On regular reduced products*

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Abstract

Assume $\langle\aleph_0, \aleph_1\rangle \to \langle\lambda, \lambda^+\rangle$. Assume M is a model of a first order theory T of cardinality at most λ^+ in a language $\mathcal{L}(T)$ of cardinality $\leq \lambda$. Let N be a model with the same language. Let Δ be a set of first order formulas in $\mathcal{L}(T)$ and let D be a regular filter on λ. Then M is Δ-embeddable into the reduced power N^λ/D, provided that every Δ-existential formula true in M is true also in N. We obtain the following corollary: for M as above and D a regular ultrafilter over λ, M^λ/D is λ^{++}-universal. Our second result is as follows: For $i < \mu$ let M_i and N_i be elementarily equivalent models of a language which has cardinality $\leq \lambda$. Suppose D is a regular filter on λ and $\langle\aleph_0, \aleph_1\rangle \to \langle\lambda, \lambda^+\rangle$ holds. We show that then the second player has a winning strategy in the Ehrenfeucht-Fraisse game of length λ^+ on $\prod_i M_i/D$ and $\prod_i N_i/D$. This yields the following corollary: Assume GCH and λ regular (or just $\langle\aleph_0, \aleph_1\rangle \to \langle\lambda, \lambda^+\rangle$ and $2^\lambda = \lambda^+$). For L, M_i and N_i be as above, if D is a regular filter on λ, then $\prod_i M_i/D \cong \prod_i N_i/D$.

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1 Introduction

Suppose M is a first order structure and F is the Frechet filter on ω. Then the reduced power M^ω/F is $\aleph_1$-saturated and hence $\aleph_2$-universal ([6]). This was generalized by Shelah in [10] to any filter F on ω for which B^ω/F is $\aleph_1$-saturated, where B is the two element Boolean algebra, and in [8] to all regular filters on ω. In the first part of this paper we use the combinatorial principle $\Box^{b^*}_\lambda$ of Shelah [11] to generalize the result from ω to arbitrary λ, assuming $\langle\aleph_0,\aleph_1\rangle\to\langle\lambda,\lambda^+\rangle$. This gives a partial solution to Conjecture 19 in [3]: if D is a regular ultrafilter over λ, then for all infinite M, the ultrapower M^λ/D is λ^{++}-universal.

The second part of this paper addresses Problem 18 in [3], which asks if it is true that if D is a regular ultrafilter over λ, then for all elementarily equivalent models M and N of cardinality $\le\lambda$ in a language of cardinality $\le\lambda$, the ultrapowers M^λ/D and N^λ/D are isomorphic. Keisler [7] proved this for good D assuming $2^\lambda=\lambda^+$. Benda [1] weakened "good" to "contains a good filter". We prove the claim in full generality, assuming $2^\lambda=\lambda^+$ and $\langle\aleph_0,\aleph_1\rangle\to\langle\lambda,\lambda^+\rangle$.

Regarding our assumption $\langle\aleph_0,\aleph_1\rangle\to\langle\lambda,\lambda^+\rangle$, by Chang's Two-Cardinal Theorem ([2]) $\langle\aleph_0,\aleph_1\rangle\to\langle\lambda,\lambda^+\rangle$ is a consequence of $\lambda=\lambda^{<\lambda}$. So our Theorem 2 settles Conjecture 19 of [3], and Theorem 13 settles Conjecture 18 of [3], under GCH for λ regular. For singular strong limit cardinals $\langle\aleph_0,\aleph_1\rangle\to\langle\lambda,\lambda^+\rangle$ follows from $\Box_\lambda$ (Jensen [5]). In the so-called Mitchell's model ([9]) $\langle\aleph_0,\aleph_1\rangle\not\to\langle\aleph_1,\aleph_2\rangle$, so our assumption is independent of ZFC.

2 Universality

Definition 1 *Suppose Δ is a set of first order formulas of the language L. The set of* Δ-existential *formulas is the set of formulas of the form*

$$\exists x_1 ... \exists x_n(\phi_1\wedge ...\wedge\phi_n),$$

where each ϕ_i is in Δ. The set of weakly Δ-existential *formulas is the set of formulas of the above form, where each ϕ_i is in Δ or is the negation of a formula in Δ. If M and N are L-structures and $h: M\to N$, we say that h is a* Δ-homomorphism *if h preserves the truth of Δ-formulas. If h preserves also the truth of negations of Δ-formulas, it is called a* Δ-embedding.

