G-compactness and Families of Quasiconformal Mappings

22/04/2013
Barcelona

G-convergence

- Italian school of PDEs for second-order equations of divergence-type, Spagnolo, Marino, De Georgi 60s and 70s;

(DIV) $$\operatorname{div}(A \nabla u) = \operatorname{div} h$$

$\begin{bmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{bmatrix}$; $\boxed{u \in W^{1,2}_{loc}(\Omega, \mathbb{R})}$; $\boxed{h \in L^2_{loc}(\Omega, \mathbb{C})}$; $\Omega \subset \mathbb{C}$

measurable, symmetric $a_{12} = a_{21}$, uniformly elliptic $\frac{|\xi|^2}{\sqrt{K}} \leq \langle A(z)\xi, \xi \rangle \leq \sqrt{K}\,|\xi|^2$, $\xi \in \mathbb{C}$

Defn $\operatorname{div}(A_j \nabla)$ <u>G-converges</u> to $\operatorname{div}(A\nabla)$ if

Given $u^j \longrightarrow u$ weakly in $W^{1,2}(\Omega, \mathbb{R})$ and $h^j \to h$ strongly in $L^2(\Omega, \mathbb{C})$ such that $\operatorname{div}(A_j \nabla u^j) = \operatorname{div} h^j$

Then $\operatorname{div}(A \nabla u) = \operatorname{div} h$.

- Route to PDEs in planar theory of qc maps is made via Beltrami equation

(*) $$\frac{\partial f}{\partial \bar z} = \mu(z) \frac{\partial f}{\partial z} \quad \text{for a.e. in } \Omega,$$

where $|\mu(z)| \leq k < 1$ measurable, $f \in W^{1,2}_{loc}(\Omega)$

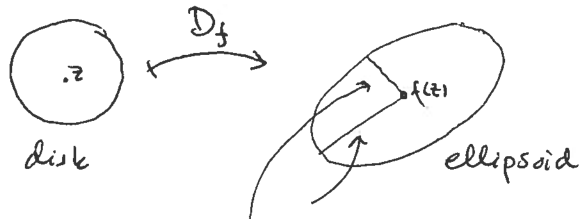
$z = x + iy$

$\partial_{\bar z} = \frac{1}{2}(\partial_x + i\partial_y) \longleftarrow$ Cauchy-Riemann operator

$\partial_z = \frac{1}{2}(\partial_x - i\partial_y) \longleftarrow$ its formal adjoint

- if f is homeomorphism, it is called <u>quasiconformal</u>

- infinitesimally



$$\frac{|\partial_z f| + |\partial_{\bar z} f|}{|\partial_z f| - |\partial_{\bar z} f|} \le \frac{1+k}{1-k}$$

- nonhomeomorphic solutions $\quad h \circ f \quad$ (Stoïlow factorization)

(captures geometric properties, broadly) $\qquad \uparrow$ analytic

- Connection between divergence-type and Beltrami equations

(DIV) $\iff$ $\partial_{\bar z} f - \mu(z)\,\partial_z f - \nu(z)\,\overline{\partial_z f} = a(z)\,h(z) + b(z)\,\overline{h(z)}$

$f = u + iv$ ← stream function $\qquad * (A \nabla u - H) = \nabla v$

↑ Hodge star $\begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}$ $\quad$ ↻ $90°$

μ and ν are given by A.

$$|\mu| + |\nu| \le k = \frac{K-1}{K+1}$$

<u>Defn</u> (G-convergence for Beltrami operators)

$$\mathcal{B}_j = \partial_{\bar z} - \mu^j(z)\,\partial_z - \nu^j(z)\,\overline{\partial_z} \xrightarrow{G} \partial_{\bar z} - \mu(z)\,\partial_z - \nu(z)\,\overline{\partial_z} = \mathcal{B}$$

if for every $f_j \to f$ weakly in $W^{1,2}_{loc}(\Omega, \mathbb{C})$

such that $\mathcal{B}_j f_j = h_j$ converges strongly in $L^2_{loc}(\Omega, \mathbb{C})$,

$$\mathcal{B}_j f_j = \mathcal{B} f$$

↑ introduced the idea

<u>Theorem</u> (Giannetti, Iwaniec, Kovalev, Moscariello, Sbordone 2004)
+ Alessandrini, Nesi 2001 + Astala, Jääskeläinen 2001 + Bojarski, D'Onofrio, Iwaniec, Sbordone 2005

$$\{\, \partial_{\bar z} - \mu(z)\,\partial_z - \nu(z)\,\overline{\partial_z} \;:\; |\mu| + |\nu| \le k < 1 \,\}$$

is G-compact

Example that gives the idea to the general case

$$\boxed{\partial_{\bar z} - \mu_j \partial_z}$$

Let Φ^j be K-quasiconformal solution to

$$\partial_{\bar z} \Phi^j = \mu^j(z)\, \partial_z \Phi^j$$

s.t. $\Phi^j(0)=0$, $\Phi^j(1) = 1$. Its unique (see Stoïlow)

$\{\phi(0)=0,\ \phi(1)=1,\ \phi\ K\text{-qc}\}$ is a normal family. Thus we have a subsequence $\Phi^{j_k} \longrightarrow \Phi$ uniformly on compact sets (Φ K-qc, $\Phi(0)=0$, $\Phi(1)=1$).

