $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a homeomorphic solution to $(*)$. Then any other solution $g \in W^{1,2}_{\mathrm{loc}}(\Omega)$ to $(*)$ can be written as

$$g = F \circ f,$$

where $F \in W^{1,2}_{\mathrm{loc}}(\Omega)$ solves the *reduced Beltrami equation*

$$\boxed{\bar\partial F(z) = \lambda(z)\ \mathrm{Im}(\partial F(z)), \qquad |\lambda(z)| \leqslant \kappa < 1,}$$

for almost every $z \in \Omega$. *Generalized Stoïlow Factorization*, Astala, Iwaniec, and Martin (2009); $\kappa = 2k/(1+k^2)$.

Distortion Inequalities

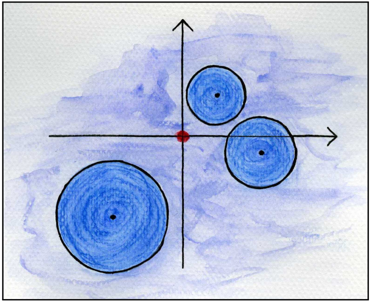
The classical Beltrami equation $\bar{\partial} f(z) = \mu(z)\, \partial f(z)$, implies

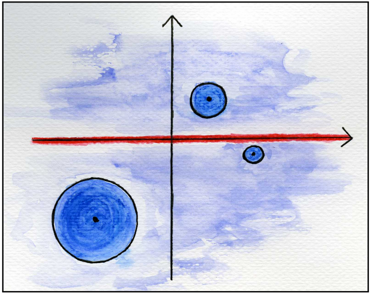
$$|\bar{\partial} f(z)| \leqslant k\, |\partial f(z)|.$$

The reduced Beltrami equation $\bar{\partial} F(z) = \lambda(z)\ \operatorname{Im}(\partial F(z))$, implies

$$|\bar{\partial} F(z)| \leqslant k\, |\operatorname{Im}(\partial F(z))|.$$

- F is K-quasiregular with $K = \dfrac{1+k}{1-k}$.
- $\operatorname{Im}\big(\partial F\big)$ is a null Lagrangian.





Reduced Beltrami Equation

Theorem (Jääskeläinen, 2010)

Suppose $f : \Omega \to \mathbb{C}$, $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$, *is a solution to the reduced Beltrami equation*

$$\overline{\partial} f(z) = \lambda(z)\ \operatorname{Im}(\partial f(z)), \quad |\lambda(z)| \leqslant k < 1, \quad a.e.\ z \in \Omega.$$

If solution is not flat, i.e., $f(z) = a\,z + b$, *where* $a \in \mathbb{R}$ *and* $b \in \mathbb{C}$, *then*

$$\operatorname{Im}(\partial f) \neq 0 \qquad \textit{almost everywhere in } \Omega.$$

- Similar role as that of the Jacobian of a general quasiregular map $J(z, f) = |\overline{\partial} f(z)|^2 - |\partial f(z)|^2 \neq 0$.
- Giannetti, Iwaniec, Kovalev, Moscariello, and Sbordone (2004) proved for homeomorphisms of the plane $\mathbb{C}$, when $k < 1/2$.
- Alessandrini and Nesi (2009), Astala and Jääskeläinen (2009) for homeomorphisms of the plane $\mathbb{C}$

Key Ideas of the Proof

Theorem (Jääskeläinen, 2010)

Suppose $f:\Omega\to\mathbb{C}$, $f\in W^{1,2}_{loc}(\Omega)$, is a solution to the reduced Beltrami equation $\overline{\partial} f(z)=\lambda(z)\ \mathrm{Im}(\partial f(z)),\quad |\lambda(z)|\leqslant k<1,\quad$ *a.e.* $z\in\Omega$. *If solution is not flat, i.e., $f(z)=a\,z+b$, where $a\in\mathbb{R}$ and $b\in\mathbb{C}$, then* $\mathrm{Im}(\partial f)\neq 0$ *almost everywhere in Ω.*

- A weak reverse Hölder inequality holds for $\partial_y(\mathrm{Re} f)$.
- Thus zeros of $\partial_y(\mathrm{Re} f)$ are of infinite order.
- $\partial_y(\mathrm{Re} f)$ and $\mathrm{Im}(\partial f)$ have same zeros.
- Studying the smoothness around a point for the reduced quasiregular mapping f at the zeros of $\mathrm{Im}(\partial f)$.

