RICCI CURVATURE, HARNACK FUNCTIONS, AND PICARD TYPE THEOREMS FOR QUASIREGULAR MAPPINGS

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1. Introduction

In the real n-space $\mathbb{R}^n$, or on Riemannian n-manifolds, $n \geq 2$, quasiregular maps serve in a geometric sense as a counterpart of holomorphic functions of one complex variable. For completeness, let us recall the definition of a quasiregular map of an open set $G \subset \mathbb{R}^n$. Suppose that $f : G \to \mathbb{R}^n$ is a continuous map. Then f is called quasiregular (qr) if

$$f \in W^{1,n}_{\mathrm{loc}}(G) \tag{1.1}$$

and there exists $K,\ 1 \leq K < \infty$, such that

$$|f'(x)|^n \leq K J_f(x) \quad \text{a.e.} \tag{1.2}$$

Above $W^{1,n}_{\mathrm{loc}}(G)$ is the Sobolev space of locally L^n-integrable maps whose distributional first order partial derivatives are also locally L^n-integrable. Furthermore, $f'(x)$ is the formal derivative of f at x defined by means of the partial derivatives, $|f'(x)|$ is the operator norm of $f'(x)$, and $J_f(x) = \det f'(x)$. These are uniquely defined almost everywhere by (1.1). It turns out that quasiregular mappings are differentiable almost everywhere and a nonconstant quasiregular map is discrete, open, and sense-preserving. For these and other basic properties of qr maps we refer to books [Re] by Yu. G. Reshetnyak and [R6] by the second author. The definition extends in a straightforward manner to the case $f : M \to N$, where M and N are connected oriented Riemannian n-manifolds. In this paper f is called K-quasiregular if it satisfies the above condition with K.

A Picard-type theorem on omitted values holds for entire quasiregular maps of $\mathbb{R}^n$ in the following form; see [R1].

1.3. Theorem. *For each $n \geq 3$ and each $K \geq 1$ there exists a positive integer $q_0(n, K)$ such that every K–quasiregular map $f : \mathbb{R}^n \to \mathbb{R}^n \setminus \{a_1, \dots, a_q\}$, where $q \geq q_0(n, K)$ and $a_1, \dots, a_q$ are distinct, is constant.*

The above theorem is known to be sharp at least in $\mathbb{R}^3$ in the sense that for each positive integer p there exists a nonconstant $K(p)$-quasiregular map $\mathbb{R}^3 \to \mathbb{R}^3$ omitting at least p points; [R5]. These results have been extended to get a defect relation in the spirit of Ahlfors' theory of covering surfaces [R2] and [R7] together with its realization [R7].

The original proof of 1.3 makes use of both of the main tools in the theory of qr maps, namely, the method of extremal length and the nonlinear potential theory. Proofs which

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use only the extremal length method are presented for instance in [R2, R3, R4, R6]. In these methods, relatively sharp estimates for the modulus of certain path families are needed both in the domain and in the range of the map. This is the main difficulty that makes it hard to generalize the method beyond $\mathbb{R}^n$. A purely potential theoretic proof of 1.3 was given by A. Eremenko and J. Lewis [EL]. Recently Lewis [Le] presented a still simplified proof. His argument is based on a uniform Harnack inequality on the domain of f and a limit function technique of Eremenko and Sodin. The main advantage of the methods of Eremenko and Lewis is that after a construction of certain n-harmonic functions with singularities in the range one can accomplish the proof by working solely in the domain of f. This makes it possible to use the method in proving Picard-type theorems for qr maps between other spaces than $\mathbb{R}^n$. Such generalizations were proved in [HR1-2] for qr maps $f : X \to M$, where X is either $\mathbb{R}^n$ or the n-dimensional Heisenberg group, $n = 2m + 1$, equipped with a left-invariant Riemannian metric and M is an open manifold with an arbitrary Riemannian metric obtained from a compact oriented n-manifold by deleting q points. Lewis' method was also applied in [HH] to get Picard's theorem for quasiregular maps on H-type Carnot groups. In this paper we use Lewis' argument without the limit function technique and we prove Picard's theorem in a quite general setting (Theorem 3.1). As a by-product we obtain a Liouville-type result for $\mathcal{A}$-harmonic functions (Corollary 2.15).