Set $\mu(z) = \dfrac{\partial_{\bar z}\Phi(z)}{\partial_z \Phi(z)}$.

Claim. $\dfrac{\partial}{\partial \bar z} - \mu \dfrac{\partial}{\partial z}$ is the G-limit of $\dfrac{\partial}{\partial \bar z} - \mu_j \dfrac{\partial}{\partial z}$.

Suppose $f^j \longrightarrow f$ in $W^{1,2}_{loc}(\Omega, \mathbb{C})$ s.t. $h_j := \partial_{\bar z} f^j - \mu^j(z)\, \partial_z f^j \longrightarrow h$ strongly in $L^2_{loc}(\Omega, \mathbb{C})$

Goal $\partial_{\bar z} f - \mu(z)\, \partial_z f = h \iff \underbrace{\partial_z \Phi\, \partial_{\bar z} f - \partial_{\bar z}\Phi\, \partial_z f}_{(\Phi\, \partial_{\bar z} f)_z - (\Phi\, \partial_z f)_{\bar z}} = \partial_z \Phi\, h$ in the sense of distributions

$\forall \eta \in C_0^\infty(\Omega, \mathbb{C})$

$$\int_\Omega (\Phi_z f_{\bar z} - \Phi_{\bar z} f_z)\,\eta = \int_\Omega (\eta_{\bar z} f_z - \eta_z f_{\bar z})\, \Phi \quad \text{(null Lagrangian)}$$

$$\overset{\text{Goal}}{=} \int_\Omega \eta\, h\, \Phi_z .$$

for each j we have

$$\int_\Omega \left(\eta_{\bar z} \boxed{f^j_z} - \eta_z \boxed{f^j_{\bar z}}\right) \boxed{\bar\Phi^j} = \int_\Omega \eta \boxed{h^j} \boxed{\bar\Phi^j_z}$$

Passing to limit

$f^j_z \xrightarrow{\text{weakly}} f_z$, $\quad f^j_{\bar z} \xrightarrow{\text{weakly}} f_{\bar z}$, $\quad \bar\Phi^j \xrightarrow{\text{uniformly on compact set}} \bar\Phi$, $\quad h^j \xrightarrow{\text{strongly in } L^2} h$, $\quad \bar\Phi^j_z \xrightarrow{\text{weakly}} \bar\Phi_z$

$\bar\Phi^j f^j_z \, ; \, \bar\Phi^j f^j_{\bar z}$ <u>ok</u> $\qquad\qquad h^j \bar\Phi^j_z$ <u>ok</u>

$\downarrow \; L^1_{loc} \qquad\qquad\qquad \downarrow \; L^1_{loc}$

$\bar\Phi f_z \quad \bar\Phi f_{\bar z} \qquad\qquad\qquad h \bar\Phi_z$

□ claim

G-limits are not <u>always</u> easy to predict.

$$\frac{\partial}{\partial \bar z} - k e^{inx} \frac{\partial}{\partial z} \xrightarrow{\;G\;} \frac{\partial}{\partial \bar z} - k^2 \frac{\partial}{\partial z}$$

weakly to 0

Proof of Theorem

1. $\partial_{\bar z} - \mu^j(z)\,\partial_z - \nu^j(z)\,\overline{\partial_z}\,, \quad |\mu^j| + |\nu^j| \le k$

Find $\Phi^j, \Psi^j : \mathbb{C} \to \mathbb{C}$ K-qc s.t.

$$(*)\quad \begin{cases} \partial_{\bar z}\Phi^j - \mu^j(z)\,\partial_z\Phi^j + \overline{\nu^j(z)}\;\overline{\partial_z\Phi^j} = 0 \\ \partial_{\bar z}\Psi^j - \mu^j(z)\,\partial_z\Psi^j + \overline{\nu^j(z)}\;\overline{\partial_z\Psi^j} = 0 \end{cases} \qquad \begin{array}{l} \phi(0) = 0 \\ \phi(1) = 1 \\ \psi(0) = 0 \\ \psi(1) = i \end{array}$$

<u>Existence</u> by Schrauder's fixed point thm for annulus $z e^{c\phi(z) - c\phi(1)}$, then go to limit

<u>Uniqueness</u> factorization $\Psi = F \circ \Phi$ $\leftarrow$ reduced qc

+ behaviour near ∞

2. Take limits

$\Phi^{j_k} \longrightarrow \Phi$ locally uniformly
Φ K-qc, $\Phi(0) = 0$, $\Phi(1) = 1$

$\psi^{j_k} \longrightarrow \psi \qquad \psi$ K-qc, $\psi(0) = 0$, $\psi(1) = i$

3. ϕ^j, ψ^j generate $\mathbb{R}$-linear family of qc mappings, i.e., $\alpha\phi^j + \beta\psi^j$ is qc $\alpha, \beta \in \mathbb{R}$ (when $\alpha = 0 = \beta$ we have constant)
(uniqueness of normalized solutions gives injectivity)

$\Longrightarrow$ Φ and ψ generate $\mathbb{R}$-linear family of qc mappings!

