NONLINEAR BELTRAMI EQUATIONS AND POSITIVE JACOBIANS

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JOINT WORK WITH ASTALA, CLOP, FARACO, KOSKI

Theorem A (Positive Jacobian). A homeomorphic solution $f \in W^{1,2}_{loc}(\Omega, \mathbb{C})$ to the nonlinear Beltrami equation

$$f_{\bar z} = \mathcal{H}(z, f_z) \quad \text{a.e.}$$

has a positive Jacobian

$$J(z, f) > 0, \text{ everywhere.}$$

- ellipticity is coded in the k-Lipschitz property

$$\mathcal{H}(z, 0) \equiv 0$$

$$|\mathcal{H}(z, \xi_1) - \mathcal{H}(z, \xi_2)| \leq k|\xi_1 - \xi_2|, \quad k < 1$$

- C^1 in ξ-variable (uniformly), i.e.,

$$\mathcal{H}_\xi(z, \xi), \ \mathcal{H}_{\bar\xi}(z, \xi)$$

continuous in (z, ξ)

- Hölder continuous in z-varible:

$$z \mapsto \mathcal{H}(z, \xi) \in C^\alpha_{loc}(\Omega)$$

Complexification:

$$f_{\bar z} = \tfrac{1}{2}(f_x + i f_y)$$

$$f_z = \tfrac{1}{2}(f_x - i f_y)$$

<u>Linear case:</u> $H(z,\xi) = \mu(z)\,\xi + \nu(z)\,\overline{\xi}$, $|\mu(z)| + |\nu(z)| \le k$

$\nu, \mu \in C^{\alpha}_{loc}(\Omega) \Rightarrow$ homeomorphic solution has positive Jacobian (Renelt '82)
Bers-Nirenberg '55 representation thm.

...

Let $f(z) = z|z|^{K-1} = z(z\overline{z})^{\frac{K-1}{2}}$, $K \ge 1$

$$\left.\begin{aligned} f_{\overline{z}} &= \frac{K-1}{2}|z|^{K-1}\frac{z}{\overline{z}} \\ f_z &= \frac{K+1}{2}|z|^{K-1} \end{aligned}\right\} \quad \mu(z) = \frac{K-1}{K+1}\frac{z}{\overline{z}}$$

NOT Hölder continuous at 0

$J(z,f) = |f_z|^2 - |f_{\overline{z}}|^2$; $J(0,f) = 0$

$f \in W^{1,1}_{loc}$ $\mathcal{L} = \frac{\partial \phi}{\partial f_z} F_z + \frac{\partial \phi}{\partial f_{\overline{z}}} F_{\overline{z}} + \dots$

$\Phi(z, f, f_z, f_{\overline{z}}) = 0$

Petrovski ellipticity $\Rightarrow$ $f_{\overline{z}} = H(z, f, f_z)$

H is k-Lipschitz on f_z

<u>$\mathcal{H}$-equation</u> • governs effectively all nonlinear planar elliptic systems (homotopic to C-R-operator $\partial_{\overline{z}}$)

- introduced by Bojarski, Iwaniec Fos
  Astala-Iwaniec-Saksman 2001
- L^p-theory
- ellipticity + $W^{1,2}_{loc}(\Omega) \Rightarrow$ quasiregularity, i.e.

$$|f_{\overline{z}}| \le k|f_z|$$

if the map is homeomorphism then it is called quasiconformal

Quasiconformality (starting 30s: Grötzsch ('28), Ahlfors, Teichmüller)

50s $\longrightarrow$ Morrey, Lehto, Virtanen, Gehring, Bers, Väisälä

- f quasiconformal $\Rightarrow$ the inverse $g = f^{-1}$ is quasiconformal, i.e., $|g_{\bar{w}}| \leq k\,|g_w|$, $g \in W^{1,2}_{loc}(\Omega)$ (qs)

- ~~Lusin property $|E| = 0 \Rightarrow |f(E)| = 0$ ($W^{1,2}_{loc}$-homeomorphism)~~