2. Harnack functions

Throughout this section M will be a complete Riemannian manifold. Following Lewis [Le] we call a continuous function $u : M \to \mathbb{R}$ a Harnack function with constant ϑ if

$$M(h, x, r) := \sup_{B(x,r)} h \leq \vartheta \inf_{B(x,r)} h$$

holds in each ball $B(x, r) \subset M$ whenever h is nonnegative in $B(x, 2r)$ and has the form $h = \pm u + a$ for some $a \in \mathbb{R}$. We are going to use the following lemma which is proved in [Le] in the case $M = \mathbb{R}^n$. The proof applies here verbatim.

2.1. Lemma. *Let u be a Harnack function in M with a constant ϑ. Suppose that $u(x_0) = 0$ and that $R \geq 0$. Then there are r, $0 < r < R$, $x_1 \in B(x_0, 2R)$, and a constant $c_0 = c_0(\vartheta) \geq 2$ such that $u(x_1) = 0$ and that*

$$c_0^{-1} M(u, x_0, R) \leq M(u, x_1, 10r) \leq c_0 M(u, x_1, r) .$$

In this section we show that $\mathcal{A}$-harmonic functions in M are Harnack functions under very weak conditions on M. Let us first recall the definition of $\mathcal{A}$-harmonic functions. Let G be an open subset of M and let $TG = \cup_{x \in G} T_x M$. Suppose that we are given a map $\mathcal{A} : TG \to TG$ such that $\mathcal{A}_x = \mathcal{A} \mid T_x M : T_x M \to T_x M$ is continuous for a.e. $x \in G$ and that the map $x \mapsto \mathcal{A}_x(X)$ is a measurable vector field whenever X is. We assume further that there are $1 < p < \infty$ and $0 < \alpha \leq \beta < \infty$ such that

$$\big\langle \mathcal{A}_x(h), h \big\rangle \geq \alpha |h|^p \tag{2.2}$$

and

$$|\mathcal{A}_x(h)| \leq \beta |h|^{p-1} \tag{2.3}$$

for a.e. $x \in G$ and for all $h \in T_x M$; further for a.e. $x \in G$

$$\big\langle \mathcal{A}_x(h) - \mathcal{A}_x(k), h - k \big\rangle > 0 \tag{2.4}$$

whenever $h \neq k$, and

$$\mathcal{A}_x(\lambda h) = |\lambda|^{p-2} \lambda \mathcal{A}_x(h) \tag{2.5}$$

whenever $\lambda \in \mathbb{R} \setminus \{0\}$.

We say that $\mathcal{A}$ is of type p if it satisfies conditions (2.2)–(2.5) with the constant p. The class of all such $\mathcal{A}$ will be denoted by $\mathcal{A}_p(G)$.

A function $u \in W^{1,p}_{\mathrm{loc}}(G)$ is a (weak) solution of the equation

$$-\operatorname{div} \mathcal{A}_x(\nabla u) = 0 \tag{2.6}$$

in G if

$$\int_G \big\langle \mathcal{A}_x(\nabla u), \nabla \varphi \big\rangle \, dx = 0$$

for all $\varphi \in C_0^\infty(G)$. Continuous solutions of (2.6) are called $\mathcal{A}$–harmonic (of type p). In the special case $\mathcal{A}_x(h) = |h|^{p-2} h$, $\mathcal{A}$-harmonic functions will be called p-harmonic. We refer to [HKM] for the nonlinear potential theory based on $\mathcal{A}$-harmonic functions.

In the rest of the section we suppose that a doubling condition and a weak $(1,p)$-Poincaré inequality hold in M. By these we mean that there are two constants C_d and C_p such that, for all balls $B = B(o,r) \subset M$,

$$|B(o,2r)| \leq C_d |B(o,r)|\,, \tag{D}$$

and

$$\fint_B |u(x) - u_B| \, dx \leq C_p r \left(\fint_{2B} |\nabla u(x)|^p \, dx \right)^{1/p} \tag{P_p}$$

whenever $u \in C^\infty(2B)$. Here and in what follows u_B is the mean-value of u over B. By a recent result of Hajłasz and Koskela [HK], the two conditions above imply that there are constants $\lambda > 1$ and c such that a Sobolev-Poincaré inequality