Wronsky-type Theorem

Theorem (Jääskeläinen, 2010)

Suppose $\Phi, \Psi \in W^{1,2}_{\mathrm{loc}}(\Omega)$ *are solutions to*

$$\overline{\partial} f(z) = \mu(z)\, \partial f(z) + \nu(z)\, \overline{\partial f(z)}, \qquad |\mu(z)| + |\nu(z)| \leqslant k < 1,$$

for almost every $z \in \Omega$. *Solutions* Ψ *and* Φ *are* $\mathbb{R}$*-linearly independent if and only if complex gradients* $\partial\,\Phi$ *and* $\partial\,\Psi$ *are pointwise independent almost everywhere, i.e.,* $\mathrm{Im}(\partial\,\Phi\, \overline{\partial\,\Psi}) \neq 0$ *almost everywhere in* Ω.

- a linear family of quasiregular mappings has a *unique* associated $\mathbb{R}$-linear Beltrami equation (the singular set has measure zero)
- the homeomorphic case with the ideas developed in Giannetti, Iwaniec, Kovalev, Moscariello, and Sbordone, (2004): the family of Beltrami differential operators (with fixed $0 \leqslant k < 1$) is G-compact.

Nonlinear Beltrami Equation

$$\overline{\partial} f(z) = \mathcal{H}(\ z\ ,\ \partial f(z)\ ) \qquad \text{a.e.}$$

(H1) measurable

(H2) $k(z)$-Lipschitz, $0 \leqslant k(z) \leqslant k < 1$

(H3) homogeneity: $\mathcal{H}(z, 0) \equiv 0$

(H2): Note that the difference $f - g$ of two solutions (f and g) to the nonlinear Beltrami equation does not necessarily solve the same equation but it still is quasiregular.

(H3): Constants are solutions.

Uniqueness of Normalized Solutions

Homeomorphic $W^{1,2}_{\mathrm{loc}}$-solution $f : \mathbb{C} \to \mathbb{C}$ is a *normalized solution* if $f(0) = 0$ and $f(1) = 1$.

For classical Beltrami equation $\bar{\partial} f(z) = \mu(z)\, \partial f(z)$ the normalized solution is unique by Stoïlow factorization, that is, every solution can be factorized as

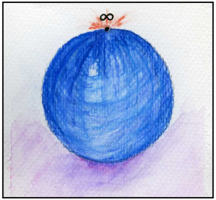
$$g = h \circ f$$

where h is *holomorphic/analytic* and f is a *normalized solution*.

For reduced Beltrami equation $\bar{\partial} F(z) = \lambda(z)\ \mathrm{Im}(\partial F(z))$, the normalized solution is unique by Astala, Iwaniec, and Martin (2009); study $z \mapsto \dfrac{f(z) - t\, z}{1 - t},\ t \in [0, 1)$.

Uniqueness in Nonlinear Systems

$$\overline{\partial} f(z) = \mathcal{H}(\ z\ ,\ \partial f(z)\ ) \qquad \text{a.e.} \qquad (*)$$



(H1) measurable

(H2) $k(z)$-Lipschitz, $0 \leqslant k(z) \leqslant k < 1$

(H3) homogeneity: $\mathcal{H}(z, 0) \equiv 0$

Theorem (Astala, Clop, Faraco, Jääskeläinen, Székelyhidi, 2011)

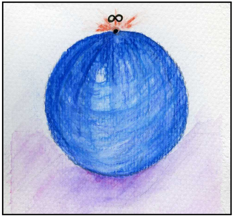
Suppose $\mathcal{H} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ satisfies (H1)–(H3) for some $k < 1$. If

$$\limsup_{|z|\to\infty} k(z) < 3 - 2\sqrt{2} = 0.17157..., \quad \textit{i.e.,} \quad \limsup_{|z|\to\infty} K(z) < \sqrt{2},$$

then the nonlinear Beltrami equation $(*)$ admits a unique homeomorphic solution $f \in W^{1,2}_{loc}(\mathbb{C})$ normalized by $f(0) = 0$ and $f(1) = 1$. The bound is sharp: for each $k(z) > 3 - 2\sqrt{2}$ near ∞, there are counterexamples.