Theorem 2 *Assume $\langle\aleph_0,\aleph_1\rangle \to \langle\lambda,\lambda^+\rangle$. Let M be a model of a first order theory T of cardinality at most λ^+, in a language L of cardinality $\leq \lambda$ and let N be a model with the same language. Let Δ be a set of first order formulas in L and let D be a regular filter on λ. We assume that every weakly Δ-existential sentence true in M is true also in N. Then there is a Δ-embedding of M into the reduced power N^λ/D.*

By letting Δ be the set of all first order sentences, we get from Theorem 2:

Corollary 3 *Assume $\langle\aleph_0,\aleph_1\rangle \to \langle\lambda,\lambda^+\rangle$. If M is a model with language $\leq \lambda$, and D is a regular ultrafilter on λ, then M^λ/D is λ^{++}-universal, i.e. if M' is of cardinality $\leq \lambda^+$, and $M' \equiv M$, then M' is elementarily embeddable into the ultrapower M^λ/D.*

We can replace "weakly Δ-existential" by "Δ-existential" in the Theorem, if we only want a Δ-homomorphism.

The idea behind the proof of Theorem 2 is roughly as follows: suppose $M = \{a_\zeta : \zeta < \lambda^+\}$. We associate to each $\zeta < \lambda^+$ finite sets $u_i^\zeta \subseteq \zeta$, $i < \lambda$, and represent the formula set Δ as a union of finite sets Δ_i. The proof involves a simultaneous recursion over λ^+ and λ. At stage i, for each $\zeta < \lambda^+$ we consider the Δ_i-type of those elements a_τ of the model whose indices lie in the set u_i^ζ, $\zeta < \lambda^+$. This will yield a witness $f_\tau(i)$ in N at stage i, τ. Naturally, the sets u_i^ζ have to have some coherence properties in order for this to work. Our embedding is then given by $a_\tau \mapsto \langle f_\tau(i) : i < \lambda\rangle/D$.

We need first an important lemma, reminiscent of Proposition 5.1 in [11]:

Lemma 4 *Assume $\langle\aleph_0,\aleph_1\rangle \to \langle\lambda,\lambda^+\rangle$. Let D be a regular filter on λ. There exist sets u_i^ζ and integers n_i for each $\zeta < \lambda^+$ and $i < \lambda$ such that for each i, ζ*

(i) $|u_i^\zeta| < n_i$

(ii) $u_i^\zeta \subseteq \zeta$

(iii) *Let B be a finite set of ordinals and let ζ be such that $B \subseteq \zeta < \lambda^+$. Then $\{i : B \subseteq u_i^\zeta\} \in D$*

(iv) *Coherency: $\gamma \in u_i^\zeta \Rightarrow u_i^\gamma = u_i^\zeta \cap \gamma$*

Assuming the lemma, and letting $M = \{a_\zeta : \zeta < \lambda^+\}$ we now define, for each ζ, a function $f_\zeta : \lambda \mapsto N$.

Let $\Delta = \{\phi_\alpha : \alpha < \lambda\}$ and let $\{A_\alpha : \alpha < \lambda\}$ be a family witnessing the regularity of D. Thus for each $i < \lambda$, the set $w_i = \{\alpha : i \in A_\alpha\}$ is finite. Let $\Delta_i = \{\phi_\alpha : \alpha \in w_i\}$, and let u_i^ζ, n_i be as in the lemma.

We define a sequence of formulas essential to the proof: suppose $\zeta < \lambda^+$ and $i < \lambda$. Let $m_i^\zeta = |u_i^\zeta|$ and let

$$u_i^\zeta = \{\xi_{\zeta,i,1}, ..., \xi_{\zeta,i,m_i^\zeta}\}$$

be the increasing enumeration of u_i^ζ. (We adopt henceforth the convention that any enumeration of u_i^ζ that is given is the increasing enumeration.) Let $\bar{\theta}_i^\zeta$ be the Δ_i-type of the tuple $\langle a_{\xi_{\zeta,i,1}}, ..., a_{\xi_{\zeta,i,m_i^\zeta}}\rangle$ in M. (So every $\phi(x_1, ..., x_{m_i^\zeta}) \in \Delta_i$ or its negation occurs as an element of $\bar{\theta}_i^\zeta$, according to whether $\phi(a_{\xi_{\zeta,i,1}}, ..., a_{\xi_{\zeta,i,m_i^\zeta}})$ or $\neg\phi(a_{\xi_{\zeta,i,1}}, ..., a_{\xi_{\zeta,i,m_i^\zeta}})$ holds in M.) We define the formula θ_i^ζ for each i by downward induction on m_i^ζ as follows:
Case 1: $m_i^\zeta + 1 = n_i$. Let $\theta_i^\zeta = \bigwedge \bar{\theta}_i^\zeta$.
Case 2: $m_i^\zeta + 1 < n_i$. Let θ_i^ζ be the conjunction of $\bar{\theta}_i^\zeta$ and all formulas of the form $\exists x_{m_i^\epsilon} \theta_i^\epsilon(x_1, ..., x_{m_i^\zeta}, x_{m_i^\epsilon})$, where ϵ satisfies $u_i^\epsilon = u_i^\zeta \cup \{\zeta\}$ and hence $m_i^\epsilon = m_i^\zeta + 1$. If no such ϵ exists, θ_i^ζ is just the conjunction of $\bar{\theta}_i^\zeta$.

