Uniqueness in Nonlinear Beltrami Equations

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Question

When a normalized homeomorphic solution to an equation is unique?

- What normalization do we choose?
- What is the equation?
- As a reminder, a map is homeomorphic, if it is bijective and the map and its inverse are continuous.

Analytic Maps

A homeomorphic analytic map on the complex plane $f : \mathbb{C} \to \mathbb{C}$ is of the form $f(z) = a\,z + b$, where $a, b \in \mathbb{C}$.

The analytic mapping is unique, if we *normalize*, for example, $0 \mapsto 0$ and $1 \mapsto 1$. In this case, $f(z) = z$.

Analytic mapping $f(z) = u(z) + i\,v(z)$ solves *the Cauchy-Riemann equations*, that is, $z = x + iy$

$$\partial_x u(z) = \partial_y v(z) \qquad \partial_y u(z) = -\partial_x v(z)$$

or with the complex notation

$$\overline{\partial}\, f(z) = 0$$

where $\overline{\partial} = \frac{1}{2}(\partial_x + i\,\partial_y)$.

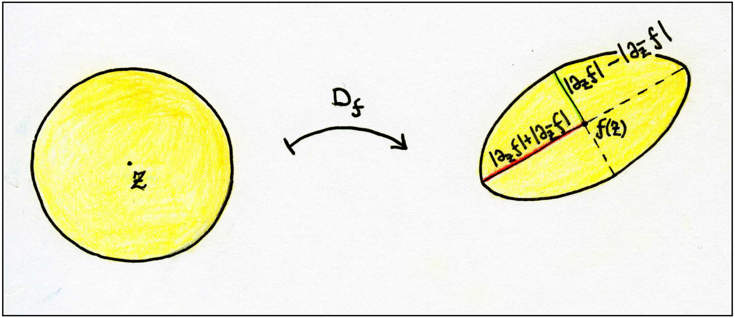
Almost Analytic Maps

A homeomorphism $f : \mathbb{C} \to \mathbb{C} \in W^{1,2}_{\text{loc}}(\mathbb{C}, \mathbb{C})$ is *K-quasiconformal* if

$$|\overline{\partial} f(z)| \leqslant k\, |\partial f(z)|, \qquad K = \frac{1+k}{1-k},$$

where $\overline{\partial} = \frac{1}{2}(\partial_x + i\,\partial_y)$, $\partial = \frac{1}{2}(\partial_x - i\,\partial_y)$, $z = x + i\,y$.

Infinitesimally a quasiconformal function maps circles into ellipses.



Linear Beltrami Equations

The classical Beltrami equation

$$\overline{\partial} f(z) = \mu(z)\, \partial f(z), \qquad |\mu(z)| \leqslant k < 1, \quad \text{a.e.} \tag{$*$}$$

is $\mathbb{C}$-linear. A general *linear Beltrami equation* takes the form

$$\overline{\partial} f(z) = \mu(z)\, \partial f(z) + \nu(z)\, \overline{\partial f(z)}, \qquad |\mu(z)| + |\nu(z)| \leqslant k < 1, \tag{$**$}$$

for almost every $z \in \mathbb{C}$.

Homeomorphic $W^{1,2}_{\text{loc}}$-solution $f : \mathbb{C} \to \mathbb{C}$ is a *normalized solution* if $f(0) = 0$ and $f(1) = 1$.

For the classical Beltrami equation $(*)$ the normalized solution is unique by the Stoïlow factorization, that is, every solution can be factorized as

$$g = P \circ f$$

where P is *analytic* and f is a *normalized solution*.

Linear Beltrami Equations

The classical Beltrami equation

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Homeomorphic $W^{1,2}_{\text{loc}}$-solution $f : \mathbb{C} \to \mathbb{C}$ is a *normalized solution* if $f(0) = 0$ and $f(1) = 1$.

