REDUCED QUASIREGULAR MAPPINGS

Aalto
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linear

Elliptic PDEs in the complex plane

complexification / notation: $z = x + iy$, $x, y \in \mathbb{R}$

$$\partial_{\bar z} f(z) = \frac{1}{2}\left(\partial_x f(z) + i\,\partial_y f(z)\right) \quad \leftarrow \text{Cauchy-Riemann operator}$$

$$\partial_z f(z) = \frac{1}{2}\left(\partial_x f(z) - i\,\partial_y f(z)\right) \quad \leftarrow \text{formal adjoint}$$

$f \in W^{1,2}_{loc}(\Omega, \mathbb{C})$ (natural domain of definition)

open, connected = domain
$\Omega \subset \mathbb{C}$

Classical Beltrami equation

$$\partial_{\bar z} f(z) = \mu(z)\, \partial_z f(z) \qquad \text{(B)}$$

measurable, $\|\mu\|_\infty \le k < 1$

- $\mathbb{C}$-linear
- two-dimensional K-quasiregular mappings; $K = \frac{1+k}{1-k}$
- if $f \in W^{1,2}_{loc}(\mathbb{C}, \mathbb{C})$ is a homeomorphic solution to (B), then any other solution takes the form

$$g = H \circ f,$$

where H is holomorphic/analytic (Stoilow factorization)

$\mathbb{R}$-linear Beltrami equation

$$\partial_{\bar z} f(z) = \mu(z)\, \partial_z f(z) + \nu(z)\, \overline{\partial_z f(z)} \qquad (R)$$

measurable, $|\mu(z)| + |\nu(z)| \leq k < 1$

- homotopic to C-R operator
- all first-order uniformly elliptic / linear equations homotopic to C-R can be represented in this form
- solutions are qr
- arise, for instance, in:
  - divergence-type (conservation laws, equations of motion & state)

$$\operatorname{div} A(z) \nabla u = 0, \qquad u \in W^{1,2}_{Loc}(\Omega, \mathbb{R})$$

$A : \Omega \to \mathbb{R}^{2\times 2}$ measurable, uniformly elliptic

Conjugate/stream function $v \in W^{1,2}_{Loc}(\Omega, \mathbb{R})$ with $\nabla v = * A(z) \nabla u$

$*$ Hodge-operator rotation by $90°$ $*(x,y) = (-y, x)$

$f = u + iv$ solves (R)

(" $\Rightarrow$ " Ω simply connected)

  - hodograph transformation of the complex gradient of a p-harmonic function ($\operatorname{div} |\nabla u|^{p-2} \nabla u = 0$)

$$\frac{\partial h}{\partial \bar\xi} = \left(\frac{1}{2} - \frac{1}{p}\right)\left(\frac{\xi}{\bar\xi}\frac{\partial h}{\partial \xi} + \frac{\bar\xi^n}{\xi^n}\overline{\frac{\partial h}{\partial \xi}}\right)$$

- parametrization? $H \circ f$ generally fails to satisfy (R)

  H holomorphic

- Generalized Stoïlow factorization $\longrightarrow$

Generalized Stoilow
Astala–Iwaniec–Martin 2009

Let $f \in W^{1,2}_{loc}(\Omega, \mathbb{C})$ be a homeomorphic solution to

$$\partial_{\bar z} f(z) = \mu(z)\, \partial_z f(z) + \nu(z)\, \overline{\partial_z f(z)},$$

then any other solution $\underbrace{}_{g \in W^{1,2}_{loc}(\Omega, \mathbb{C})}$ to the equation takes the form

$$g = F \circ f,$$

where

$$\partial_{\bar w} F = \lambda(w)\, \operatorname{Im} \partial_w F, \qquad w \in f(\Omega)$$

and

$$\lambda(w) = \frac{-2i\,\nu(z)}{1 + |\nu(z)|^2 - |\mu(z)|^2}, \qquad z = f^{-1}(w)$$

Converse direction holds, too.