Uniqueness in Nonlinear Systems

$$\overline{\partial} f(z) = \mathcal{H}(\ z\ ,\ \partial f(z)\ ) \qquad \text{a.e.} \qquad (*)$$



(H1) measurable

(H2) $k(z)$-Lipschitz, $0 \leqslant k(z) \leqslant k < 1$

(H3) homogeneity: $\mathcal{H}(z, 0) \equiv 0$

(H4) $\mathcal{H}(z, 1) \equiv 0$

Theorem (Astala, Clop, Faraco, Jääskeläinen, Székelyhidi, 2011)

Suppose $\mathcal{H} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ satisfies conditions (H1)–(H4) for some $k < 1$. If

$$\limsup_{|z|\to\infty} k(z) < \frac{1}{3} \quad (3 - 2\sqrt{2} = 0.17157\ldots) \qquad \limsup_{|z|\to\infty} K(z) < 2 \quad (\sqrt{2}),$$

then the function $f(z) = z$ is the unique normalized homeomorphic solution $f \in W^{1,2}_{loc}(\mathbb{C})$ to the nonlinear Beltrami equation $(*)$. The bound is sharp.

Idea of the Proof

Let f, g be two normalized solutions, $0 \mapsto 0$, $1 \mapsto 1$.

$$|P(h(z))| = |f(z) - g(z)| \leqslant C|z|^{\|K(z)\|} \leqslant C|h^{-1}(h(z))|^{\|K(z)\|} \leqslant C|h(z)|^{\|K(z)\|^2}$$

- P is *analytic*, h is normalized and $\|K(z)\|$-quasiconformal, Stoïlow factorization (difference is $\|K(z)\|$-quasiregular)
- f and g are $\|K(z)\|$-quasiconformal
- h^{-1} is $\|K(z)\|$-quasiconformal

P is polynomial with at least two zeros, $z = 0$ and $z = 1$. Hence degree $\geqslant 2$.

Near ∞, maps f, g, and h are K_0-quasiconformal for any $K_0 < \sqrt{2}$. For large $|z|$,

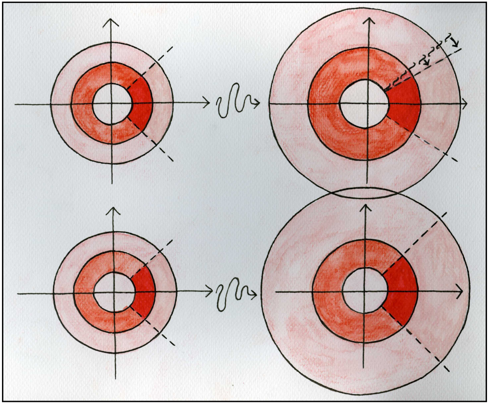
$$\frac{1}{C}|z|^{2/K_0} \leqslant |P(h(z)|) = |f(z) - g(z)| \leqslant C|z|^{K_0}.$$