For the linear equation ($**$), the normalized solution is unique by Astala, Iwaniec, and Martin (2009); the factorization by the reduced Beltrami equation, that is, every solution can be factorized as

$$g = P \circ f$$

where P is *a reduced quasiregular map* and f is a *normalized solution*.

Existence of a Normalized Solution

There exists a homeomorphic solution $f : \mathbb{C} \to \mathbb{C} \in W^{1,2}_{\text{loc}}(\mathbb{C}, \mathbb{C})$ normalized by $f(0) = 0$ and $f(1) = 1$ to

$$\overline{\partial} f(z) = \mathcal{H}(\, z \,,\, f(z) \,,\, \partial f(z)\,) \qquad \text{a.e.}$$

- k-Lipschitz, $0 \leqslant k < 1$
- homogeneity: $\mathcal{H}(z, 0) \equiv 0$

- Iwaniec, 1976 ($\mathcal{H}$ is measurable in z and continuous in f)
- Astala, Iwaniec, and Martin, 2009 (the so-called Lusin measurability of $\mathcal{H}$)
- Is the solution unique?

Normalized Solution Is Not Always Unique

Note that, no matter how small is the distortion, the uniqueness of normalized solutions need not hold even for the quasilinear Beltrami equation.

$$\overline{\partial} f(z) = \mu(z, f(z))\, \partial f(z)$$

Choose $0 < k < 1$ and let $f_t(z) = t\,\bar{z} + (1-t)\,z$, where $0 < t \leqslant k/(1+k)$. Next, set

$$\mu(z, \zeta) = \begin{cases} \dfrac{|\zeta - z|}{|\zeta - \bar{z}|}, & 0 \leqslant |\zeta - z| \leqslant k|\zeta - \bar{z}| \\ k, & \text{otherwise.} \end{cases}$$

Without the f Dependence

$$\bar{\partial} f(z) = \mathcal{H}(\, z \,,\, \partial f(z) \,) \qquad \text{a.e.}$$

(H1) measurable

(H2) $k(z)$-Lipschitz, $0 \leqslant k(z) \leqslant k < 1$

(H3) homogeneity: $\mathcal{H}(z, 0) \equiv 0$

More tools: Note that the difference $f - g$ of two solutions (f and g) to the nonlinear Beltrami equation does not necessarily solve the same equation but it still is quasiregular.

$$\left|\bar{\partial}(f(z) - g(z))\right| = \left|\mathcal{H}(z, \partial f(z)) - \mathcal{H}(z, \partial g(z))\right| \leqslant k \left|\partial(f(z) - g(z))\right|$$

Uniqueness in Nonlinear Equations

$$\overline{\partial} f(z) = \mathcal{H}(\, z \,,\, \partial f(z) \,) \qquad \text{a.e.} \qquad (*)$$

(H1) measurable

(H2) $k(z)$-Lipschitz, $0 \leqslant k(z) \leqslant k < 1$

(H3) homogeneity: $\mathcal{H}(z, 0) \equiv 0$

Theorem (Astala, Clop, Faraco, Jääskeläinen, Székelyhidi, 2012)

Suppose $\mathcal{H} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ satisfies (H1)–(H3) for some $k < 1$. If

$$\limsup_{|z|\to\infty} k(z) < 3 - 2\sqrt{2} = 0.17157\ldots, \quad \textit{i.e.,} \quad \limsup_{|z|\to\infty} K(z) < \sqrt{2},$$

then the nonlinear Beltrami equation $(*)$ admits a unique homeomorphic solution $f \in W^{1,2}_{loc}(\mathbb{C})$ normalized by $f(0) = 0$ and $f(1) = 1$. The bound is sharp: for each $k(z) > 3 - 2\sqrt{2}$ near ∞, there are counterexamples.

