

## Exercise Fail Whiten: Introduction

In the prewhitening technique, Discrete Fourier Transform (DFT) detects one signal at the time. This approach does not always work, because DFT can not detect two signals, if their frequencies  $f_1$  and  $f_2$  are too close to each other. The criterion for DFT detection of both signals is

$$|f_2 - f_1| > f_0. \quad (1)$$

where  $f_0 = 1/\Delta T$  and  $\Delta T = t_n - t_1$  is the time span of data. If this criterion is not fulfilled, the two DFT periodogram peaks at frequencies  $f_1$  and  $f_2$  merge into only one peak. Naturally, this  $f_0$  limit can be reached only, if the quality and quantity of data is sufficient. For example, [1] Loumos & Deeming (1978) argued that for real data this limit can exceed  $f_0$ .

In this exercise, we analyse simulated data that contains two sinusoidal signals. The model used in creating these simulated data is the sum of two sinusoidal light curves

$$s_1(t) = a_1 \sin(2\pi f_1 t) \quad (2)$$

$$s_2(t) = a_2 \sin(2\pi f_2 t). \quad (3)$$

The peak to peak amplitudes are  $A_1 = 2a_1$  and  $A_2 = 2a_2$ . The frequencies are fixed to  $f_1 = P_1^{-1}$  and  $f_2 = P_2^{-1}$ , where the periods are  $P_1 = 2.^d39$  and  $P_2 = 2.^d41$ . The interference pattern of the sum curve

$$s(t) = s_1(t) + s_2(t) \quad (4)$$

repeats itself during the lap cycle period

$$P_{\text{lap}} = |P_1^{-1} - P_2^{-1}|^{-1} = |f_1 - f_2|^{-1} \approx 288. \quad (5)$$

If  $f_1 \approx f_2$ , the solution for the sum curve  $s(t)$  depends *only* on the amplitudes  $a_1$  and  $a_2$ . The solutions are different for

$$a_1 = a_2 \quad (6)$$

$$a_1 \neq a_2. \quad (7)$$

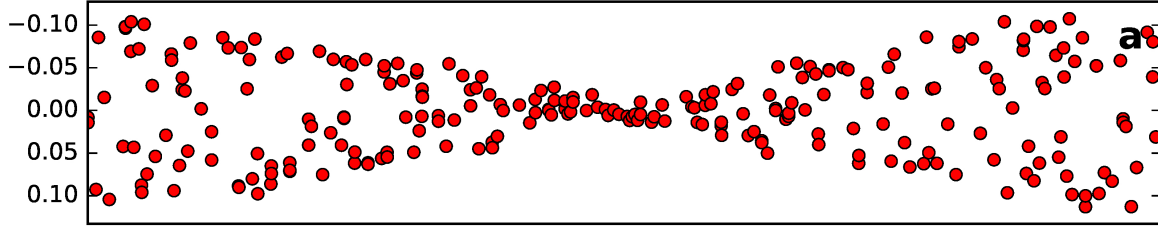


Figure 1: (a) Simulated data having equal amplitudes  $a_1 = a_2 = 0.05$  and  $\Delta T = P_{\text{lap}}$ .

### Case $a_1 = a_2$

If Eq. 6 is true, then

$$s(t) = s_1(t) + s_2(t) = a_a(t) \sin(2\pi f_a t), \quad (8)$$

where the amplitude is

$$a_a(t) = a_1 \sqrt{2 + 2 \cos[2\pi(f_1 - f_2)t]} \quad (9)$$

and the frequency is

$$f_a = (f_1 + f_2)/2. \quad (10)$$

This frequency of the sum  $s(t)$  remains constant  $f = f_a$ . The amplitude  $a_a(t)$  of Eq. 9 varies between  $a_{a,\text{min}} = 0$  and  $a_{a,\text{max}} = 2a_1$  during the lap cycle  $P_{\text{lap}}$ .

Finally, an abrupt  $f_a/2$  phase shift of  $s(t)$  occurs when  $a_a(t)$  goes to zero. One arbitrary sample of such simulated data is shown in Figure 1, where the  $\Delta T$  time span of data is equal to the lap cycle period  $P_{\text{lap}}$ .