An easy induction, based on the fact that there is a uniform bound n_i on the sizes of the sets u_i^ζ, shows that for a fixed $i < \lambda$, the cardinality of the set $\{\theta_i^\zeta : \zeta < \lambda^+\}$ is finite.

Let $i < \lambda$ be fixed. We define $f_\epsilon(i)$ for $\epsilon \in u_i^\zeta$ by induction on $\zeta < \lambda^+$ in such a way that the following condition remains valid:

(IH) If $\zeta^* < \zeta$ and $u_i^{\zeta^*} = \{r_{\epsilon_1}, ..., r_{\epsilon_k}\}$, then $N \models \theta_i^{\zeta^*}(f_{\epsilon_1}(i), ..., f_{\epsilon_k}(i))$.

Actually, $f_\epsilon(i)$ gets defined once and for all at the first stage ζ such that $\epsilon \in u_i^\zeta$. To define $f_\epsilon(i)$ for $\epsilon \in u_i^\zeta$, we consider different cases:
Case 1: $n_i = m_i^\zeta + 1$.
Case 1.1: $n_i = 1$. Then there is nothing to prove, since u_i^ζ is empty.
Case 1.2: $n_i > 1$. Let $u_i^\zeta = \{\xi_1, \ldots, \xi_{m_i^\zeta}\}$. Since $m_i^\zeta + 1 = n_i$, the formula θ_i^ζ is the Δ_i-type of the elements $\{a_{\xi_1}, \ldots, a_{\xi_{m_i^\zeta}}\}$. By assumption $\gamma = \xi_{m_i^\zeta}$ is the maximum element of u_i^ζ. We note that for $\epsilon \in u_i^\zeta \cap \gamma$, $f_\epsilon(i)$ is already

defined. By coherency, $u_i^\gamma = u_i^\zeta \cap \gamma = \{\xi_1, \ldots, \xi_{m_i^\zeta - 1}\}$. Since $\gamma < \zeta$, we know by the induction hypothesis that

$$N \models \theta_i^\gamma(f_{\xi_1}(i), \ldots, f_{\xi_{m_i^\zeta - 1}}(i)).$$

As $u_i^\zeta = u_i^\gamma \cup \{\gamma\}$ and $m_i^\gamma < n_i - 1$, the formula θ_i^γ contains the formula $\exists x_{m_i^\zeta} \theta_i^\zeta(x_1, \ldots, x_{m_i^\zeta})$ as a conjunct. Thus

$$N \models \exists x_{m_i^\zeta} \theta_i^\zeta(f_{\xi_1}(i), \ldots, f_{\xi_{m_i^\zeta - 1}}(i), x_{m_i^\zeta}).$$

Now let $b \in N$ witness this formula and set $f_\gamma(i) = b$.
Case 2: $m_i^\zeta + 1 < n_i$. Let $u_i^\zeta = \{\xi_1, \ldots, \xi_{m_i^\zeta}\}$. We have that $M \models \theta_i^\zeta(a_{\xi_1}, \ldots, a_{\xi_{m_i^\zeta}})$, and therefore $M \models \exists x_{m_i^\zeta} \theta_i^\zeta(a_{\xi_1}, \ldots, a_{\xi_{m_i^\zeta - 1}}, x_{m_i^\zeta})$. Let $\gamma = \max(u_i^\zeta) = \xi_{m_i^\zeta}$. By coherency $u_i^\gamma = u_i^\zeta \cap \gamma$ and therefore since $\gamma < \zeta$ again by the induction hypothesis we have that

$$N \models \theta_i^\gamma(f_{\xi_1}(i), \ldots, f_{\xi_{m_i^\zeta - 1}}(i)).$$

But then as in case 1.2 we can infer that

$$N \models \exists x_{m_i^\zeta} \theta_i^\zeta(f_{\xi_1}(i), \ldots, f_{\xi_{m_i^\zeta - 1}}(i), x_{m_i^\zeta}).$$

As in case 1 choose an element $b \in N$ to witness this formula and set $f_\gamma(i) = b$.