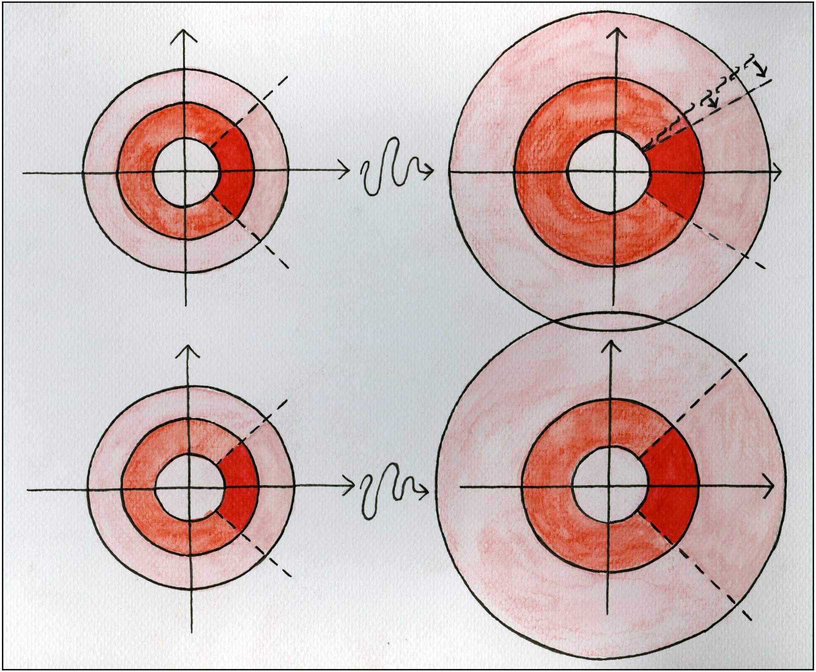
Thus $K_0 \geqslant \sqrt{2}$, which is contradiction.

Counterexamples

$$f_t(z) = \begin{cases} (1+t)\, z\, |z|^{\sqrt{2}-1} - t\, (z\,|z|^{1/\sqrt{2}-1})^2, & \text{for } |z| > 1, \\ (1+t)\, z - t\, z^2, & \text{for } |z| \leqslant 1, \end{cases}$$

$$g_t(z) = \begin{cases} (1+t)\, z\, |z|^{\sqrt{2}-1} - t\, z\, |z|^{1/\sqrt{2}-1}, & \text{for } |z| > 1, \\ z, & \text{for } |z| \leqslant 1. \end{cases}$$





Counterexamples

$$f_t(z) = \begin{cases} (1+t)\, z\,|z|^{\sqrt{2}-1} - t\,(z\,|z|^{1/\sqrt{2}-1})^2, & \text{for } |z|>1, \\ (1+t)\,z - t\,z^2, & \text{for } |z| \leqslant 1, \end{cases}$$

$$g_t(z) = \begin{cases} (1+t)\, z\,|z|^{\sqrt{2}-1} - t\,z\,|z|^{1/\sqrt{2}-1}, & \text{for } |z|>1, \\ z, & \text{for } |z| \leqslant 1. \end{cases}$$

Define for fixed $z \notin \partial\mathbb{D}$

$$\mathcal{H}(z\,,0) = 0, \qquad \mathcal{H}(z\,,\partial f(z)) = \overline{\partial} f(z), \qquad \mathcal{H}(z\,,\partial\, g(z)) = \overline{\partial}\, g(z).$$

The map $\mathcal{H}(z\,,\cdot) : \{0, \partial f(z), \partial\, g(z)\} \to \mathbb{C}$ is k_0-Lipschitz, where $k_0 = \max\{k, k_f, k_g\} \to 3 - 2\sqrt{2}$ as $t \to 0$.

By the *Kirszbraun extension theorem*, the mapping can be *extend*ed to a k_0-Lipschitz map $\mathcal{H}(z,\cdot) : \mathbb{C} \to \mathbb{C}$. From an abstract use of the Kirszbraun extension theorem, however, it is not entirely clear that the obtained map $\mathcal{H}$ is measurable in z, i.e., (H1) is satisfied.

Most General Nonlinear Beltrami Equation

$$\overline{\partial} f(z) = \mathcal{H}(z\,, f\,, \partial f(z)) \qquad \text{a.e.}$$

Note that, no matter how small is the distortion, the uniqueness of normalized solutions need not hold for the general nonlinear Beltrami equation.

Choose $0 < k < 1$ and let $f_t(z) = t\,\bar{z} + (1-t)\,z$, where $0 < t \leqslant k/(1+k)$. Next, set

$$\mathcal{H}(z\,,\,\zeta\,,\,w) = \mu(z\,,\,\zeta)\,w, \qquad \mu(z\,,\,\zeta) = \begin{cases} \dfrac{|\zeta - z|}{|\zeta - \bar{z}|}, & 0 \leqslant |\zeta - z| \leqslant k|\zeta - \bar{z}| \\ k, & \text{otherwise.} \end{cases}$$

Thank You!

"Geometry is just plane fun." – Unknown