Uniqueness in Nonlinear Equations

$$\overline{\partial} f(z) = \mathcal{H}(\ z\ ,\ \partial f(z)\ ) \qquad \text{a.e.} \qquad (*)$$

(H1) measurable

(H2) $k(z)$-Lipschitz, $0 \leqslant k(z) \leqslant k < 1$

(H3) homogeneity: $\mathcal{H}(z, 0) \equiv 0$

Corollary

Suppose $\mathcal{H} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ satisfies (H1)–(H3) for some $k < 1$. If

$$\limsup_{|z|\to\infty} k(z) < 3 - 2\sqrt{2} = 0.17157..., \quad \textit{i.e.,} \quad \limsup_{|z|\to\infty} K(z) < \sqrt{2},$$

then the nonlinear Beltrami equation $(*)$ admits a unique homeomorphic solution $f \in W^{1,2}_{loc}(\mathbb{C})$ normalized by $f(\alpha) = a$ and $f(\beta) = b$. Moreover, the difference of two homeomorphic solutions is injective, i.e., quasiconformal.

Idea of the Proof

Let f, g be two normalized solutions, $0 \mapsto 0$, $1 \mapsto 1$.

$$|P(h(z))| = |f(z) - g(z)| \leqslant C|z|^{\|K(z)\|} \leqslant C|h^{-1}(h(z))|^{\|K(z)\|} \leqslant C|h(z)|^{\|K(z)\|^2}$$

- P is *analytic*, h is normalized and $\|K(z)\|$-quasiconformal, Stoïlow factorization (difference is $\|K(z)\|$-quasiregular)
- f and g are $\|K(z)\|$-quasiconformal

$$\frac{1}{C}|z|^{1/\|K(z)\|} \leqslant |f(z)| \leqslant C|z|^{\|K(z)\|}, \qquad |z| \geqslant 1$$

- h^{-1} is $\|K(z)\|$-quasiconformal

P is polynomial with at least two zeros, $z = 0$ and $z = 1$. Hence degree $\geqslant 2$.

Idea of the Proof

Let f, g be two normalized solutions, $0 \mapsto 0$, $1 \mapsto 1$. We have $P(h(z)) = f(z) - g(z)$, where

- h is normalized and $\|K(z)\|$-quasiconformal
- f and g are $\|K(z)\|$-quasiconformal

$$\frac{1}{C}|z|^{1/\|K(z)\|} \leqslant |f(z)| \leqslant C|z|^{\|K(z)\|}, \qquad |z| \geqslant 1$$

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P is polynomial with at least two zeros, $z = 0$ and $z = 1$. Hence degree $\geqslant 2$.

Near ∞, maps f, g, and h are K_0-quasiconformal for any $K_0 < \sqrt{2}$. For large $|z|$,

$$\frac{1}{C}|z|^{2/K_0} \leqslant |P(h(z))| = |f(z) - g(z)| \leqslant C|z|^{K_0}.$$

Thus $K_0 \geqslant \sqrt{2}$, which is contradiction.

