UNIQUENESS IN NON-LINEAR BELTRAMI EQUATIONS

18.10.11 Syracuse

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QUESTION

$f: \mathbb{C} \to \mathbb{C}$ is a homeomorphism, $f \in W^{1,2}_{loc}(\mathbb{C})$, and it solves non-linear Beltrami ~~equation~~ equation, when does two points define it uniquely?

representative for the equation
~~if there is~~ regularity between solutions, ~~[illegible]~~
G-compactness problems, qr-families

Elliptic PDEs in the complex plane

complexification/notation: $z = x + iy$, $x, y \in \mathbb{R}$

$\bar\partial f(z) = \frac{1}{2}(\partial_x f(z) + i \partial_y f(z))$ Cauchy-Riemann operator

$\partial f(z) = \frac{1}{2}(\partial_x f(z) - i \partial_y f(z))$ formal adjoint

1) Classical Beltrami equation

(B) $\bar\partial f(z) = \mu(z)\, \partial f(z)$, μ measurable, $\|\mu\|_\infty \le k < 1$

- $\mathbb{C}$-linear, ~~[illegible]~~ characterizes 2-dimensional K-quasiregular mappings, $K = \frac{1+k}{1-k}$
- if $f \in W^{1,2}_{loc}(\mathbb{C}, \mathbb{C})$ is a homeomorphic solution to (B), natural domain of definition
then any other solution g takes the form

$$g = H \circ f,$$

where H is holomorphic; Stoilow factorization

- If we normalize $f(0) = 0$, $f(1) = 1$, f is uniquely defined.

i) Uniformly elliptic non-linear equations

$$(*) \qquad \bar\partial f(z) = \mathcal{H}(z, f, \partial f)$$

homotopic to Cauchy-Riemann component, i.e., can be continuously deformed to $\bar\partial f$.

$$f \in W^{1,2}_{loc}(\mathbb{C}, \mathbb{C})$$

Thm (EXISTENCE OF SOLUTIONS)

$(*)$ admits normalized homeomorphic solution

$$\mathcal{H} : \mathbb{C} \times \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$$

f is qr $\Leftarrow$

1° $|\mathcal{H}(z, f, w_1) - \mathcal{H}(z, f, w_2)| \leq k|w_1 - w_2|$, $\quad 0 < k < 1$

contraction in ∂f-variable, uniform ellipticity

2° $\mathcal{H}(z, f, 0) \equiv 0$

homogeneity

constants are solutions $\Leftarrow$ $\Downarrow$

3° Lusin-measurable

$\mathcal{H} : Z_j \times F_j \times W_j \to \mathbb{C}$ is continuous

and increasing sequence of compact sets

$A_1 \subset A_2 \subset \dots \subset \mathbb{C}$, $\bigcup_j A_j$ has full measure

Astala, Iwaniec, Martin 2009

Bojarski, Iwaniec 1974, Iwaniec 1976 $\mathcal{H}$ measurable in z and continuous in f

Rem Lusin-measurability $\Rightarrow$ $z \mapsto \mathcal{H}(z, f(z), \partial f(z))$ is measurable

Rem ~~[illegible]~~ ~~[illegible]~~ $f(0) = 0$, $f(1) = 1$ is not the key point, as one can normalize by $f(a) = A$, $f(b) = B$. Using similarities returns the question to 0 and 1-case

Question of uniqueness is more subtle even for a system

$(**)$ $\qquad \bar\partial f(z) = \mathcal{H}(z, \partial f(z))$

1° $|\mathcal{H}(z,\omega_1) - \mathcal{H}(z,\omega_2)| \le k(z)|\omega_1 - \omega_2|, \quad 0 \le k(z) \le k < 1$

2° $\mathcal{H}(z,0) \equiv 0$

3° $z \mapsto \mathcal{H}(z,\omega)$ measurable

Rem More tools the difference # of solutions is $K(z)$-qr (does not necessarily solve $(**)$)

As mentioned, f is $K(z)$-qr, $K = \frac{1+k}{1-k}$?.

All solutions do not have the same μ is (B).

Example $|\bar\partial f(z)| = |\mathcal{H}(z, \partial f)| \le k \operatorname{dist}(\partial f(z), \Gamma)$,
where Γ is closed and $\mathcal{H}(z, \Gamma) \equiv 0$.

Thm If

$$\limsup_{|z|\to\infty} k(z) < 3 - 2\sqrt{2} = 0.17157\ldots,$$

then $(**)$ admits a unique homeomorphic solution $f \in W^{1,2}_{loc}(\mathbb{C}, \mathbb{C})$ normalized by $f(0) = 0$ and $f(1) = 1$.

Moreover, k is sharp, i.e., if $k > 3 - 2\sqrt{2}$, there exists $\mathcal{H}$ with 1°–3° which admits at least two normalized solutions.