It remains to be shown that the mapping $a_\zeta \mapsto \langle f_\zeta(i) : i < \lambda \rangle / D$ satisfies the requirements of the theorem, i.e. we must show, for all ϕ such that $\phi \in \Delta$ or $\neg\phi \in \Delta$:

$$M \models \phi(a_{\xi_1}, \ldots, a_{\xi_k}) \Rightarrow \{i : N \models \phi(f_{\xi_1}(i), \ldots, f_{\xi_k}(i))\} \in D.$$

So let such a ϕ be given, and suppose $M \models \phi(a_{\xi_1}, \ldots, a_{\xi_k})$. Let $I_\phi = \{i : N \models \phi(f_{\xi_1}(i), \ldots, f_{\xi_k}(i))\}$. We wish to show that $I_\phi \in D$. Let $\alpha < \lambda$ so that ϕ is ϕ_α or its negation. It suffices to show that $A_\alpha \subseteq I_\phi$. Let $\zeta < \lambda^+$ be such that $\{\xi_1, ..., \xi_n\} \subseteq \zeta$. By Lemma 4 condition (iii), $\{i : \{\xi_1, ..., \xi_n\} \subseteq u_i^\zeta\} \in D$. So it suffices to show

$$A_\alpha \cap \{i : \{\xi_1, ..., \xi_n\} \subseteq u_i^\zeta\} \subseteq I_\phi.$$

Let $i \in A_\alpha$ such that $\{\xi_1, ..., \xi_n\} \subseteq u_i^\zeta$. By the definition of θ_i^ζ we know that $N \models \theta_i^\zeta(f_{\xi_1}(i), \ldots, f_{\xi_k}(i))$. But the Δ_i-type of the tuple $\langle a_{\xi_1}, \ldots, a_{\xi_k} \rangle$ occurs as a conjunct of θ_i^ζ, and therefore $N \models \phi(f_{\xi_1}(i), \ldots, f_{\xi_k}(i))$ □

3 Proof of Lemma 4

We now prove Lemma 4. We first prove a weaker version in which the filter is not given in advance:

Lemma 5 *Assume* $\langle\aleph_0,\aleph_1\rangle \to \langle\lambda,\lambda^+\rangle$. *Then there exist sets* $\langle u_i^\zeta : \zeta < \lambda^+, i < cof(\lambda)\rangle$, *integers* n_i *and a regular filter* D *on* λ, *generated by* λ *sets, such that (i)-(iv) of Lemma 4 hold.*

Proof. By [11, Proposition 5.1, p. 149] the assumption $\langle\aleph_0,\aleph_1\rangle \to \langle\lambda,\lambda^+\rangle$ is equivalent to:

$\Box_\lambda^{b^*}$: There is a λ^+-like linear order L, sets $\langle C_a^\zeta : a \in L, \zeta < cf(\lambda)\rangle$, equivalence relations $\langle E^\zeta : \zeta < cf(\lambda)\rangle$, and functions $\langle f_{a,b}^\zeta : \zeta < \lambda, a \in L, b \in L\rangle$ such that

(i) $\bigcup_\zeta C_a^\zeta = \{b : b <_L a\}$ (an increasing union in ζ).

(ii) If $b \in C_a^\zeta$, then $C_b^\zeta = \{c \in C_a^\zeta : c <_L b\}$.

(iii) E^ζ is an equivalence relation on L with $\leq \lambda$ equivalence classes.

(iv) If $\zeta < \xi < cf(\lambda)$, then E^ξ refines E^ζ.

(v) If $aE^\zeta b$, then $f_{a,b}^\zeta$ is an order-preserving one to one mapping from C_a^ζ onto C_b^ζ such that for $d \in C_a^\zeta, dE^\zeta f_{a,b}^\zeta(d)$.

(vi) If $\zeta < \xi < cf(\lambda)$ and $aE^\xi b$, then $f_{a,b}^\zeta \subseteq f_{a,b}^\xi$.

(vii) If $f_{a,b}^\zeta(a_1) = b_1$, then $f_{a_1,b_1}^\zeta \subseteq f_{a,b}^\zeta$.

(viii) If $a \in C_b^\zeta$ then $\neg E^\zeta(a,b)$.