- Hölder-continuous with exponent $\frac{1}{K} = \frac{1-k}{1+k}$

- Improved regularity: $f \in W^{1,p}_{loc}(\Omega)$, $p(K) > 2$

  Bojarski '58

  Astala's area distortion '94 $\quad p < \frac{2K}{K-1} = 1 + \frac{1}{k}$

- ~~Lusin property ($\Rightarrow$ $W^{1,2}_{loc}$-homeomorphism) $|E| = 0 \Leftrightarrow |f(E)| = 0$ ... $J(z,f)$ ... a.e.~~

- Lusin property $\quad |E| = 0 \Leftrightarrow |f(E)| = 0$

  ($W^{1,2}_{loc}$-homeomorphism "$\Rightarrow$")

  Thus $\quad J(z,f) > 0$ a.e.

Theorem B (Schauder estimates, Hölder continuous derivatives)

Let $f \in W^{1,2}_{loc}(\Omega, \mathbb{C})$ be a solution to

$$f_{\bar z} = \mathcal{H}(z, f_z) \quad \text{a.e.}$$

ellipticity

Hölder continuity in z

Then $f \in C^{1,\gamma}_{loc}(\Omega, \mathbb{C})$, $\gamma \leq \min\{\alpha, \frac{1}{K}\}$. $\quad \left(\frac{1}{K} = \frac{1-k}{1+k}\right)$

[$\gamma = \alpha$ if $\alpha < \frac{1}{K}$]

__Proof__ (in the autonomous case)

[Let $F \in W^{1,2}_{loc}(\Omega, \mathbb{C})$, $F_{\bar z} = \mathcal{H}(F_z)$; $\mathcal{H}(0) = 0$, $\mathcal{H}$ is k-Lipschitz

Then $F \in W^{2,p}_{loc}(\Omega, \mathbb{C}) \subset C^{1,\gamma}_{loc}(\Omega, \mathbb{C})$, $\gamma = 1 - \frac{2}{p}$]

Sobolev embedding

$p(k) > 2$

[Higher integrability]

$\gamma = 1 - \frac{2}{p} < 1 - \frac{2(K-1)}{2K} = \frac{1}{K}$

Let $U \Subset \Omega$, Ω bounded

Differential quotients

$$F^h(z) := \frac{F(z+h) - F(z)}{h}, \quad z \in U', \quad U \Subset U' \Subset \Omega$$

slightly bigger

$h \in \mathbb{C}$, $|h| \leq d(U', \partial\Omega)$

$$|F^h_{\bar z}| = \frac{|F_{\bar z}(z+h) - F_{\bar z}(z)|}{|h|} = \frac{1}{|h|}\left|\mathcal{H}(F_z(z+h)) - \mathcal{H}(F_z(z))\right|$$

$$\leq k \left| \frac{F_z(z+h) - F_z(z)}{h} \right| = k\,|F^h_z|.$$

Thus $F^h : U' \to \mathbb{C}$ is $K = \frac{1+k}{1-k}$ - qr.

Caccioppoli estimate

$$\int_{U''} |D_z F^h|^p \leq C(U', U, p) \left[\int_{U'} |F^h|^2 \right]^{\frac{p}{2}}$$

Thus

$$\int_{U''} \left| \frac{F_{\bar z}(z+h) - F_{\bar z}(z)}{h} \right|^p + \int_{U''} \left| \frac{F_z(z+h) - F_z(z)}{h} \right|^p$$

$$\leq C(U', U, p) \left[\int_{U'} \left| \frac{f(z+h) - F(z)}{h} \right|^2 \right]^{\frac{p}{2}}$$

$$\leq C(U', p) \left(\int_{\Omega} |D_z F|^2 \right)^{\frac{p}{2}}$$

does not depend on h

$$\Longrightarrow \quad F_z, F_{\bar z} \in W^{1,p}(U)$$

$\square$

General case (Theorem B) by fixing the coefficients, i.e., studying $f - F$, where

$$\begin{cases} F_{\bar z} = \mathcal{H}(z_0, F_z) & \text{on } \mathbb{D}(z_0, R) \\ \operatorname{Re}(f - F) = 0 & \text{on } \partial\mathbb{D}(z_0, R) \end{cases}$$

(Schauder's (30s) idea)

Proof of Theorem A

" $H(|f_{\bar z}|)$ "
"

Easy case: $f_{\bar z} = k|f_z|$, $f \in W^{1,2}_{loc}(\Omega, \mathbb{C})$.