Remarks i) 1° is for some $k<1$ (bounds only at ∞)

ii) in terms of qc-distortion bound $\limsup\limits_{|z|\to\infty} K(z) < \sqrt{2}$,

iii) No uniqueness for general systems $\bar\partial f = \mathcal{H}(z, f, \partial f)$ no matter how small the k is!

Proof. Assume there exist two normalized homeomorphic solutions $f, g \in W^{1,2}_{loc} \in W^{1,2}_{loc}(\mathbb{C}, \mathbb{C})$ to $(**)$.

Let $K_\infty = \limsup_{|z|\to\infty} K(z) < \sqrt{2}$. $\boxed{\text{Claim}}$ For any $K_0 > K_\infty$.

$$|f(z)| \le C (1+|z|)^{K_0} \quad \text{and} \quad |g(z)| \le C (1+|z|)^{K_0}$$

Indeed, f, g are $K(z)$-qc and we can decompose

$$f = H \circ F,$$

where H, F are normalized qc homeomorphisms and

$$\mu_F = \chi_{\mathbb{C} \setminus \mathbb{D}(0,R)} \mu_f .$$

Further, we may choose R so large that F is K_0-qc in $\mathbb{C}$. Then, by quasisymmetry,

$$\frac{1}{C_{K_0}} |z|^{1/K_0} \le |F(z)| \le C_{K_0} |z|^{K_0}, \quad |z| \ge 1.$$

Since H is conformal near ∞, $H(z) = Cz + O(\frac{1}{z})$, the claim follows.

The difference $f-g$ is quasiregular, but not necessarily injective. By Stoilow factorization

$$f-g = P \circ h,$$

where P is holomorphic and h is normalized $K(z)$-qc.

$$|P(h(z))| = |f(z) - g(z)| \overset{\text{Claim}}{\leq} C |z|^{K_0}$$

$$= C |h^{-1}(h(z))|^{K_0} \leq C |h(z)|^{K_0 \|K\|_\infty},$$

h^{-1} is $\|K(z)\|_\infty$ - qc

$|z| \geq 1$.

Hence P is a polynomial.

Two zeros, at $z=0$ and $z=1$, imply $\deg(P) \geq 2$.

As above, we can decompose $h = H_1 \circ F_1$.

Thus $\frac{1}{C} |z|^{1/K_0} \leq |h(z)|$.

Combining estimates, for $|z|$ large,

$$\frac{1}{C} |z|^{2/K_0} \leq |P(h(z))| = |f(z) - g(z)| \leq C |z|^{K_0}.$$

This implies $K_0 \geq \sqrt{2}$ $\lightning$

Counterexample for $k > 3 - 2\sqrt{2}$

For any $0 < t < 1$, set

$$F_t(z) = \begin{cases} (1+t)\, z/|z| - t z^2, & |z| > 1 \\ (1+t)\, z - t z^2, & |z| \le 1 \end{cases}$$

$$G_t(z) = \begin{cases} (1+t)\, z/|z| - t z, & |z| > 1 \\ z, & |z| \le 1 \end{cases}$$

Remarks i) Both normalized at $z=0$ and $z=1$.

ii) modifications of radial stretching $z \mapsto z/|z|$

iii) difference is a polynomial $t(z-z^2)$ vanishing at $z=0$ and $z=1$.

Next, reduce distortion constant by composing with extra qc-factor

$$\varphi(z) = \begin{cases} z|z|^{\frac{1}{\sqrt{2}}-1}, & |z| > 1 \\ z, & |z| \le 1 \end{cases}$$

$$f_t(z) = F_t \circ \varphi$$

$$g_t(z) = G_t \circ \varphi$$

$\boxed{t \to 0}$ $K_{f_t} \to \sqrt{2}$, $K_{g_t} \to \sqrt{2}$

$$K_{f_t - g_t} \Rightarrow \sqrt{2}$$

Define for fixed $z \notin \partial\mathbb{D}$

$$w \longmapsto \mathcal{H}(z, w) \quad \text{as}$$

$$\mathcal{H}(z,0) \equiv 0, \quad \mathcal{H}(z, \partial f_t(z)) = \bar{\partial} f_t(z), \quad \mathcal{H}(z, \partial g_t(z)) = \bar{\partial} g_t(z)$$

max ξ $\quad$ ξ - Lipschitz

Extend by Kirszbraun extension theorem to whole plane.

Constructive proof shows that the extension is measurable in z-variable.

More symmetries

- $\mathcal{H}(z, 1) \equiv 0 \implies$ the identity is a solution
  - unique if $k(z) < \frac{1}{3}$ near ∞
  - unique if there exists a path γ between 0 and 1 s.t. $\mathcal{H}(z, \gamma(t)) \in L^{p_0}(\mathbb{C})$, $p_0 < 2$, uniformly in t
- $\mathbb{R}$-homogeneous in the ~~[illegible]~~ second variable
- $\bar{\partial} f = \mathcal{H}(\partial f)$ homeomorphic solutions are affine