More on regular reduced products

Juliette Kennedy *
Department of Mathematics
University of Helsinki
Helsinki, Finland

Saharon Shelah †
Institute of Mathematics
Hebrew University
Jerusalem, Israel

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Abstract

The authors show, by means of a finitary version $\square^{fin}_{\lambda,D}$ of the combinatorial principle $\square^{b^*}_{\lambda}$ of [6], the consistency of the failure, relative to the consistency of supercompact cardinals, of the following: for all regular filters D on a cardinal λ, if M_i and N_i are elementarily equivalent models of a language of size $\leq \lambda$, then the second player has a winning strategy in the Ehrenfeucht-Fraïssé game of length λ^+ on $\prod_i M_i/D$ and $\prod_i N_i/D$. If in addition $2^\lambda = \lambda^+$ and $i < \lambda$ implies $|M_i|+|N_i| \leq \lambda^+$ this means that the ultrapowers are isomorphic. This settles negatively conjecture 18 in [1].

The problem of when two elementarily equivalent structures have isomorphic ultrapowers was prominent in the model theory of the 1960's. Keisler [2] proved, assuming GCH, that elementarily equivalent structures have isomorphic ultrapowers. Keisler's proof depended on GCH both on the question of existence of good ultrafilters and on limiting the size of the ultraproducts. More exactly, Keisler considered a language of size λ, models M of size $\leq \lambda^+$ and a λ^+-good countably incomplete ultrafilter D on λ. He proved

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that M^λ/D is λ^+-saturated. Under the weaker assumption that D is regular he proved that M^λ/D is λ^+-universal, i.e. every $N \equiv M^\lambda/D$ can be elementarily embedded into it.

Shelah [5] improved the result by eliminating GCH: two structures M and N are elementarily equivalent if and only if, for some λ and some regular ultrafilter D on λ the structures M^λ/D and N^λ/D are isomorphic. This left open the following question, asked by Chang and Keisler as Conjecture 18 in [1]:

> Let M and N be structures of cardinality $\leq \lambda$ in a language of size $\leq \lambda$ and let D be a regular ultrafilter over λ. If $M \equiv N$, then $M^\lambda/D \cong N^\lambda/D$.

The Conjecture is a natural one as most of the model theory regarding ultrapowers is centered on the regular ultrafilters. Also at the time of Keisler's question GCH was generally considered a reasonable assumption for the question.

Also the Conjecture is formulated for models of size $\leq \lambda$, perhaps for accidental reasons, but it seems more natural if M and N have cardinality $\leq \lambda^+$.

Conjecture 19 of [1], which we also address in this paper, is:

> If D is a regular ultrafilter over λ, then for all infinite M, M^λ/D is λ^{++}-universal.

In [4] the authors proved that the transfer principle $(\aleph_1, \aleph_0) \to (\lambda^+, \lambda)$ implies for all regular filters D on λ

$(1)_D$ For all M in a language of size $\leq \lambda$, M^λ/D is λ^{++}-universal.

$(2)_D$ If M_i and N_i are elementarily equivalent models of a language of size $\leq \lambda$, then the second player has a winning strategy in the Ehrenfeucht-Fraïssé game of length λ^+ on $\prod_i M_i/D$ and $\prod_i N_i/D$.

Assuming $2^\lambda = \lambda^+$, $(2)_D$ is equivalent to:

$(2')_D$ For M_i, N_i as in $(2)_D$ of cardinality $\leq \lambda^+$, $\prod_i M_i/D \cong \prod_i N_i/D$.

We note that regularity is necessary for (1_D). I.e.