Idea: Derive equation for the inverse $g = f^{-1}$ and use Hölder continuity of the derivative. (Theorem B)

$$f_{\bar z} = \frac{-g_{\bar w}}{J_g}, \quad f_z = \frac{\overline{g_w}}{J_g}, \quad w = f(z)$$

Now, we have the same equation for $-g$

$$-g_{\bar w} = k|g_w|, \quad g \in W^{1,2}_{loc}(f(\Omega)) \text{ (inverse of the qc)}$$

Schauder estimates in the autonomous case give $g \in C^{1,\gamma}_{loc}(f(\Omega))$

$\Rightarrow$ J_g (locally) bounded ; $J_f(z) = \frac{1}{J_g(w)} > 0$.

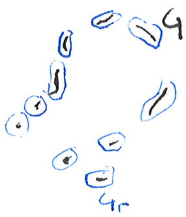
Let $\Omega \subset \mathbb{C}$ be bounded.

Null set of the Jacobian

$$G = \{ z \in \Omega : J(z, f) = 0 \}$$

has zero Lebesgue measure (qc)
Lusin property

We now assume that $G \neq \emptyset$ and find a contradiction.



For each $r > 0$ define

$$G_r = \{ z \in \Omega : 0 < d(z, G) < r \}.$$

Let $g = f^{-1}$. Then g is qc and $g \in W^{1,2}_{loc}(\Omega)$.

Lemma For $r > 0$ small enough, g solves

$$g_{\bar{\omega}} = \mathcal{H}^*(g, g_\omega) \quad \text{a.e. in } f(G_r).$$

$\mathcal{H}^*(g, \xi)$ elliptic + Hölder
($+ C^1$ in ξ-variable)

has zero measure (Lusin)

$\Rightarrow$ g $\in W^{1,2}_{loc}(\Omega)$ solves the equation a.e. in $f(G_r) \cup f(G)$

$$| \mathcal{H}^*(g(\omega_1), \xi) - \mathcal{H}^*(g(\omega_2), \xi) | \leq C_1 | g(\omega_1) - g(\omega_2) |^\alpha$$

g $\frac{1}{K}$-Hölder (qc) $\leq C_2 | \omega_1 - \omega_2 |^{\alpha/K}$

Now, Theorem B gives that

$$g \in C^{1,\gamma'}_{loc}(f(G_r) \cup f(G)) \quad \text{for some} \quad \gamma' > \gamma$$

Thus the Jacobian of g is (locally) bounded. As

$$f_{\bar z} = \frac{-g_{\bar w}}{J_g}, \quad f_z = \frac{\overline{g_w}}{J_g}, \quad w = f(z);$$

and $f \in C^{1,\gamma}_{loc}(\Omega)$ (by Theorem B)

$$|g_w| \geq \sqrt{J_g} = \frac{1}{\sqrt{J_f}} \geq \frac{1}{\underbrace{|f_z|}_{=|f_z - 0|}} \geq \frac{1}{C_1 \, \mathrm{dist}(z, G)^{\gamma}}$$

$$\geq \frac{C_2}{r^{\gamma}}, \qquad w \in f(G_r)$$

which is a contradiction (J_g blows up!) as $w \to f(G)$.

$\boxed{\text{Proof of Lemma.}}$ $\qquad$ Goal: $g_{\bar w} = \mathcal{H}^*(g, g_w)$

$$f_{\bar z} = \mathcal{H}(z, f_z)$$

can be expressed as

$$(*) \qquad -g_{\bar w} = (|g_w|^2 - |g_{\bar w}|^2)\,\mathcal{H}\left(g, \frac{\overline{g_w}}{|g_w|^2 - |g_{\bar w}|^2}\right)$$

The point is that in $f(G_r)$ we can solve $g_{\bar w}$ in terms of g and g_w.