- $W^{1,2}_{loc}$-solutions to $\boxed{\partial_{\bar z} f(z) = \lambda(z)\, \operatorname{Im} \partial_z f}$, $|\lambda| \le k < 1$

  are called reduced quasiregular mappings

  (not every qr is rqr: $z \mapsto z + b\bar z$, $b \in \mathbb{D}$)

- λ does not depend on the derivatives of f

  (e.g., Hölder regularity of $\mu, \nu \longrightarrow \lambda$)

  in general, λ depends on f

- EXAMPLES

  – $z \mapsto az + b$, $a \in \mathbb{R}$, $b \in \mathbb{C}$

  is always a solution

  (if a homeomorphic solution $f: \mathbb{C} \to \mathbb{C}$ fixes two points,

  then $f(z) \equiv z$, AIM09)

  $$\frac{f(z) - tz}{1-t}$$ normalized $0 \mapsto 0$, $1 \mapsto 1$, infinity behaviour $\Rightarrow$ qc

- constant dilatation: $\partial_{\bar z} f(z) = K \,\mathrm{Im}\, \partial_z f(z)$

$\Longleftrightarrow \quad f = f_0^{-1} \circ H \circ f_0$, H holomorphic/analytic

$f_0 = z + k\bar z$, $\quad K = \dfrac{2ki}{1+|k|^2}$.

- radial stretching:

$f_1(z) = z|z|^{K-1}$, $\quad f_2(z) = z|z|^{1/K-1}$

rotated solves reduced, i.e., if_j solves w. $\lambda_j = i\mu_j$

$\mu_j(z) = (-1)^{j-1} \dfrac{K-1}{K+1} \dfrac{z}{\bar z}$

- Remarks - reduced qr has same Hölder-regularity as qr, i.e., $C^{1/K}$ (cannot be improved in general)

  - probably no invariant semi-group structure:
    - for constant dilatation $f_0^{-1} \Gamma f_0$ (yes) ↑ analytic
    - $f_1 \circ f_2 \circ \dots \circ f_1$ (rotated radial stretchings) is not uniformly qr (Sullivan–Tukia–type)

  - linear mappings are the only ones with $\mathrm{Im}\, f_z \equiv 0$:

$\boxed{\text{non-vanishing}}$ Suppose that $f: \Omega \to \mathbb{C}$, $f \in W^{1,2}_{loc}(\Omega, \mathbb{C})$ is a reduced qr mapping. If it is not flat, i.e., $f(z) = az + b$, $a \in \mathbb{R}$, then

$$\mathrm{Im}\, f_z \neq 0 \quad \text{a.e.}$$

- for global homeomorphic reduced qc map $f : \mathbb{C} \to \mathbb{C}$, $\operatorname{Im} \partial_z f(z)$ does <u>not</u> change sign (Bojarski - D'Onofrio - Iwaniec - Sbordone 2005): study

$$\operatorname{Im}\left(\frac{f(z) - f(w)}{z - w}\right) : \quad \mathbb{C} \times \mathbb{C} \setminus \{(z,w) : z = w\} \longrightarrow \mathbb{R}$$

If real, on some points then $f(z) = tz + f(0)$

otherwise, continuous, non-vanishing, real-valued function on connected domain $\Rightarrow$ does not change sign

- non-vanishing history for $\mathbb{C} \to \mathbb{C}$ homeo: Giannetti - Iwaniec - Kovalev - Moscariello - Sbordone 2004 $k \leq \frac{1}{2}$

  for all $k < 1$, Alessandrini - Nesi 2009, Astala - Jääskeläinen 2009

- homeo $\Rightarrow$ does not change sign

- does not change sign $\Rightarrow$ homeo ?

<u>Injectivity of qr mappings</u>

$\boxed{\text{injectivity vs range of } Df \text{ Iwaniec - Kovalev - Onninen 2009}}$

If $f : \Omega \to \mathbb{C}$ non-constant qr and $\operatorname{Im} f_z \geq 0$ a.e., then f is a local homeo

If $\Omega = \mathbb{C}$, the f is a homeomorphism

Suprisingly, approximating RQR $f : \Omega \longrightarrow \mathbb{C}$ (bounded domain, a thick boundary) with $\operatorname{Im}(\partial_z f) \geq 0$ a.e is constant or injective

technical detail: if f solves reduced, then $g := -if$ solves $\partial_{\bar z} g = -i\lambda(z)\, \mathrm{Re}\, \partial_z g$;
conversely ig solves $\partial_{\bar z} f = \lambda(z)\, \mathrm{Im}\, \partial_z f$

idea of the Proof of IKo2009

local structure of integral curves of continuous planar vector field near its critical point $\boxed{\frac{dz}{dt} = f(z)}$