Lemma 1 *For any filter D on an infinite cardinal λ, if for all infinite N the structure N^λ/D is λ^+-universal, then D is regular.*

Proof. For $k = 1, 2$ let $\mathcal{M}_k = \langle M_k, P_i^k \rangle_{i<\lambda}$, where $M_1 = \lambda$, $M_2 = \lambda + 1$ and the P_i^k are defined as follows. Let $k = 1$ and let D_0 be a regular filter on λ. Let $\{A_i\} \subseteq D_0$ witness the regularity of D_0. Thus for $\alpha < \lambda, \{i < \lambda | \alpha \in A_i\}$ is finite. Set $P_i^1 = A_i$. Now let $k = 2$. Let $P_i^2 = A_i \cup \{\lambda\}$. Now let D be any ultrafilter on λ and suppose there is an elementary embedding g mapping M_2 into M_1^λ/D. Let $g(\lambda) = [f]$ and let $X_\alpha = \{i | f(i) \in P_\alpha^1\}$. For each $\alpha < \lambda, a \in P_\alpha^2$ implies $X_\alpha \in D$. It is easy to see that $\{X_\alpha\}_{\alpha<\lambda}$ is a regular family in D. $\square$

We note that a similar argument can be used to show that $(1)_D$ fails if the language of M has size λ^+.

This was a partial answer to the above Conjectures 18 and 19. In this paper we show the converse for singular strong limit λ. Under GCH this is necessary as by Chang's Two Cardinal Theorem the transfer principle $(\aleph_1, \aleph_0) \to (\lambda^+, \lambda)$ can only fail, in the presence of GCH, for singular λ. It is known that $(\aleph_1, \aleph_0) \not\twoheadrightarrow (\aleph_{\omega+1}, \aleph_\omega)$ + GCH is consistent relative to the consistency of a supercompact cardinal. It follows that the statement $(2)_D$is independent of ZFC relative to the consistency of supercompact cardinals. On the other hand $(\aleph_1, \aleph_0) \to (\lambda^+, \lambda)$ holds for all λ if $V = L$.

In fact we show more. The results of [4] were obtained using a finitary version, denoted here by $\square_{\lambda,D}^{fin}$, of the combinatorial principle $\square_\lambda^{b^*}$ from [6], which is equivalent for all λ to $(\aleph_1, \aleph_0) \to (\lambda^+, \lambda)$. We showed in [4] that whenever λ is singular strong limit, $\square_{\lambda,D}^{fin}$ is actually equivalent to $\square_\lambda^{b^*}$, using Theorem 2.3 and Remark 2.5 of [6]. Thus $\square_{\lambda,D}^{fin}$ is equivalent to $(\aleph_1, \aleph_0) \to (\lambda^+, \lambda)$, again for λ singular strong limit. The consistency of e.g. GCH+$\neg\square_\lambda^{b^*}$ follows. Precisely we showed:

Lemma 2 *[4] Let D be a regular filter on λ where λ is a singular strong limit cardinal. Then $\square_{\lambda,D}^{fin}$ is equivalent to $\square_\lambda^{b^*}$.*

The final piece needed for obtaining the transfer principle from $(1)_D$, $(2)_D$ and $(2')_D$ for singular strong limit λ involves proving their equivalence with the principle $\square_{\lambda,D}^{fin}$, which equivalence is proved for all λ (Theorem 4).

We note that $(2)_D$ is more robust than what was originally conjectured, i.e. we have given a condition on when player II has a winning strategy in the EF game of length λ^+ on the two structures $\prod_i M_i/D$ and $\prod_i N_i/D$. Thus again if λ is singular strong limit, then the model theoretic $(1)_D$ and $(2)_D$ are equivalent to the set theoretic $(\aleph_1, \aleph_0) \to (\lambda^+, \lambda)$.

We need the following definition, from Lemma 4 of [4]:

Definition 3 *Let D be a regular filter on λ. If there exist sets u_i^ζ and integers n_i for each $\zeta < \lambda^+$ and $i < \lambda$ such that for each i, ζ*

(i) $|u_i^\zeta| < n_i$

(ii) $u_i^\zeta \subseteq \zeta$

(iii) *Let B be a finite set of ordinals and let ζ be such that $B \subseteq \zeta < \lambda^+$. Then $\{i : B \subseteq u_i^\zeta\} \in D$*

(iv) *Coherence: $\gamma \in u_i^\zeta \Rightarrow u_i^\gamma = u_i^\zeta \cap \gamma$,*

then we say that $\square^{fin}_{\lambda,D}$ holds.

Our main result:

Theorem 4 *Assume $\lambda \geq \aleph_0$ and D is a regular filter on λ. Then the following conditions are equivalent:*

(i) $\square^{fin}_{\lambda,D}$.