This is not enough to prove Lemma 5, so we have to work a little more. Let

$$\Xi_\zeta = \{a/E^\zeta : a \in L\}.$$

We assume, for simplicity, that $\zeta \neq \xi$ implies $\Xi_\zeta \cap \Xi_\xi = \emptyset$. Define for $t_1, t_2 \in \Xi_\zeta$:

$$t_1 <_\zeta t_2 \iff (\exists a_1 \in t_1)(\exists a_2 \in t_2)(a_1 \in C_{a_2}^\zeta).$$

Proposition 6 $\langle\Xi_\zeta, <_\zeta\rangle$ *is a tree order with* $cf(\lambda)$ *as the set of levels.*

Proof. We need to show (a) $t_1 <_\zeta t_2 <_\zeta t_3$ implies $t_1 <_\zeta t_3$, and (b) $t_1 <_\zeta t_3$ and $t_2 <_\zeta t_3$ implies $t_1 <_\zeta t_2$ or $t_2 <_\zeta t_1$ or $t_1 = t_2$. For the first, $t_1 <_\zeta t_2$ implies there exists $a_1 \in t_1$ and $a_2 \in t_2$ such that $a_1 \in C^\zeta_{a_2}$. Similarly $t_2 <_\zeta t_3$ implies there exists $b_2 \in t_2$ and $b_3 \in t_3$ such that $b_2 \in C^\zeta_{b_3}$. Now $a_2 E^\zeta b_2$ and hence we have the order preserving map $f^\zeta_{a_2,b_2}$ from $C^\zeta_{a_2}$ onto $C^\zeta_{b_2}$. Recalling $a_1 \in C^\zeta_{a_2}$, let $f^\zeta_{a_2,b_2}(a_1) = b_1$. Then by (vi), $a_1 E^\zeta b_1$ and hence $b_1 \in t_1$. But then $b_1 \in C^\zeta_{b_2}$ implies $b_1 \in C^\zeta_{b_3}$, by coherence and the fact that $b_2 \in C^\zeta_{b_3}$. But then it follows that $t_1 <_\zeta t_3$.

Now assume $t_1 <_\zeta t_3$ and $t_2 <_\zeta t_3$. Let $a_1 \in t_1$ and $a_3 \in t_3$ be such that $a_1 \in C^\zeta_{a_3}$, and similarly let b_2 and b_3 be such that $b_2 \in C^\zeta_{b_3}$. $a_3 E^\zeta b_3$ implies we have the order preserving map $f^\zeta_{a_3,b_3}$ from $C^\zeta_{a_3}$ to $C^\zeta_{b_3}$. Letting $f^\zeta_{a_3,b_3}(a_1) = b_1$, we see that $b_1 \in C^\zeta_{b_3}$. If $b_1 <_L b_2$, then we have $C^\zeta_{b_2} = C^\zeta_{b_3} \cap \{c : c < b_2\}$ which implies $b_1 \in C^\zeta_{b_2}$, since, as $f^\zeta_{a_3,b_3}$ is order preserving, $b_1 <_L b_2$. Thus $t_1 <_\zeta t_2$. The case $b_2 <_L b_1$ is proved similarly, and $b_1 = b_2$ is trivial. □

For $a <_L b$ let

$$\xi(a,b) = \min\{\zeta : a \in C^\zeta_b\}.$$

Denoting $\xi(a,b)$ by ξ, let

$$tp(a,b) = \langle a/E^\xi, b/E^\xi \rangle.$$

If $a_1 <_L \ldots <_L a_n$, let

$$tp(\langle a_1, \ldots, a_n \rangle) = \{\langle l, m, tp(a_l, a_m)\rangle | 1 \le l < m \le n\}$$

and

$$\Gamma = \{tp(\vec{a}) : \vec{a} \in {}^{<\omega}L\}.$$

For $t = tp(\vec{a}), \vec{a} \in {}^nL$ we use n_t to denote the length of $\vec{a}$.

Proposition 7 *If $a_0 <_L \ldots <_L a_n$, then*

$$\max\{\xi(a_l, a_m) : 0 \le l < m \le n\} = \max\{\xi(a_l, a_n) : 0 \le l < n\}.$$

Proof. Clearly the right hand side is $\le$ the left hand side. To show the left hand side is $\le$ the right hand side, let $l < m < n$ be arbitrary. If $\xi(a_l, a_n) \le \xi(a_m, a_n)$, then $\xi(a_l, a_m) \le \xi(a_m, a_n)$. On the other hand, if $\xi(a_l, a_n) > \xi(a_m, a_n)$, then $\xi(a_l, a_m) \le \xi(a_l, a_n)$. In either case $\xi(a_l, a_m) \le \max\{\xi(a_k, a_n) : 0 \le k < n\}$. □

Let us denote $\max\{\xi(a_l, a_n) : 0 \leq l < n\}$ by $\xi(\vec{a})$. We define on Γ a two-place relation $\leq_\Gamma$ as follows:

$$t_1 <_\Gamma t_2$$

if there exists a tuple $\langle a_0, \ldots a_{n_{t_2}-1}\rangle$ realizing t_2 such that some subsequence of the tuple realizes t_1.