Existence: $\left| \underbrace{|g_{\bar w}| = (|g_w|^2 - |g_{\bar w}|^2)}_{\text{RHS of } (*) \text{ depends only on } |g_{\bar w}|,\, g_w,\, g} \left|\mathcal{H}\left(g, \frac{\overline{g_w}}{|g_w|^2 - |g_{\bar w}|^2}\right)\right| \right|$

enough to solve $|g_{\bar w}|$ in terms of g and g_w.

$$\varphi(s) = s - (t^2 - s^2)\left|\mathcal{H}\left(g, \frac{\bar\xi}{t^2 - s^2}\right)\right|.$$

We will show that $\underline{\varphi(s) = 0}$ for some $s \in [0, kt]$ $\left(s = |g_{\bar w}| \leq k|g_w| = kt\right)$.

φ continuous, k-Lipschitz $\mathcal{H}$

$$\varphi(s) \geq s - kt \Rightarrow \varphi(kt) \geq 0,$$

$$\varphi(0) = -t^2\left|\mathcal{H}\left(g, \frac{\bar\xi}{t^2}\right)\right| \leq 0.$$

$\nearrow \mathcal{U}^*(g, \Xi) = \zeta$

<u>Uniqueness</u>: $s = |\zeta|$

$$\psi(\zeta) = (t^2 - s^2)\,\mathcal{H}\left(g, \frac{\Xi}{t^2 - s^2}\right)$$

Enough to show that ψ is contraction, $\left[\,|D_\zeta \psi| < 1,\ \psi \in C^1\,\right]$

since then

$\zeta + \psi(\zeta)$ must be

injective.

assumption on $\mathcal{H}$

$$\psi \in C^1, \quad \psi(\zeta) = \psi(\bar\zeta)$$

Thus it will be enough to show

$$|\psi_\zeta(\zeta)| \leq \varepsilon \quad \left(\text{implies } |\psi_{\bar\zeta}(\zeta)| \leq \varepsilon\right).$$

Now,

$$\psi_\zeta = -\bar\zeta\left(\mathcal{H}\left(g, \underbrace{\frac{\Xi}{t^2 - s^2}}_{\tau}\right) - \tau\,\mathcal{H}_\tau(g, \tau) - \bar\tau\,\mathcal{H}_{\bar\tau}(g, \tau)\right)$$

Mean-value theorem + $|\tau| \leq \dfrac{t}{t^2 - s^2} \leq \dfrac{t}{(1-k)t^2}$

$$\lesssim \text{“}\frac{1}{t}\text{”} = \text{“}\frac{1}{|g_w|}\text{”}$$

SMALL $(f(G_r))$

+ C^1-assumption

$\Rightarrow |\psi_\zeta(\zeta)| \leq \varepsilon$; for $r > 0$ small enough

Regularity:

Regularity of $\mathcal{H}$

$+ \ |\psi_\zeta| \le \varepsilon$

$+$ differentiating $\mathcal{H}^*$

$\square$

Thm (Autonomous case) $F \in W^{1,2}_{loc}(\Omega, \mathbb{C})$ homeomorphic,

$$F_{\bar z} = \mathcal{H}(F_z) \quad \text{a.e.}$$

$\mathcal{H}$ k-Lipschitz, $\mathcal{H}(0) = 0$. [No C^1-assumption]

Then

$$J(z, F) > 0 \text{ everywhere}$$

Proof. Differential quotients

$$F^h(z) = \frac{F^h(z+h) - F^h(z)}{h} \quad \text{are qr}$$

Thus $F^h = H^h \circ \Phi^h$

analytic (H^h), normalized qc (Φ^h)

One can show that

$\{H^n\}$, $\{\Phi^n\}$ are normal families

$\Rightarrow$ limits exists and are analytic and normalized qc, respectively

$\Rightarrow$ $\partial_x F = H \circ \Phi$ does not have zeros by Hurwitz theorem, since H^n does not have zeros (F^n homeo).