ExerciseLinearNonlinear: Introduction

Definition: The model $g(t)$ has p free parameters $[\bar{\beta} = \beta_1, \beta_2, ..., \beta_p]$. This model is **linear**, **if all** $i = 1, ..., p$ model partial derivatives

$$\frac{\partial g(t)}{\partial \beta_i}$$

do not contain any free parameter $\beta_1, ..., \beta_p$. The model is **non-linear**, **if any** of these partial derivatives contains any free parameter $\beta_1, ..., \beta_p$.

DCM model is

$$g(t) = g(t, K_1, K_2, K_3) = h(t) + p(t),$$

where

$$\begin{aligned} h(t) &= h(t, K_1, K_2) = \sum_{i=1}^{K_1} h_i(t) \\ h_i(t) &= \sum_{j=1}^{K_2} \mathbf{B_{i,j}} \cos(2\pi j\ \mathbf{f_i}\ t) + \mathbf{C_{i,j}} \sin(2\pi j\ \mathbf{f_i}\ t) \\ p(t) &= p(t, K_3) = \sum_{k=0}^{K_3} p_k(t) \\ p_k(t) &= \mathbf{M_k} \left[\frac{2t}{\Delta T}\right]^k. \end{aligned}$$

The free parameters are $\bar{\beta} = [\beta_1, \beta_2, ..., \beta_p] = [B_{1,1}, C_{1,1}, f_1, ..., B_{K_1,K_2}, C_{K_1,K_2}, f_{K_1}, M_0, ..., M_{K_3}]$. They are highlighted above in **blue** colour. The number of free parameters $\bar{\beta}$ in this DCM model is

$$p = K_1 \times (2K_2 + 1) + K_3 + 1.$$

Partial derivatives of K_1 frequencies f_i

In all partial derivative equations below, the terms equal to zero are highlighted in **blue** colour.

The $g(t)$ model partial derivative with respect to frequency f_1 is

$$\begin{aligned}
\frac{\partial g(t)}{\partial f_1} &= \frac{\partial h(t)}{\partial f_1} + \frac{\partial p(t)}{\partial f_1} = \frac{\partial h(t)}{\partial f_1} + 0 \\
&= \frac{\partial}{\partial f_1}\left[\sum_{j=1}^{K_2} B_{i,j}\cos(2\pi j f_i t) + C_{i,j}\sin(2\pi j f_i t)\right] \text{ where } i = 1 \\
\Rightarrow \frac{\partial g(t)}{\partial f_1} &= \frac{\partial}{\partial f_1}\left[\sum_{j=1}^{K_2} B_{1,j}\cos(2\pi j f_1 t) + C_{1,j}\sin(2\pi j f_1 t)\right] = ... \qquad (1)
\end{aligned}$$

There is no need to continue, because it is certain that this partial derivative **contains** the free parameter f_1, as well the free parameters $B_{1,j}$ and $C_{1,j}$, where $j = 1, ..., K_2$. This means that the $g(t)$ model is **non-linear**. If there are more signals ($K_1 > 1$), the $g(t)$ model partial derivatives with respect to these other frequencies $f_2, ..., f_{K_1}$ **also contain** free parameters.

Partial derivatives of $K_1 \times 2K_2$ sine and cosine curve amplitudes

The $g(t)$ model partial derivative with respect to amplitude $B_{1,1}$ is

$$\begin{aligned}
\frac{\partial g(t)}{\partial B_{1,1}} &= \frac{\partial h(t)}{\partial B_{1,1}} + \frac{\partial p(t)}{\partial B_{1,1}} = \frac{\partial h(t)}{\partial B_{1,1}} + 0 \\
&= \frac{\partial}{\partial B_{1,1}}\left[\sum_{j=1}^{K_2} B_{1,j}\cos(2\pi j f_1 t) + C_{1,j}\sin(2\pi j f_1 t)\right] \\
&= \sum_{j=1}^{K_2} \frac{\partial}{\partial B_{1,1}}\left[B_{1,j}\cos(2\pi j f_1 t)\right] + \frac{\partial}{\partial B_{1,1}}\left[C_{1,j}\sin(2\pi j f_1 t)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{K_2} \frac{\partial}{\partial B_{1,1}} \left[B_{1,j} \cos\left(2\pi j f_1 t\right)\right] + 0 \\
&= \cos\left(2\pi j f_i t\right), \text{where } i = 1 \text{ and } j = 1 \\
\Rightarrow \frac{\partial g(t)}{\partial B_{1,1}} &= \cos\left(2\pi f_1 t\right) \qquad (2)
\end{aligned}
$$

The $g(t)$ partial derivatives with respect to all other free parameter $B_{i,j}$ or $C_{i,j}$ amplitudes give a similar term, which **contains** only one free parameter, the frequency f_i.