$$\left(\fint_B |u(x) - u_B|^{\lambda p} \, dx \right)^{1/(\lambda p)} \leq cr \left(\fint_B |\nabla u(x)|^p \, dx \right)^{1/p} \tag{2.7}$$

holds for all balls $B = B(o,r)$ and for all $u \in C^\infty(B)$; see also [SC]. As a consequence of (2.7) we have the following Sobolev and Poincaré inequalities

$$\left(\fint_B |v(x)|^{\lambda p} \, dx \right)^{1/(\lambda p)} \leq cr \left(\fint_B |\nabla v(x)|^p \, dx \right)^{1/p} \tag{2.8}$$

for every $v \in C_0^\infty(B)$, and

$$\left(\fint_B |u(x) - u_B|^p \, dx\right)^{1/p} \le cr \left(\fint_B |\nabla u(x)|^p \, dx\right)^{1/p} \tag{2.9}$$

whenever $u \in C^\infty(B)$.

Harnack's inequalities are commonly proved by a Moser-type iteration method where the Sobolev and Poincaré inequalities are involved. A third inequality which is needed in the iteration is a Caccioppoli-type estimate; see [H2] for the proof. Assume that $G \subset M$ is open and u is a positive $\mathcal{A}$-harmonic function in G, where $\mathcal{A} \in \mathcal{A}_p(M)$ satisfies (2.2) – (2.5) with constants α and β. If we then set $v = u^{q/p}$, where $q \in \mathbb{R} \setminus \{0, p-1\}$, we have

$$\int_G \eta^p |\nabla v|^p \, dx \le \left(\frac{\beta |q|}{\alpha |q - p + 1|}\right)^p \int_G v^p |\nabla \eta|^p \, dx \tag{2.10}$$

whenever $\eta \in C_0^\infty(G)$ is nonnegative. Now the Harnack inequality reads as follows.

2.11. Theorem. *Suppose that M satisfies (D) and (P_p). Then there is a constant $C_0 = C_0(C_d, C_p, \lambda, p, \beta/\alpha)$ such that*

$$\sup_{x \in B} u(x) \le C_0 \inf_{x \in B} u(x) \tag{2.12}$$

whenever u is a positive $\mathcal{A}$-harmonic function of type p in G and $B = B(o, R)$ is a ball such that $4B \subset G$.

Proof. We will not give all details since the proof is almost the same as that in [H2]. Let $R \le t < t' \le \frac{3}{2}R$, and write $t_i = t + (t' - t)2^{-i}$ and $B_i = B(o, t_i)$ for every $i = 0, 1, \dots$. Then we choose a non-negative $\eta_i \in C_0^\infty(B_i)$ such that $\eta_i = 1$ in B_{i+1} and $|\nabla \eta_i| \le 2(t_i - t_{i+1})^{-1} = 2^{i+2}(t' - t)^{-1}$. Next we choose $q_0 \in \mathbb{R} \setminus \{0\}$ such that, for every i,

$$|q_0 \lambda^i - p + 1| \ge \frac{(p-1)(\lambda - 1)}{\lambda + 1}, \tag{2.13}$$

where λ is the constant in (2.7). Set $q_i = q_0 \lambda^i$. Applying the Sobolev inequality (2.8) to $\eta_i u^{q_i/p}$ and using the doubling property and the Caccioppoli estimate (2.10) we obtain

$$\left(\fint_{B_{i+1}} u^{q_i \lambda} \, dx\right)^{1/\lambda} \le c 2^{ip} \left(\frac{t}{t' - t}\right)^p \left(\left(\frac{|q_0 \lambda^i|}{|q_0 \lambda^i - p + 1|}\right)^p + 1\right) \fint_{B_i} u^{q_i} \, dx \, .$$