### Case $a_1 < a_2$

If Eq. 7 is true, we assume that  $a_1 < a_2$ . In this case, the sum is

$$s(t) = s_1(t) + s_2(t) = a_b(t) \sin[2\pi f_2 t + \phi_b(t)], \quad (11)$$

where the amplitude is

$$a_b(t) = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos[2\pi(f_1 - f_2)t]} \quad (12)$$

and the phase shift is

$$\phi_b(t) = \arctan \left[ \frac{a_1 \sin [2\pi(f_1 - f_2)t]}{a_1 \cos [2\pi(f_1 - f_2)t] + a_2} \right]. \quad (13)$$

The  $[\phi_b(t)]$  units are radians.

The amplitude  $a_b(t)$  of Eq. 12 varies regularly between

$$a_{b,\min} = a_2 - a_1 \quad (14)$$

and

$$a_{b,\max} = a_1 + a_2 \quad (15)$$

during the lap cycle period  $P_{\text{lap}}$ .

The phase modulation  $\phi_b(t)$  induces frequency changes of  $s(t)$  between

$$f_{b,\max} = f_2 + \frac{a_1(f_1 - f_2)}{a_2 + a_1} \quad (16)$$

at the  $a_{b,\max}$  epochs, and

$$f_{b,\min} = f_2 - \frac{a_1(f_1 - f_2)}{a_2 + a_1} \quad (17)$$

at the  $a_{b,\min}$  epochs. These frequency changes of  $s(t)$  at both sides of  $f_2$  are repeated regularly during  $P_{\text{lap}}$ . The frequency  $f_2$  of the stronger  $s_2(t)$  sinusoid ( $a_1 < a_2$ ) dominates these frequency changes of  $s(t)$ .

The maximum  $\phi_b(t)$  phase shift is

$$\phi_{b,\max} = \arcsin (a_1/a_2) \quad (18)$$

to both directions. The  $[\phi_{b,\max}]$  units are radians. For  $a_1 = a_2$ , the maximum phase shift  $\phi_{b,\max}$  is  $\pi/2$  radians, or equivalently  $(\pi/2)/(2\pi) = 1/4$  in dimensionless phase units with  $f_2$ . Since this shift at  $a_{b,\min}$  occurs to both directions, its full range is

$$\Delta\phi_b = [\arcsin (a_1/a_2)]/\pi \quad (19)$$

in dimensionless phase units. This gives  $\Delta\phi_b = 0.5$  when  $a_1 = a_2$ . No phase shift can exceed this limit. This abrupt phase shift occurs also when  $a_1 \neq a_2$ , but it is smaller. For example, the  $a_1 = 0.048$  and  $a_2 = 0.050$  combination causes an abrupt  $\Delta\phi_b = 0.41$  phase shift at  $a_{b,\min}$ .

In conclusion, the *observed* amplitude, period and minimum of the sum  $s(t)$  fluctuate regularly with the lap period cycle  $P_{\text{lap}}$  when the condition of Eq. 7 is true.

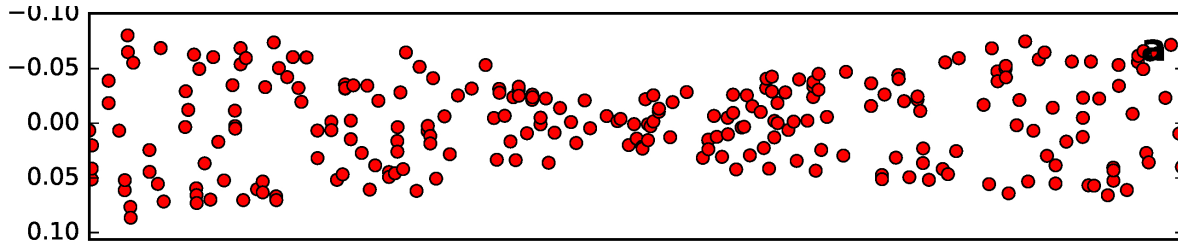


Figure 2: Simulated data having unequal  $a_1 = 0.025$  and  $a_2 = 0.050$ , and  $\Delta T = P_{\text{lap}}$

One arbitrary sample of simulated data, where  $a_1 < a_2$ , is shown in Figure 2. The  $\Delta T$  time span of data is equal to the lap cycle period  $P_{\text{lap}}$ .

### ExerciseFailWhiten: Problem

Two different samples of simulated data are created using the model of Eq. 4. The periods are  $P_1 = 2.39$  and  $P_2 = 2.41$ . The amplitudes are  $a_1 = a_2 = 0.05$ . The two different samples of simulated data are given in the home-page files

[ExerciseFailWhiten1.dat](#)

[ExerciseFailWhiten2.dat](#)

These data are analysed with [ExercisePreWhiten.py](#) pre-whitening technique model solution program. You have already received this program from your assistant. In the current exercise, you do not have apply this program, but you should remember/understand how it works.