Clearly, $\langle \Gamma, \leq_\Gamma\rangle$ is a directed partial order.

Proposition 8 *For $t \in \Gamma$, $t = tp(b_0, \ldots b_{n-1})$ and $a \in L$, there exists at most one $k < n$ such that $b_k E^{\xi(b_0,\ldots,b_{n-1})} a$.*

Proof. Let $\zeta = \xi(b_0, \ldots, b_{n-1})$ and let $b_{k_1} \neq b_{k_2}$ be such that $b_{k_1} E^\zeta a$ and $b_{k_2} E^\zeta a$, $k_1, k_2 \leq n-1$. Without loss of generality, assume $b_{k_1} < b_{k_2}$. Since E^ζ is an equivalence relation, $b_{k_2} E^\zeta b_{k_1}$ and thus we have an order preserving map $f^\zeta_{b_{k_2}, b_{k_1}}$ from $C^\zeta_{b_{k_2}}$ to $C^\zeta_{b_{k_1}}$. Also $b_{k_1} \in C^\zeta_{b_{k_2}}$, by the definition of ζ and by coherence, and therefore $f^\zeta_{b_{k_2}, b_{k_1}}(b_{k_1}) E^\zeta b_{k_1}$. But this contradicts (viii), since $f^\zeta_{b_{k_2}, b_{k_1}}(b_{k_1}) \in C^\zeta_{b_{k_1}}$. $\square$

Definition 9 *For $t \in \Gamma$, $t = tp(b_0, \ldots b_{n-1})$ and $a \in L$ suppose there exists $k < n$ such that $b_k E^{\xi(b_0,\ldots,b_{n-1})} a$. Then let $u^a_t = \{f^{\zeta(b_0,\ldots,b_{n-1})}_{a,b_k}(b_l) : l < k\}$ Otherwise, let $u^a_t = \emptyset$.*

Finally, let D be the filter on Γ generated by the λ sets

$$\Gamma_{\geq t^*} = \{t \in \Gamma : t^* <_L t\}.$$

We can now see that the sets u^a_t, the numbers n_t and the filter D satisfy conditions (i)-(iv) of Lemma 4 with L instead of λ^+: Conditions (i) and (ii) are trivial in this case. Condition (iii) is verified as follows: Suppose B is finite. Let $a \in L$ be such that $(\forall x \in B)(x <_L a)$. Let $\vec{a}$ enumerate $B \cup \{a\}$ in increasing order and let $t^* = tp(\vec{a})$. Clearly

$$t \in \Gamma_{\geq t^*} \Rightarrow B \subseteq u^a_t.$$

Condition (iv) follows directly from Definition 9 and Proposition 8.

To get the Lemma on λ^+ we observe that since L is λ^+-like, we can assume that $\langle \lambda^+, <\rangle$ is a submodel of $\langle L, <_L\rangle$. Then we define $v^\alpha_t = u^\alpha_t \cap \{\beta : \beta < \alpha\}$.

Conditions (i)-(iv) of Lemma 5 are still satisfied. Also having D a filter on Γ instead of on λ is immaterial as $|\Gamma| = \lambda$. $\Box$

Now back to the proof of Lemma 4. Suppose u_i^ζ, n_i and D are as in Lemma 5, and suppose D' is an arbitrary regular filter on λ. Let $\{A_\alpha : \alpha < \lambda\}$ be a family of sets witnessing the regularity of D', and let $\{Z_\alpha : \alpha < \lambda\}$ be the family generating D. We define a function $h : \lambda \to \lambda$ as follows. Suppose $i < \lambda$. Then let

$$h(i) \in \bigcap \{Z_\alpha | i \in A_\alpha\}.$$

Now define $v_\alpha^\zeta = u_{h(\alpha)}^\zeta$. Define also $n_\alpha = n_{h(\alpha)}$. Now the sets v_α^ζ and the numbers n_α satisfy the conditions of Lemma 4. $\Box$

4 Is $\Box_\lambda^{b^*}$ needed for Lemma 5?

In this section we show that the conclusion of Lemma 5 (and hence of Lemma 4) implies $\Box_\lambda^{b^*}$ for singular strong limit λ. By [11, Theorem 2.3 and Remark 2.5], $\Box_\lambda^{b^*}$ is equivalent, for singular strong limit λ, to the following principle:

$\mathcal{S}_\lambda$: There are sets $\langle C_a^i : a < \lambda^+, i < cf(\lambda)\rangle$ such that

- (i) If $i < j$, then $C_a^i \subseteq C_a^j$.
- (ii) $\bigcup_i C_a^i = a$.
- (iii) If $b \in C_a^i$, then $C_b^i = C_a^i \cap b$.
- (iv) $\sup\{otp(C_a^i) : a < \lambda^+\} < \lambda$.