By iteration we obtain

$$\left(\fint_{B_j} \left(u^{q_0}\right)^{\lambda^j} dx\right)^{1/\lambda^j} \le 2^{pS_j'} \left(\frac{ct}{t' - t}\right)^{pS_j} \prod_{i=0}^{j-1} \left(\frac{|q_0 \lambda^i|^p}{|q_0 \lambda^i - p + 1|^p} + 1\right)^{1/\lambda^i} \fint_{B_0} u^{q_0} \, dx \, ,$$

where $S_j = \sum_{i=0}^{j} \lambda^{-i}$ and $S'_j = \sum_{i=0}^{j}(i+1)\lambda^{-i}$. By the condition (2.13), the product above has an upper bound which depends only on λ and p. Letting $j \to \infty$ we get

$$\sup_{B(o,t)} u^{q_0} \le c \left(\frac{t}{t'-t} \right)^{p\lambda/(\lambda-1)} \fint_{B(o,t')} u^{q_0}\, dx$$

provided (2.13) holds. We get rid of the restriction (2.13) by observing that, for every $q > 0$, there exists $q_0 \in [q/\lambda, q]$ such that (2.13) holds. On the other hand, (2.13) is true for every $q_0 < 0$. Hence

$$\sup_{B(o,t)} u^{q} \le c \left(\frac{t}{t'-t} \right)^{\xi} \fint_{B(o,t')} u^{q}\, dx$$

for every $q \ne 0$, where $\xi = p\lambda^2/(\lambda-1)$. Using the general John-Nirenberg lemma from [BG, Theorem 4] we get

$$\sup_{B(o,R)} u \le C \exp\bigl(cA(u)\bigr) \inf_{B(o,R)} u\,,$$

where

$$A(u) = \sup_{R \le r \le 3R/2} \inf_{a \in \mathbb{R}} \fint_{B(o,r)} |\log u(x) - a|\, dx\,.$$

The Poincaré inequality (P_p) implies that

$$\inf_{a \in \mathbb{R}} \fint_{B(o,r)} |\log u(x) - a|\, dx \le C_p r \left(\fint_{B(o,2r)} |\nabla \log u(x)|^p\, dx \right)^{1/p}.$$

To estimate further the right hand side, we use the inequality

$$\int_{B(o,2r)} |\nabla \log u(x)|^p\, dx \le c \int_{B(o,4R)} |\nabla \eta(x)|^p\, dx\,,$$

where $\eta \in C_0^\infty(B(o,4R))$ such that $\eta \ge 1$ in $B(o,2r)$; see e.g. [HKM]. Choosing η such that $|\nabla \eta| \le c/r$ and using the doubling property once again we finally obtain

$$\inf_{a \in \mathbb{R}} \fint_{B(o,r)} |\log u(x) - a|\, dx \le cr\bigl(|B(o,2r)|^{-1} r^{-p} |B(o,4R)|\bigr)^{1/p} \le c\,.$$

This completes the proof of Harnack's inequality.

A standard consequence of Harnack's inequality is the following Hölder continuity estimate for $\mathcal{A}$-harmonic functions; see e.g. [HKM]. If u is $\mathcal{A}$-harmonic in G, then

$$\operatorname{osc}\bigl(u; B(x,r)\bigr) \le c\bigl(r/R\bigr)^{\gamma} \operatorname{osc}\bigl(u; B(x,R)\bigr) \tag{2.14}$$

whenever $B(x,R) \subset G$ and $r \le R$. The constants c and γ depend only on C_d, C_p, λ, p, and β/α. Another consequence of Harnack's inequality is the following Liouville-type result.

2.15. Corollary. *Suppose that M satisfies (D) and (P_p). Then every positive $\mathcal{A}$-harmonic function of type p in M must be constant.*

As an example of a manifold M which satisfies the two conditions (D) and (P_p), we consider here a complete, noncompact Riemannian n-manifold M whose Ricci curvature is nonnegative. The doubling condition holds with $C_d = 2^n$ as a consequence of the Bishop-Gromov comparison theorem [CGT]

$$\frac{|B(o,R)|}{|B(o,r)|} \leq \left(R/r\right)^n, \quad 0 < r \leq R.$$

On the other hand, Buser's isoperimetric inequality [Bu] implies that (P_1) holds on M. Hence a global Harnack's inequality holds on M for any positive $\mathcal{A}$-harmonic function. As a consequence we obtain that every positive $\mathcal{A}$-harmonic function in M is constant. Recall that a celebrated theorem of Yau states that every positive harmonic function on such M is constant; see [Y]. We refer to [H1–4] and [HR3] for further results on the existence of nonconstant $\mathcal{A}$-harmonic functions with various properties. Further examples of manifolds which satisfy conditions (D) and (P_1) can be found if one uses results in [CS1]. They showed that (D) and (P_1) are invariant under quasi-isometries (also called rough isometries) between Riemannian manifolds of bounded local geometry. During the preparation of this article we received a manuscript [CS2] of Coulhon and Saloff-Coste where they also prove a global Harnack inequality for p-harmonic functions on manifolds that satisfy the doubling condition and a Poincaré inequality.