idea of the Proof of IKo2009 when $\Omega = \mathbb{C}$

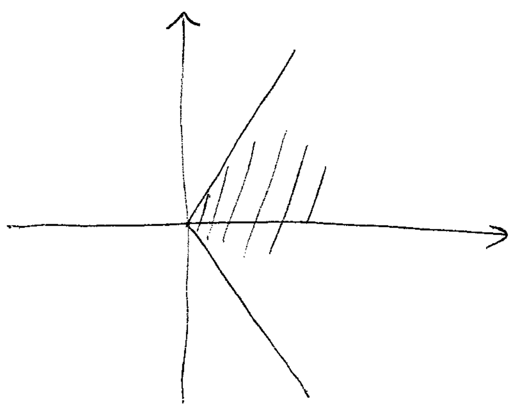
study $f^\lambda(z) = f(z) + i\lambda z$

to show : $\boxed{f^\lambda \quad qc}$

- $$|\partial_{\bar z} f^\lambda| \le k\, \mathrm{Im}(\partial_z f^\lambda) = k\Big[\underbrace{(\mathrm{Im}\,\partial_z f) + \lambda}_{>0}\Big]$$

- $$\frac{1}{J(z, f^\lambda)} \in L^\infty(\mathbb{C})$$

idea of the Proof of Thm [illegible]



- $f : \Omega \to \mathbb{C}$ is δ-monotone $(0 < \delta \le 1)$ if for every $z, w \in \Omega$

$$\langle h(z) - h(w), z - w\rangle \ge \delta\, |h(z) - h(w)|\, |z - w|$$

$\Leftrightarrow$

$$|\partial_{\bar z} f| \le \underbrace{\sqrt{1-\delta^2}\, \mathrm{Re}(\partial_z f) - \delta\, |\mathrm{Im}\, \partial_z f|}_{} = \mathrm{dist}(\partial_z f, \Gamma_\delta)$$

$$\Gamma_\delta := \left\{ (\sqrt{1-\delta^2} + i\delta)t \;;\; t \in \mathbb{R} \right\}$$

- Approximate reduced qr $f : \Omega \to \mathbb{C}$ with δ-monotone $[(\mathrm{Re}\, \partial_z f) \geq 0$ a.e.$]$
  - δ-monotone mappings are homeomorphisms $\Rightarrow$ local uniform limit is a constant or a homeomorphism

How to find?

- Riemann–Hilbert problem

$$\begin{cases} \partial_{\bar z} f_\delta (z) = \lambda(z)\, \mathrm{dist}(\partial_z f_\delta(z), \Gamma_\delta) \\ \mathrm{Re}\, f_\delta = \mathrm{Re}\, f \quad \text{on } \partial\Omega \end{cases}$$

- f_δ are δ-monotone if $\mathrm{Re}\, \partial_z f_\delta \geq 0$ a.e.

  non-vanishing of $\mathrm{Re}\, \partial_z f$ (compare $\mathrm{Im}\, \partial_z f \neq 0$ a.e.)

[Wronsky-type thm]

Suppose $\Phi, \psi \in W^{1,2}_{loc}(\Omega, \mathbb{C})$ are solutions to $\mathbb{R}$-linear B-eq

$$\partial_{\bar z} f = \mu(z)\, \partial_z f + \nu(z)\, \overline{\partial_z f}$$

ψ and ϕ are $\mathbb{R}$-linearly independent iff

$\partial_z \phi$ and $\partial_z \psi$ are pointwise independent a.e.,

i.e., $\mathrm{Im}(\partial_z \phi\, \overline{\partial_z \psi}) \neq 0$ a.e.

- Proof: Generalized Stoïlow factorization and nonvanishing of $\mathrm{Im}\, \partial_z F_h$ for reduced qr

Linear families of quasiregular mappings

- a linear family $\mathcal{F} \subset W^{1,2}_{loc}(\Omega, \mathbb{C})$ of K-quasiregular mappings

$$\mathcal{F} = \left\{ \sum_{i \in \mathcal{I}} a_i f_i \ : \ a_i \in \mathbb{R} \right\}$$

$\mathcal{I}$ countable, f_i linearly independent

- Not every linear combination of qr mappings is qr: $k\bar{z} + z$, $k\bar{z} - z$

- if mappings are solution to the same $\mathbb{R}$-linear equation, then the combination is qr

- conversely, for every family $\mathcal{F}$ there is a unique $\mathbb{R}$-linear equation

$$\partial_{\bar{z}} f = \mu(z)\, \partial_z f + \nu(z)\, \overline{\partial_z f}$$

such that every $f \in \mathcal{F}$ solves the equation

- existence (2-dim case) [If F consists of qc:s and constants, the family is two-dimensional]