## Results for ExerciseFailWhiten1.dat

Using the following `dft.dat` control file for `ExercisePreWhiten.py`

```
1 = Tag           = Whatever
2 = file1         = ExerciseFailWhitenData1.dat
3 = K1           = 2
4 = K3           = 0
5 = PMIN         = 2.0
6 = PMAX         = 3.0
7 = OFAC         = 40.
```

gives these results

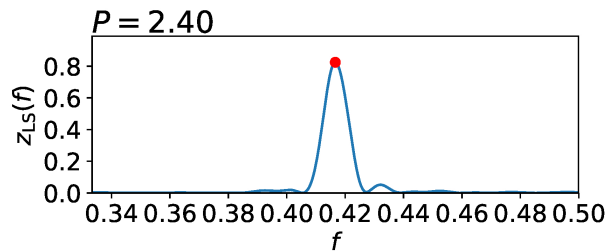


Figure 3: First signal detected from original data.

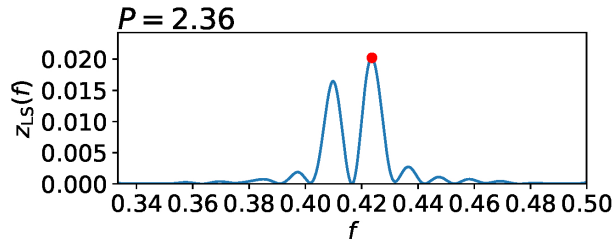


Figure 4: Second signal detected from first residuals.

**This prewhitening analysis fails:**

**Neither one** of the correct periods  $P_1 = 2.39$  or  $P_2 = 2.41$  is detected!

## Results for ExerciseFailWhiten2.dat

Using the following `dft.dat` control file for `ExercisePreWhiten.py`

```
1 = Tag           = Whatever
2 = file1         = ExerciseFailWhitenData2.dat
3 = K1           = 2
4 = K3           = 0
5 = PMIN         = 2.0
6 = PMAX         = 3.0
7 = OFAC         = 40.
```

gives these results

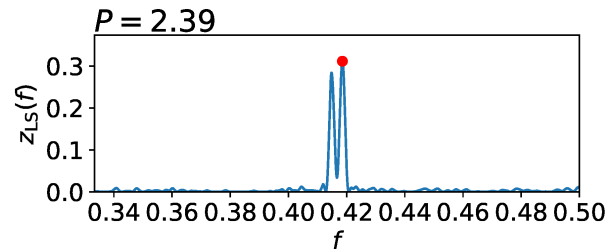


Figure 5: First signal detected from original data.

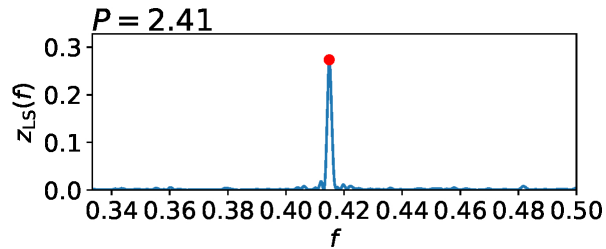


Figure 6: Second signal detected from first residuals.

**This prewhitening analysis succeeds:**

**Both** of the correct periods  $P_1 = 2.39$  or  $P_2 = 2.41$  are detected!

## Questions

1. Figures 3 and 4: What prevents the detection of the correct  $P_1 = 2.39$  and  $P_2 = 2.41$  periods?
2. Figure 3: What causes the detection of the wrong  $P_1 = 2.40$  period?
3. Figures 5 and 6: What allows the detection of the correct  $P_1 = 2.39$  and  $P_2 = 2.41$  periods?

Answer each question with less than 50 words. Give the equation(-s) that explain the above results, like: “... the relation in Eq. 7 predicts that  $P_2 = 2.36$  ...” Write your answers to a pdf-file. Send this file to the assistant. Each correct answer gives one point. Therefore, you can get one bonus point from this exercise **(3/2pts)**.

**Tip:** Get **ExerciseFailWhiten1.dat** and **ExerciseFailWhiten2.dat** data files from the home-page. Look at them carefully. The correct answers can be deduced from these files.

## References

- [1] G. L. Loumos and T. J. Deeming. Spurious results from Fourier analysis of data with closely spaced frequencies. *Ap&SS*, 56:285–291, July 1978.