3. Picard's theorem

Let M and N be noncompact oriented Riemannian n-manifolds. We assume that M is complete, admits a uniform Harnack inequality for $\mathcal{A}$-harmonic functions of type n with constant $\vartheta = \vartheta(\beta/\alpha)$, and has the following covering property: for each $0 < k < 1$ there exists $m = m(k)$ such that every ball $B(o,r) \subset M$ can contain at most m disjoint balls of radius kr. Furthermore, we assume that N has at least 2 ends, i.e. there exists a compact set $C \subset N$ such that $N \setminus C$ has at least 2 unbounded components V_1 and V_2.

3.1. Theorem. *Let M and N be as above. For every $K \geq 1$ there exists $q = q(K)$ such that every K-quasiregular map $f : M \to N$ must be constant if N has at least q ends.*

Remark. In the theorem above, the dimension n and the functions $\beta/\alpha \mapsto \vartheta(\beta/\alpha)$ and $k \mapsto m(k)$ are supposed to be fixed. Of course, q depends on these, too. We remark here that the above assumptions on M are satisfied if the two conditions (D) and (P_n) hold on M, in particular, if M has nonnegative Ricci curvature.

We are going to prove the above theorem by using the close connection between quasiregular maps and $\mathcal{A}$–harmonic functions. Namely, let $G' \subset N$ be open and let $f : G \to G'$ be quasiregular. Suppose that u is an $\mathcal{A}$–harmonic function in G', with $\mathcal{A} \in \mathcal{A}_n(G')$. Then $u \circ f$ is $f^\# \mathcal{A}$-harmonic in G, where $f^\# \mathcal{A}$, the pullback of $\mathcal{A}$, belongs to $\mathcal{A}_n(G)$ and depends on f and $\mathcal{A}$. This important result was first proved by Reshetnyak for quasiregular maps between open sets of $\mathbb{R}^n$ in a special case, see [Re]. We use the result in the case where u is n-harmonic in G'. Then $u \circ f$ is $\mathcal{A}$–harmonic in G where $\mathcal{A}$ is defined by

$$\mathcal{A}_x(h) = \big\langle \theta(x)h, h \big\rangle^{n/2-1} \theta(x)h\,,$$

with

$$\theta(x) = \begin{cases} J_f^{2/n}(x) T_x f^{-1} T_x f^{-1*}\,, & \text{if } J_f(x) > 0; \\ \mathrm{id}\,, & \text{otherwise.} \end{cases}$$

Here $T_x f^{-1*} : T_x M \to T_{f(x)} N$ is the transpose of the linear map $T_x f^{-1}$. We can choose

$$\alpha = 1/K \quad \text{and } \beta = K^{n-1}$$

as the constants of $\mathcal{A}$.

In the proof of 3.1 we may assume that N has zero n-capacity at infinity. Otherwise, since N has at least two ends, it would be possible to construct nonconstant positive n-harmonic functions on N and further obtain nonconstant positive $\mathcal{A}$-harmonic functions on M. This would violate the uniform Harnack inequality assumption. The proof of 3.1 is then based on Lemma 2.1 and on the following lemma.

3.2. Lemma. *Assume that N has zero n-capacity at infinity. Suppose that $C \subset N$ is compact such that $N \setminus C$ has at least q unbounded components $V_1, \dots, V_q$. Then there exist n-harmonic functions v_j, $j = 2, \dots, q$, and a positive constant κ such that*

$$|v_j| \le \kappa \textit{ in } C \tag{3.3}$$

$$|v_j - v_i| \le 2\kappa \textit{ in } V_1 \tag{3.4}$$

$$\sup_{V_1} v_j = \infty \tag{3.5}$$

$$\inf_{V_j} v_j = -\infty \tag{3.6}$$

$$v_j \textit{ is bounded in } V_k,\ k \ne 1, j \tag{3.7}$$

$$v_j(x) > \kappa \Rightarrow x \in V_1 \tag{3.8}$$

$$v_j(x) < -\kappa \Rightarrow x \in V_j \tag{3.9}$$

Some functions v_j with above properties were constructed in [HR1] in a special case where N is an open manifold with an arbitrary metric obtained from a compact n-manifold by deleting q points. The construction in the above setting is similar and needs only minor changes.