(ii) If M_i and N_i, $i < \lambda$, are elementarily equivalent models of a language of cardinality $\leq \lambda$, then the second player has a winning strategy in the Ehrenfeucht-Fraïssé game of length λ^+ on $\prod_i M_i/D$ and $\prod_i N_i/D$.

(iii) If M and N are structures of a language of cardinality $\leq \lambda$, $N \equiv M^\lambda/D$ and $|N| \leq \lambda^+$, then there is a homomorphism $N \to M^\lambda/D$.

(iv) If Δ is a set of quantifier-free formulas and M^λ/D satisfies every existential Δ-sentence (i.e. a sentence of the form $\exists\vec{x}(\phi_1 \wedge ... \wedge \phi_n)$, where each ϕ_i is in Δ) true in N, $|N| \leq \lambda^+$, then there is a Δ-homomorphism $N \to M^\lambda/D$, i.e. a homomorphism $N \to M^\lambda/D$ which preserves Δ formulas.

Additionally, if D is an ultrafilter, then $(i) - (iv)$ are equivalent to

(v) If M is a structure in a language of cardinality $\leq \lambda$, then M^λ/D is λ^{++}-universal.

Moreover in (ii), (iii) *and* (iv) *we can equivalently assume that the models* M_i, N_i, M *and* N *have cardinality* $\leq \lambda^+$.

Proof. $(i) \to (ii)$, $(i) \to (iii)$ and $(i) \to (iv)$ follow from the "Δ-existential" version of Theorem 2 of [4] which gives a Δ-homomorphism of N into M^λ/D for any set Δ of first-order formulas such that every Δ-existential sentence true in N is true in M. If D is an ultrafilter, [4] gives $(i) \to (v)$ and, on the other hand, $(v) \to (iii)$ is straightforward.

$(ii) \to (iii)$: It follows from $N \equiv M^\lambda/D$ that there are $M_0 \equiv M$ and a homomorphism $N \to M_0$ (i.e. a mapping from N to M_0 which respects the functions and relations of N) such that M_0 has cardinality $\leq \lambda^+$. By (ii) the second player has a winning strategy in the Ehrenfeucht-Fraïssé game of length λ^+ on M_0^λ/D and M^λ/D. Using this winning strategy we get easily a homomorphism $N \to M^\lambda/D$, by composing the appropriate mappings.

$(iii) \to (iv)$: Let M and N be as in (iv). Let N^* be an expansion of N obtained by giving a name to every Δ-definable relation. Let $(M^\lambda/D)^*$ be obtained similarly from M^λ/D. Let $M_0^* \equiv (M^\lambda/D)^*$ such that there is a homomorphism $N^* \to M_0^*$. (The existence of such M_0^* follows from the fact that we can find $M_0^* \equiv (M^*)^\lambda/D$ and a homomorphism $N^* \to M_0^*$. But then $(M^*)^\lambda/D$ is canonically embeddable into $(M^\lambda/D)^*$). By (iii) there is a homomorphism $M_0^* \to M^\lambda/D$. Thus there is a Δ-homomorphism $N \to M^\lambda/D$.

$(iv) \to (i)$: Let λ, D be given and let τ denote a language of cardinality $\leq \lambda$. It suffices to prove the following

Claim. There exist M, N such that

a) $|M| = \lambda, |N| = \lambda^+$

b) $\tau_M = \tau_N$ and $|\tau_N| \leq \lambda$

c) $M \equiv N$

d) For $\Delta =$ the quantifier free formulas of τ_M, N has a Δ-homomorphism into M^λ/D, and hence $\Box^{fin}_{\lambda,D}$ holds.

Proof. Let $\tau^* = \{F_\alpha \mid \alpha < \lambda\} \cup \{<\}$, for F_α a unary function symbol. Let K be the family of all structures M such that

$K1)$ M is a finite τ^*-structure.

$K2)$ The universe of M is $\{0, 1, \ldots, k-1\}$, for some $k \in \mathbb{N}$, $k \neq 0$, denoted $\eta(M)$.

$K3)$ $M \models \forall x(F_\alpha(x) \leq x)$ for all $\alpha < \lambda$.