Thus it suffices to prove:

Proposition 10 *Suppose the sets u_i^ζ and the filter D are as given by Lemma 5 and λ is a limit cardinal. Then $\mathcal{S}_\lambda$ holds.*

Proof. Suppose $\mathcal{A} = \{A_\alpha : \alpha < \lambda\}$ is a family of sets generating D. W.l.o.g., $\mathcal{A}$ is closed under finite intersections. Let λ be the union of the increasing sequence $\langle \lambda_\alpha : \alpha < cf(\lambda)\rangle$, where $\lambda_0 \geq \omega$. Let the sequence $\langle \Gamma_\alpha : \alpha < cf(\lambda)\rangle$ satisfy:

(a) $|\Gamma_\alpha| \leq \lambda_\alpha$

(b) Γ_α is continuously increasing in α with λ as union

(c) If $\beta_1, ..., \beta_n \in \Gamma_\alpha$, then there is $\gamma \in \Gamma_\alpha$ such that

$$A_\gamma = A_{\beta_1} \cap ... \cap A_{\beta_n}.$$

The sequence $\langle \Gamma_\alpha : \alpha < cf(\lambda) \rangle$ enables us to define a sequence that will witness $\mathcal{S}_\lambda$. For $\alpha < cf(\lambda)$ and $\zeta < \lambda^+$, let

$$V^\alpha_\zeta = \{\xi < \zeta : (\exists \gamma \in \Gamma_\alpha)(A_\gamma \subseteq \{i : \xi \in u^\zeta_i\})\}.$$

Lemma 11 *(1) $\langle V^\alpha_\zeta : \alpha < \lambda \rangle$ is a continuously increasing sequence of subsets of ζ, $|V^\alpha_\zeta| \leq \lambda_\alpha$, and $\bigcup \{V^\alpha_\zeta : \alpha < cf(\lambda)\} = \zeta$.*

(2) If $\xi \in V^\alpha_\zeta$, then $V^\alpha_\xi = V^\alpha_\zeta \cap \xi$.

Proof. (1) is a direct consequence of the definitions. (2) follows from the respective property of the sets u^ζ_i. □

Lemma 12 $\sup\{otp(V^\alpha_\zeta) : \zeta < \lambda^+\} \leq \lambda^+_\alpha$.

Proof. By the previous Lemma, $|V^\alpha_\zeta| \leq \lambda_\alpha$. Therefore $otp(V^\alpha_\zeta) < \lambda^+_\alpha$ and the claim follows. □

The proof of the proposition is complete: (i)-(iii) follows from Lemma 11, (iv) follows from Lemma 12 and the assumption that λ is a limit cardinal. □

More equivalent conditions for the case λ singular strong limit, D a regular ultrafilter on λ, are under preparation.

5 Ehrenfeucht-Fraïssé-games

Let M and N be two first order structures of the same language L. All vocabularies are assumed to be relational. The *Ehrenfeucht-Fraïssé-game of length γ of M and N* denoted by EFG_γ is defined as follows: There are two players called I and II. First I plays x_0 and then II plays y_0. After this I plays x_1, and II plays y_1, and so on. If $\langle (x_\beta, y_\beta) : \beta < \alpha \rangle$ has been played and $\alpha < \gamma$, then I plays x_α after which II plays y_α. Eventually a sequence $\langle (x_\beta, y_\beta) : \beta < \gamma \rangle$ has been played. The rules of the game say that both players have to play elements of $M \cup N$. Moreover, if I plays his x_β in M

(N), then II has to play his y_β in N (M). Thus the sequence $\langle (x_\beta, y_\beta) : \beta < \gamma \rangle$ determines a relation $\pi \subseteq M \times N$. Player II wins this round of the game if π is a partial isomorphism. Otherwise I wins. The notion of winning strategy is defined in the usual manner. We say that a player *wins* EFG_γ if he has a winning strategy in EFG_γ.

Note that if II has a winning strategy in EFG_γ on M and N, where M and N are of size $\leq |\gamma|$, then $M \cong N$.