(4.)

$$(*) \Rightarrow \quad \mu^j(z) = i\,\frac{\psi^j_{\bar z}\,\overline{\phi^j_z} - \overline{\psi^j_z}\,\phi^j_{\bar z}}{2\,\mathrm{Im}(\phi^j_z\,\overline{\psi^j_z})}$$

$$\nu^j(z) = i\,\frac{\psi^j_z\,\phi^j_{\bar z} - \psi^j_{\bar z}\,\phi^j_z}{2\,\mathrm{Im}(\phi^j_z\,\overline{\psi^j_z})}$$

when $M^j(z) = \begin{bmatrix} \phi^j_z & \overline{\phi^j_z} \\ \psi^j_z & \overline{\psi^j_z} \end{bmatrix}$ is invertible

Singular set: $2i \det \mathrm{Im}(\phi^j_z \overline{\psi^j_z}) = \det M^j(z) = 0$ ← one can define $\nu \equiv 0$

$$\mu = \frac{\phi_{\bar z}}{\phi_z} = \frac{\psi_{\bar z}}{\psi_z}$$

(5.) define μ and ν for ϕ and ψ similarly

- k-bound for μ and ν (family is K-qc + technical argumentation that takes care that a.e. in $|\alpha\phi_{\bar z} + \beta\psi_{\bar z}| \le k|\alpha\phi_z + \beta\psi_z|$ depends a priori on α and β)

- <u>uniqueness</u>

SINGULAR SET MUST HAVE MEASURE ZERO

[Theorem $\operatorname{Im}(\phi_z \overline{\Psi_z}) \neq 0$ a.e.

key idea: $$\operatorname{Im}(\phi_z \overline{\Psi_z}) = \frac{J(z,\phi)}{-1+|\mu|^2-|\nu|^2}\operatorname{Im}\left([\psi\circ\Phi]_{\bar z}(\phi(z))\right)$$

has same zeros

and $\omega = (\operatorname{Re}(\psi\circ\phi))_y \in L^2_{loc}(\Omega,\mathbb{R})$ is a weak solution to adjoint equation $L^*\omega = 0$, i.e.,

using correspondence between div and Beltrami

$$\int_\Omega \omega L \xi = 0 \qquad \forall \xi \in C_0^\infty(\Omega)$$

non-divergence-type operator $\sum_{i,j=1}^{2} b_{ij}(z)\frac{\partial^2}{\partial x_i x_j}$

Bojarski et al. $\overset{2005}{\Longrightarrow}$ $\operatorname{Im}(\phi_z \overline{\Psi_z}) \leq 0$

(uniqueness + studying $\operatorname{Im}\left(\frac{\phi(z)-\phi(w)}{\psi(z)-\psi(w)}\right)$)

<u>Fabes - Stroock</u>

$L^*\omega = 0$, $\omega \geq 0$ $\Rightarrow$ reverse Hölder holds, i.e.,

$$\left(\frac{1}{r^2}\int_{B(a,r)} \omega^2\right)^{1/2} \leq \frac{C}{r^2}\int_{B(a,r)} \omega$$

$\square$

6. Show that $\partial_{\bar z} - \mu\partial_{\bar z} - \nu\partial_z$ is the G-limit.

If the family is <u>not</u> injective, i.e.,

$\alpha \phi + \beta \psi$ is K-qr for all $\alpha, \beta \in \mathbb{R}$

- ω changes sign
- Fabes-Stroock $\overset{\text{changes}}{\underset{\text{to}}{\rightsquigarrow}}$ weak reverse Hölder
- One still has $\mathrm{Im}(\phi_z \overline{\psi_z}) \neq$ a.e. (Jääskeläinen 2012)

$\Rightarrow$ unique μ and ν for K-qr family

If the family is noninjective, $\exists$

is there one-to-one
$\left(\partial_{\bar z} f = \mathcal{H}(z, \partial_z f) \leftarrow \text{compare} \right)$ $\{f_a\}_{a \in \mathbb{C}}$ s.t. $f_a - f_b$ is K-qc

- the question of uniqueness
  (not true in general Astala, Clop, Faraco, Jääskeläinen, Székelyhidi Jr. 2012)
  true, e.g., $k < 3 - 2\sqrt{2}$ near ∞
- define $\partial_{\bar z} f_a = \mathcal{H}(z, \partial_z f_a)$
  the question of uniqueness of $\mathcal{H}$: $\{\partial_z f_a(z)\}_{a \in \mathbb{C}} = \mathbb{C}$?
- $\boxed{\begin{matrix} \mathcal{H}(z, \ w) \\ \uparrow \quad \uparrow \\ C^1 \quad C^1 \\ k < 3 - \sqrt{2} \end{matrix}}$ $\Longleftrightarrow$ $\{f_a(z)\}$ + regularity

Astala, Clop, Faraco, Jääskeläinen