$K4)$ If $m_1 = F^M_{\alpha_1}(m)$, $m_2 = F^M_{\alpha_2}(m)$ and $m_1 < m_2$, then there exists a $\beta < \lambda$ such that $m_1 = F^M_\beta(m_2)$.

$K5)$ If $F^M_{\alpha_2}(m_3) = m_2$, $F^M_{\alpha_1}(m_2) = m_1$ and $m_1 < m_2 < m_3$, then there exists $\alpha_3 < \lambda$ such that $F^M_{\alpha_3}(m_3) = m_1$.

$K6)$ $w(M) =_{df} \{\alpha \mid F^M_\alpha \text{ is not the identity}\}$ is finite.

$K7)$ For $m_1 < m_2 < k$ there is exactly one α such that $m_1 = F_\alpha(m_2)$).

We note that K is non-empty, taking K to be, e.g., a one element structure. Let $\{M_i \mid i < \lambda\}$ list K. We will add the M_i together into one structure. I.e., we define a model M^* for $\tau = \tau^* \cup \{E\}$ such that

$a_K)$ The universe of $M^* = \cup\{\{i\} \times M_i \mid i < \lambda\}$

$b_K)$ $E^{M^*} = \{\langle (i_1, m_1), (i_2, m_2)\rangle \mid m_1 < \eta(M_{i_1}), m_2 < \eta(M_{i_2}) \text{ and } i_1 = i_2\}$

$c_K)$ $<^{M^*} = \{\langle (i, m_1), (i, m_2)\rangle \mid m_1 < m_2 < \eta(M_i)\}$

$d_K)$ $F^{M^*}_\alpha(i, m) = \langle i, F^{M_i}_\alpha(m)\rangle$.

Now for $\rho < \lambda^+$ let h_ρ be a partial one to one function from λ onto ρ, and let $\langle a_\rho \mid \rho < \lambda^+\rangle$ be a set of new constant symbols.

Subclaim. There is N^* such that

$a)$ N^* is a τ^{**} structure of cardinality λ^+, where $\tau^{**} = \tau \cup \{a_\rho \mid \rho < \lambda^+\} \cup \{\overline{m}\}_{m \in M}$,

$b)$ $M^* \preccurlyeq N^* \restriction \tau$.

$c)$ $a_\rho E^{N^*} a_0$ for $\rho < \lambda^+$,

$d)$ $a_\rho <^{N^*} a_\xi$, for $\rho < \xi < \lambda^+$,

$e)$ $N^* \models F_j(\bar{a}_\rho) = \bar{a}_\epsilon$, if $h_\rho(j) = \epsilon$, for $j < \lambda$ and $\epsilon < \rho < \lambda^+$.

Proof of Subclaim. Let $T = Th(M^*, \overline{m})_{m\in M^*} \cup \{a_\rho E a_0\}_{\rho<\lambda^+} \cup \{a_\rho < a_\xi \mid \rho < \xi < \lambda^+\} \cup \{F_j(a_\rho) = a_\epsilon \mid h_\rho(j) = \epsilon, \epsilon < \rho < \lambda^+\}$. We claim that T is consistent. To see this, let

$$T_0 = \{\varphi_i(\vec{\overline{m}}\}_{i=1,\dots,n} \cup \{a_{\rho_i} E a_0\}_{i=1,\dots k} \cup \{a_{\epsilon_i} < a_{\zeta_i}\}_{i=1,\dots,l} \cup \{F_{\alpha_i}(a_{\mu_i}) = a_{\nu_i}\}_{i=1,\dots,m}$$

be a finite part of T, where $a_{i_1}, \dots, a_{i_k}$ and $\overline{m}_{j_1}, \dots, \overline{m}_{j_l}$ are all the parameters occurring in T_0. Let $I_0 = \{i \mid \text{for some } j = j_1 \dots, j_l,\ m_j = (i,a),\ i < \lambda,\ a < \eta(M_i)\}$. We can find M_{i_0}, $i_0 \in \lambda \setminus I_0$, such that if we interpret the constants a_ρ in M_{i_0} and $\overline{m}_{j_i}$ by $m_{j_i} \in M^*$, then this expansion of M^* is a model of T_0 and thus T_0 is consistent. The subclaim is proved.