Partial derivatives of $K_3 + 1$ polynomial trend M_k coefficients

The $g(t)$ model partial derivative with respect to any polynomial trend coefficient M_k is

$$
\begin{aligned}
\frac{\partial g(t)}{\partial M_k} &= \frac{\partial h(t)}{\partial M_k} + \frac{\partial p(t)}{\partial M_k} = 0 + \frac{\partial p(t)}{\partial M_k} = \frac{\partial}{\partial M_k}\left[\sum_{k=0}^{K_3} p_k(t)\right] \\
&= \sum_{k=0}^{K_3} \frac{\partial}{\partial M_k}\left[M_k \left(\frac{2t}{\Delta T}\right)^k\right] \\
\Rightarrow \frac{\partial g(t)}{\partial M_k} &= \left(\frac{2t}{\Delta T}\right)^k \qquad (3)
\end{aligned}
$$

The $g(t)$ partial derivatives with respect to all free parameter M_k polynomial trend coefficients **do not contain** any free parameters β_i.

Dividing free parameters into two groups

The free parameters $\bar{\beta} = [\beta_1, \beta_2, ..., \beta_p] = [B_{1,1}, C_{1,1}, f_1, ..., B_{K_1,K_2}, C_{K_1,K_2}, f_{K_1}, M_0, ..., M_{K_3}]$ can be divided into groups

$$\bar{\beta}_I = [f_1, ..., f_{K_1}] \tag{4}$$
$$\bar{\beta}_{II} = [B_{1,1}C_{1,1}, ..., B_{K_1,K_2}, C_{K_1,K_2}, M_0, ..., M_{K_3}] \tag{5}$$

The first $\bar{\beta}_I$ group contains the frequencies. The second $\bar{\beta}_{II}$ group contains the amplitudes of trigonometric functions and the polynomial trend coefficients. Let us assume that

A1. We fix frequencies $\bar{\beta}_I = [f_1, ..., f_{K_1}]$ to known **tested** numerical values.

Then, the frequencies f_i are no longer free parameters. They are constants, like the 2π constants in the $g(t)$ model. This leads to the following situation

C1. Since the tested constant frequencies are no longer free parameters, there is no need to compute the $g(t)$ model partial derivatives with respect to frequencies f_i (Eq. 1). These partial derivatives **do not** make the DCM model **non-linear**.

C2. Since tested constant f_i frequencies are not free parameters, the $g(t)$ model partial derivatives with respect to all amplitudes (Eq. 2) **contain no** free parameters, which would make the DCM model **non-linear**.

C3. The $g(t)$ model partial derivatives with respect to all polynomial $p(t)$ trend coefficients M_k **contain no** free parameters (Eq. 3), which would make DCM model **non-linear**.

Under assumption A1, the DCM model $g(t)$ is **linear**, because C1, C2 and C3 show that the model partial derivatives **do not contain** any free parameter β_i.

The above result was shortly stated in **Paper I**

"The first group of free parameters, the frequencies $\bar{\beta}_I = [f_1, ..., f_{K_1}]$, make this $g(t)$ model **non-linear***. If these $\bar{\beta}_I$ are fixed to constant known numerical values, the model becomes* **linear***, and the solution for the remaining second group of free parameters, $\bar{\beta}_{II} = [B_{1,1}C_{1,1}, ..., B_{K_1,K_2}, C_{K_1,K_2}, M_0, ..., M_{K_3}]$, is unambiguous."*

Linear and non-linear models

The crucial difference between **linear** and **non-linear** models is

- The free parameter $\bar{\beta}$ solution is **ambiguous**, if the model is **non-linear**, because this solution depends on the chosen trial value $\bar{\beta}_{\text{trial}}$. The final value $\bar{\beta}_{\text{final}}$ is obtained from an iteration beginning from $\bar{\beta}_{\text{trial}}$.
- The free parameter $\bar{\beta}$ solution is **unambiguous**, if the model is **linear**. No trial value $\bar{\beta}_{\text{trial}}$ is required.

ExerciseLinearNonlinear: Problem

Five models are given below. Are they **linear** or **non-linear**?

$g(t) = At + B$, $\bar{\beta} = [A, B]$

$g(t) = A^2t + B$, $\bar{\beta} = [A, B]$

$g(t) = M + A \sin t + B \cos t$, $\bar{\beta} = [M, A, B]$

$g(t) = M + A \sin{(t + B)}$, $\bar{\beta} = [M, A, B]$

$g(t) = M + A \cos{(ft)}$, $\bar{\beta} = [M, A, f]$

If some of these models are **non-linear**, is it possible to make them **linear**? Write your answers to a latex-file **ExLiNo.tex**, and send **ExLiNo.pdf** to the assistant.