Assume L is of cardinality $\leq \lambda$ and for each $i < \lambda$ let M_i and N_i be elementarily equivalent L-structures. Shelah proved in [12] that if D is a regular filter on λ, then Player II has a winning strategy in the game EFG_γ on $\prod_i M_i/D$ and $\prod_i N_i/D$ for each $\gamma < \lambda^+$. We show that under a stronger assumption, II has a winning strategy even in the game EFG_{λ^+}. This makes a big difference because, assuming the models M_i and N_i are of size $\leq \lambda^+$, $2^\lambda = \lambda^+$, and the models $\prod_i M_i/D$ and $\prod_i N_i/D$ are of size $\leq \lambda^+$. Then by the remark above, if II has a winning strategy in EFG_{λ^+}, the reduced powers are actually isomorphic. Hyttinen [4] proved this under the assumption that the filter is, in his terminology, semigood.

Theorem 13 *Assume $\langle \aleph_0, \aleph_1 \rangle \to \langle \lambda, \lambda^+ \rangle$. Let L be a language of cardinality $\leq \lambda$ and for each $i < \lambda$ let M_i and N_i be two elementarily equivalent L-structures. If D is a regular filter on λ, then Player II has a winning strategy in the game* EFG_{λ^+} *on $\prod_i M_i/D$ and $\prod_i N_i/D$.*

Proof. We use Lemma 4. If $i < \lambda$, then, since M_i and N_i are elementarily equivalent, Player II has a winning strategy σ_i in the game EFG_{n_i} on M_i and N_i. We will use the set u_i^ζ to put these short winning strategies together into one long winning strategy.

A "good" position is a sequence $\langle (f_\zeta, g_\zeta) : \zeta < \xi \rangle$, where $\xi < \lambda^+$, and for all $\zeta < \xi$ we have $f_\zeta \in \prod_i M_i$, $g_\zeta \in \prod_i N_i$, and if $i < \lambda$, then $\langle (f_\epsilon(i), g_\epsilon(i)) : \epsilon \in u_i^\zeta \cup \{\zeta\} \rangle$ is a play according to σ_i.

Note that in a good position the equivalence classes of the functions f_ζ and g_ζ determine a partial isomorphism of the reduced products. To see this, suppose $\langle (f_\zeta, g_\zeta) : \zeta < \xi \rangle$ is a good position, $\phi(x_1, \ldots, x_k)$ is atomic and $I_\phi = \{i : M_i \models \phi(f_{\alpha_1}(i), \ldots, f_{\alpha_k}(i))\} \in D$. We wish to show that $I'_\phi = \{i : N_i \models \phi(g_{\alpha_1}(i), \ldots, g_{\alpha_k}(i))\} \in D$. By Lemma 4, if $\gamma < \lambda^+$ is such that $B = \{\alpha_1, \ldots, \alpha_k\} \subseteq \gamma$, then $J_\gamma = \{i : B \subseteq u_i^\gamma\} \in D$. Thus $J_\gamma \cap I_\phi \in D$, and for each $i \in J_\gamma$, $\langle (f_\epsilon(i), g_\epsilon(i)) : \epsilon \in u_i^\gamma \rangle$ is part of the play according to σ_i. Thus for each such i, $i \in I_\phi \leftrightarrow i \in I'_\phi$ i.e. $J_\gamma \cap I_\phi = J_\gamma \cap I'_\phi$, whence $I'_\phi \in D$.

The strategy of player II is to keep the position of the game "good", and thereby win the game. Suppose ξ rounds have been played and II has been able to keep the position "good". Then player I plays f_ξ. We show that player II can play g_ξ so that $\langle (f_\zeta, g_\zeta) : \zeta \leq \xi \rangle$ remains "good". Let $i < \lambda$. Let us look at $\langle (f_\epsilon(i), g_\epsilon(i)) : \epsilon \in u_i^\xi \rangle$. We know that this is a play according to the strategy σ_i and $|u_i^\xi| < n_i$. Thus we can play one more move in EF_{n_i} on M_i and N_i with player I playing $f_\xi(i)$. Let $g_\xi(i)$ be the answering move of II in this game according to σ_i. The values $g_\xi(i)$, $i < \lambda$, constitute the function g_ξ. We have shown that II can maintain a "good" position. □

Corollary 14 *Assume GCH and λ regular (or just $\langle \aleph_0, \aleph_1 \rangle \to \langle \lambda, \lambda^+ \rangle$ and $2^\lambda = \lambda^+$). Let L be a language of cardinality $\leq \lambda$ and for each $i < \lambda$ let M_i and N_i be two elementarily equivalent L-structures. If D is a regular filter on λ, then $\prod_i M_i/D \cong \prod_i N_i/D$.*

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