Giannetti – Iwaniec – Kovalev – Moscariello – Sbordone 2004

Bojarski – D'Onofrio – Iwaniec – Sbordone 2005

assume ϕ and ψ are generators, goal: FIND μ, ν such that

$$\begin{cases} \partial_{\bar{z}} \phi = \mu\, \partial_z \phi + \nu\, \overline{\partial_z \phi} \\ \partial_{\bar{z}} \psi = \mu\, \partial_z \psi + \nu\, \overline{\partial_z \psi} \end{cases}$$

when $M(z) = \begin{bmatrix} \partial_z \phi(z) & \overline{\partial_z \phi(z)} \\ \partial_z \psi(z) & \overline{\partial_z \psi(z)} \end{bmatrix}$ is invertible,

μ and ν are uniquely defined

On the singular set, i.e., $\det M(z) = 2i \,\mathrm{Im}(\phi_z \overline{\psi_z}) = 0$

one can define $\nu \equiv 0$.

Singular set has measure zero, by Wronsky-type theorem

$\boxed{\text{Uniqueness}}$

- G-compactness for Beltrami operators

$$\{ \partial_{\bar z} - \mu \partial_z - \nu \overline{\partial_z} \; : \; |\mu| + |\nu| \le k < 1 \}$$

(G-I-K-M-S 04 introduced)

Idea for the Proof of nonvanishing

- SERIES REPRESENTATION

for a.e. $z_0 \in E = \{ z \in \Omega : \mathrm{Im}(\partial_z f(z)) = 0 \}$

$$f(w) = \underset{\substack{\cap \\ \mathbb{C}}}{c_0} + \underset{\substack{\cap \\ \mathbb{R}}}{c_1}(w - z_0) + \varepsilon(w) \qquad \text{near } z_0$$

c_0, c_1 depending only on z_0, f

$\varepsilon(w)$: $\int_{D(z_0,r)} |D\varepsilon| \, dm = O(r^{n+1}) \quad \forall n \in \mathbb{N}$

- $g(w) := f(w) - c_0 - c_1(w - z_0)$ solves the same reduced equation as f

$\Rightarrow$ g qr with $\int_{\mathbb{D}(z_0,r)} |Dg| = O(r^{n+1})$

- QR + Morrey

$$\underbrace{\sup_{|z_0-w|<r/2} \overbrace{|g(z_0) - g(w)|}^{\leq\, c\left(\frac{|z_0-w|}{r}\right)^{\alpha(K)}\left(\int_{\mathbb{D}(z_0,r)} |Dg|^2\right)^{1/2}}}_{\geq\, c r^{\gamma},\ \gamma>0} = O(r^{n+1})$$

- by Stoilow $g = H \circ G$

$H \uparrow O(|z - G(z_0)|^m)$, $m \geq 1$ for nonconstant

- for the series representation, a.e. $z_0 \in E$ is a zero of infinite order

$$\int_{\mathbb{D}(z_0,r)} |\partial_{\bar z} f| \leq k \int_{\mathbb{D}(z_0,r)} \operatorname{Im}(\partial_z f) \leq O(r^n)$$

and generalized Cauchy formula

$$f(w) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}(z_0,r_0)} \frac{f(z)}{z-w}\,dz + \frac{1}{\pi}\int_{\mathbb{D}(z_0,r_0)} \frac{\partial_{\bar z} f(z)}{w-z}\,dm(z)$$

+ calculation & operator theory

- zeros of infinite order

– ∴ $(\mathrm{Re}\, f)_y \in L^2_{loc}(\Omega)$ has weak-reverse Hölder

$\int_\Omega w\, L(\varphi)\, dm = 0,\ \forall \varphi \in C_0^\infty(\Omega)$

Inequality (weak solution to adjoint equation of non-divergence type uniformly elliptic operator $L = \sum_{i,j=1}^{2} a_{ij}(z) \frac{\partial^2}{x_i x_j}$)

$$\left(\frac{1}{r^2} \int_B (\mathrm{Re}\, f)_y^2 \, dm \right)^{1/2} \leq \frac{C}{r^2} \int_{2B} |(\mathrm{Re}\, f)_y| \, dm$$

– iterating ~~weak~~ reverse Hölder

– $(\mathrm{Re}\, f)_y$ and $\mathrm{Im}\, \partial_z f$ have same zeros