Proof of 3.1. Suppose that a nonconstant K-quasiregular map $f : M \to N$ exists. Then the functions $u_j = v_j \circ f$ are Harnack functions in M with constant $\vartheta = \vartheta(\beta/\alpha)$, where β/α depends only on K and n. Furthermore, all functions u_j are unbounded from both above and below. Applying Lemma 2.1 to u_2 we obtain $x_i \in M$ and $r_i > 0$, $i = 1, 2, \dots$, such that $u_2(x_i) = 0$,

$$M(u_2, x_i, 4r_i) \le c_0 M(u_2, x_i, r_i/2) \le c_0 M(u_2, x_i, r_i)\,, \tag{3.10}$$

and that $M(u_2, x_i, r_i/2) \to \infty$ as $i \to \infty$. Fix $x = x_i$ and $r = r_i$ such that $M(u_2, x, r/2)$ is so large that the estimates below hold, for instance $M(u_2, x, r/2) \geq 4\vartheta\kappa$ will do. Since $M(u_2, x, r/2)$ is large, we obtain from (3.4) and (3.8) that

$$M(u_2, x, s) - 2\kappa \leq M(u_j, x, s) \leq M(u_2, x, s) + 2\kappa \tag{3.11}$$

whenever $s \geq r/2$. Next we use [HR1, Lemma 4.2] to conclude that

$$M(u_j, x, r) \leq (\vartheta - 1)M(-u_j, x, 2r) \tag{3.12}$$

for all j. Observe that to be able to use [HR1, Lemma 4.2], we need to know that $u_j(z) = 0$ for some $z \in B(x, r)$. The existence of such a point z can be seen as follows. Assume on the contrary that $u_j > 0$ in $B(x, r)$. Then $u_j(y) \leq c u_j(x)$ for all $y \in B(x, r/2)$ by Harnack's inequality. Since $M(u_2, x, r/2)$ is large, also $u_j(y)$ is large for some $y \in B(x, r/2)$ by properties (3.4) and (3.8). We conclude that $u_j(x)$ and hence also $u_2(x)$ are large which contradicts with the fact that $u_2(x) = 0$.

Now the inequalities (3.11) and (3.12) imply that

$$M(u_2, x, r) \leq c_1 M(-u_j, x, 2r)\,, \tag{3.13}$$

where $c_1 = 2(\vartheta - 1)$. Fix $z_j \in \bar{B}(x, 2r)$ such that

$$M(-u_j, x, 2r) = -u_j(z_j)\,. \tag{3.14}$$

As in [HR1, (5.5)] we conclude from the Hölder estimate (2.14), [HR1, Lemma 4.2], and (3.11) that

$$\operatorname{osc}\big(u_j; B(z_j, \varrho)\big) \leq c_2 \big(\varrho/r\big)^{\gamma} M(u_2, x, 4r) \tag{3.15}$$

whenever $\varrho \leq r$. Now (3.13), (3.14), and (3.15) imply that

$$\max_{\bar{B}(z_j, \varrho)} u_j \leq -c_1^{-1} M(u_2, x, r) + c_2 \big(\varrho/r\big)^{\gamma} M(u_2, x, 4r)\,.$$

By (3.10), $M(u_2, x, 4r) \leq c_0 M(u_2, x, r)$, and so we may choose $\varrho > 0$ such that

$$c_2 \big(\varrho/r\big)^{\gamma} M(u_2, x, 4r) \leq (2c_1)^{-1} M(u_2, x, r)$$

and that $\varrho/r = c_3$, a constant depending only on ϑ. Hence

$$\max_{\bar{B}(z_j, \varrho)} u_j \leq -(2c_1)^{-1} M(u_2, x, r)\,.$$

Since $M(u_2, x, r)$ is large, we conclude from (3.9) that the balls $B(z_j, \varrho) \subset B(x, 4r)$ are mutually disjoint. The covering property assumption then implies that there can be at most m of them. Hence the number of ends of N is bounded by a constant which depends only on K, n, and on the functions $\beta/\alpha \mapsto \vartheta(\beta/\alpha)$ and $k \mapsto m(k)$.

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