Now let N^* be as in claim 1 and let $N = N^* \restriction \tau^*_M$. We note that the pair of structures N together with the M defined above satisfying $a_K) - d_K)$, satisfy the hypothesis of Theorem 2-(iv), i.e. $(M^*)^\lambda/D$ satisfies every existential Δ-sentence true in N where $\Delta =$ the quantifier-free formulas of τ^*_M. This is because $N \equiv M^*$ and these Δ-sentences are preserved under reduced products. Therefore by (iv) there is a Δ-homomorphism $g\colon N \to (M^*)^\lambda/D$. Let $g(a_\rho)$ be denoted by f_ρ/D.

We are now ready to define the sets u_i^ζ referred to in (i)-(iv) of the condition $\square^{fin}_{\lambda,D}$. To this end, for $\epsilon < \rho < \lambda^+$, $\rho \geq \lambda$, define

$$A_{\epsilon,\rho} = \{j < \lambda \mid M^* \models (f_\rho(j) > f_\epsilon(j)) \wedge F_{i(\epsilon,\rho)}(f_\rho(j)) = f_\epsilon(j)\}.$$

Note that if $\epsilon < \rho < \lambda^+$, $\rho \geq \lambda$, then $A_{\epsilon,\rho} \in D$, since $(M^*)^\lambda/D \models (g(a_\epsilon) < g(a_\rho)) \wedge F_{i(\epsilon,\rho)}(g(a_\rho)) = g(a_\epsilon)$. For each $\rho \in (\lambda, \lambda^+)$ and $j \leq \lambda$ define

$$W_{\rho,j} = \{\epsilon < \rho \mid \epsilon \geq \lambda \text{ and } j \in A_{\epsilon,\rho}\}.$$

First note that without loss of generality

$(*)_1$: $M^* \models f_\rho(j) E f_0(j)$ for every j, ρ.

We claim that

$(*)_2$: if we choose $u^\rho_j =_{df} W_{\rho,j}$ then $\langle u^\rho_j | \rho < \lambda^+, j < \lambda\rangle$ exemplifies $\square^{fin}_{\lambda,D}$, i.e. these objects satisfy the demands $(i) - (iv)$ of Definition 3.

By $(*)_1$ we can let $f_\rho(j) = (i_j, m(i_j, \rho))$. Clearly if we prove $(*)_2$ we are done.

(i), (ii): $W_{\rho,j}$ is a finite subset of ρ:

$$\begin{aligned}\{\epsilon < \rho \mid \epsilon \geq \lambda \text{ and } j \in A_{\epsilon,\rho}\} &= \{\epsilon < \rho \mid \epsilon \geq \lambda \text{ and } M \models F_{i(\epsilon,\rho)}(f_\rho(j)) = f_\epsilon(j)\} \\ &= \{\epsilon < \rho \mid \epsilon \geq \lambda \text{ and } M_{i_0} \models F^{M_{i_0}}_{i(\epsilon,\rho)}(f_\rho(j)) = f_\epsilon(j)\}.\end{aligned}$$

But $w(M_{i_0})$ is finite, and therefore so is $W_{\rho,j}$. Thus if n_i is taken to be $w(M_{i_o})$ then (i) and (ii) of $\Box^{fin}_{\lambda,D}$ are satisfied.