Counterexamples

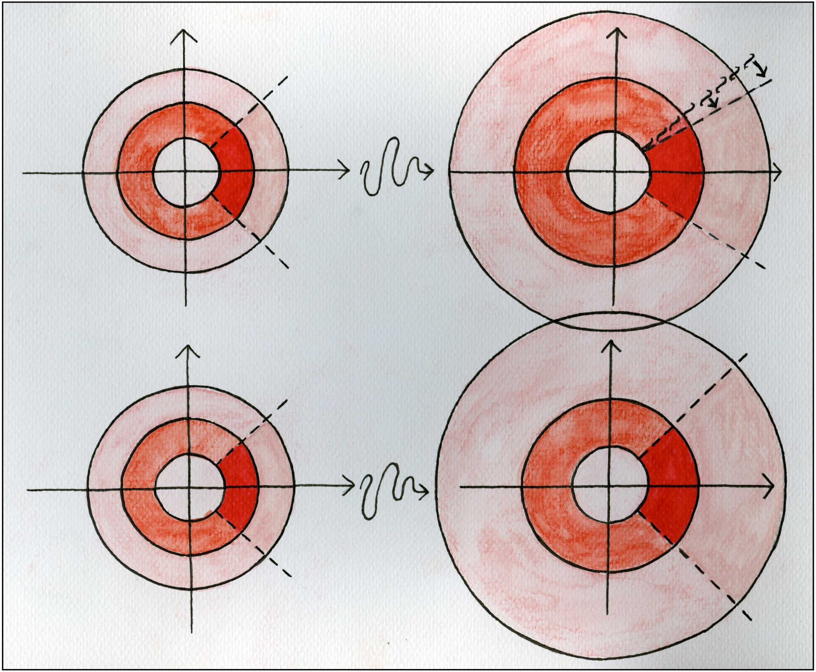
For large $|z|$,

$$\frac{1}{C}|z|^{2/\sqrt{2}} \leqslant |P(h(z))| = |f(z) - g(z)| \leqslant C|z|^{\sqrt{2}}.$$

- P is a polynomial of degree 2
- h behaves like $|z|^{1/\sqrt{2}}$, we choose the standard radial stretching $z\,|z|^{1/\sqrt{2}-1}$

$$f_t(z) = \begin{cases} (1+t)\, z\, |z|^{\sqrt{2}-1} - t\, (z\,|z|^{1/\sqrt{2}-1})^2, & \text{for } |z| > 1, \\ (1+t)\, z - t\, z^2, & \text{for } |z| \leqslant 1, \end{cases}$$

$$g_t(z) = \begin{cases} (1+t)\, z\, |z|^{\sqrt{2}-1} - t\, z\, |z|^{1/\sqrt{2}-1}, & \text{for } |z| > 1, \\ z, & \text{for } |z| \leqslant 1. \end{cases}$$



Counterexamples

$$f_t(z) = \begin{cases} (1+t)\, z\,|z|^{\sqrt{2}-1} - t\,(z\,|z|^{1/\sqrt{2}-1})^2, & \text{for } |z| > 1, \\ (1+t)\, z - t\, z^2, & \text{for } |z| \leqslant 1, \end{cases}$$

$$g_t(z) = \begin{cases} (1+t)\, z\,|z|^{\sqrt{2}-1} - t\, z\,|z|^{1/\sqrt{2}-1}, & \text{for } |z| > 1, \\ z, & \text{for } |z| \leqslant 1. \end{cases}$$

Define for fixed $z \notin \partial\mathbb{D}$

$$\mathcal{H}(z\,,0) = 0, \qquad \mathcal{H}(z\,,\partial f(z)) = \overline{\partial} f(z), \qquad \mathcal{H}(z\,,\partial\, g(z)) = \overline{\partial}\, g(z).$$

The map $\mathcal{H}(z\,,\cdot) : \{0, \partial f(z), \partial\, g(z)\} \to \mathbb{C}$ is k_0-Lipschitz, where $k_0 = \max\{k, k_f, k_g\} \to 3 - 2\sqrt{2}$ as $t \to 0$.

By the *Kirszbraun extension theorem*, the mapping can be *extend*ed to a k_0-Lipschitz map $\mathcal{H}(z, \cdot) : \mathbb{C} \to \mathbb{C}$. From an abstract use of the Kirszbraun extension theorem, however, it is not entirely clear that the obtained map $\mathcal{H}$ is measurable in z, i.e., (H1) is satisfied.

Uniqueness in Beltrami Equations

$$\overline{\partial} f(z) = \mathcal{H}(\ z\ ,\ f(z)\ ,\ \partial f(z)\ ) \qquad \text{a.e.}$$

- for the linear Beltrami equations we have uniqueness of the normalized homeomorphic solution $f : \mathbb{C} \to \mathbb{C} \in W^{1,2}_{\text{loc}}(\mathbb{C}, \mathbb{C})$
- for the nonlinear Beltrami equations without the f dependence the uniqueness holds under the explicit bound of the ellipticity near the infinity
- with the f dependence there are counterexamples for the uniqueness no matter how small the ellipticity is near the infinity

Thank You!