(iv): (coherency) holds of $W_{\rho,j}$, i.e. if $\rho_1 < \rho_2 < \lambda^+$ and $j < \lambda$ is given, if $\rho_1 \in W_{\rho_2,j}$ then $W_{\rho_1,j} = W_{\rho_2,j} \cap \rho_1$. Why? Let $\alpha \in W_{\rho_1,j}$. Then $\lambda \leq \alpha < \rho_2$ and $j \in A_{\alpha,\rho_1}$. But then $M^* \models F_{i(\alpha,\rho_1)}(f_{\rho_1}(j)) = f_\alpha(j)$ and therefore $M_{i_0} \models F^{M_{i_0}}_{i(\alpha,\rho_1)}(f_{\rho_1}(j)) = f_\alpha(j)$. $\rho_1 \in W_{\rho_2,j}$ and therefore $j \in A_{\rho_1,\rho_2}$. This means $M_{i_0} \models F^{M_{i_0}}_{i(\rho_1,\rho_2)}(f_{\rho_2}(j)) = f_{\rho_1}(j)$. By the definition of M_{i_0}, there is β such that $M_{i_0} \models F^{M_{i_0}}_\beta(f_{\rho_2}(j)) = f_\alpha(j)$. But $\beta = i(\alpha,\rho_2)$ and $j \in A_{\alpha,\rho_2}$ and therefore $\alpha \in W_{\rho_2,j}$. For the other direction suppose $\alpha \in W_{\rho_2,j} \cap \rho_1$. Then $\alpha \geq \lambda$ and $j \in A_{\alpha,\rho_2}$, i.e. $M_{i_0} \models F^{M_{i_0}}_{i(\alpha,\rho_2)}(f_{\rho_2}(j)) = f_\alpha(j)$. $\rho_1 \in W_{\rho_2,j}$ means that as before $j \in A_{\rho_1,\rho_2}$, i.e. $M_{i_0} \models F^{M_{i_0}}_{i(\rho_1,\rho_2)}(f_{\rho_2}(j)) = f_{\rho_1}(j)$. But then since $\alpha < \rho_1$, there is β such that $M_{i_0} \models F^{M_{i_0}}_\beta(f_{\rho_1}(j)) = f_\alpha(j)$, i.e. as before $j \in A_{\rho_1,j}$ and $\alpha \in W_{\rho_1,j}$.

To see that (iii) is satisfied, let $B \subseteq \lambda^+$ be a finite set of ordinals such that $B \cap \lambda = \emptyset$, and let ρ be such that $B \subseteq \rho < \lambda^+$. We wish to show that $\{j \mid B \subseteq W_{\rho,j}\} \in D$. Let $B = \{j_1, \ldots, j_n\}$. Recall that by (e) of the subclaim, $N^* \models F_{i(j,\rho)}(a_\rho) = a_j$ whence

$$C_k = \{i \mid M^* \models F_{i(j_k,\rho)}(f_\rho(i)) = f_{j_k}(i)\} \in D,$$

for each $k = 1, \ldots, n$. Also by definition if $i \in C_k$ then $i \in A_{j_k,\rho}$ whence $j_k \in W_{\rho,i}$. Thus $C_1 \cap \cdots \cap C_n \subseteq \{i \mid B \subseteq W_{\rho,j}\} \in D$. The claim is proved.

Now if we transfer the sets u^ρ_j, $\lambda \leq \rho < \lambda^+$ to all of λ^+, (iv) implies (i), proving the Claim.

The claim concerning the equivalent formulations involving models of size $\leq \lambda^+$ follows from the fact that in the derivation $(iv) \to (i)$ above we have $|M| = \lambda$ and $|N| = \lambda^+$. Thus Theorem 4 is proved. $\Box$

Corollary 5 *Assume $\lambda \geq \aleph_0$, $2^\lambda = \lambda^+$ and D is a regular filter on λ. Then the following are equivalent:*

- *(i)* $\Box^{fin}_{\lambda,D}$.
- *(ii) If M_i and N_i, $i < \lambda$, are elementarily equivalent models of a language of cardinality $\leq \lambda$, and $|M_i|$, $|N_i| \leq \lambda^+$ then $\prod_i M_i/D \cong \prod_i N_i/D$.*

Corollary 6 *GCH + the failure of properties (i)-(iv) of Theorem 4 for $\lambda = \aleph_\omega$ is consistent relative to the consistency of supercompact cardinals.*

Proof. Assume GCH and $(\aleph_1, \aleph_0) \not\twoheadrightarrow (\aleph_{\omega+1}, \aleph_\omega)$. It is well-known (cf. [1] Proposition 4.3.5) that there is a regular ultrafilter D on $\aleph_\omega$. The principle $\square^{b^*}_{\aleph_\omega}$ fails. Thus $\square^{fin}_{\aleph_\omega, D}$ fails by [4]. By Theorem 4, (i)-(v) fail for the regular ultrafilter D. $\square$

A drawback of Corollary 6 is that it deals with filters rather than ultrafilters, which was originally the most interesting case. This case will be dealt with in a work in preparation.

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