p-CAPACITY AND p-HYPERBOLICITY OF SUBMANIFOLDS

ILKKA HOLOPAINEN†, STEEN MARKVORSEN#, AND VICENTE PALMER*

Abstract. We use explicit solutions to a drifted Laplace equation in warped product model spaces as comparison constructions to show p-hyperbolicity of a large class of submanifolds for $p \geq 2$. The condition for p-hyperbolicity is expressed in terms of upper support functions for the radial sectional curvatures of the ambient space and for the radial convexity of the submanifold. In the process of showing p-hyperbolicity we also obtain explicit lower bounds on the p-capacity of finite annular domains of the submanifolds in terms of the drifted 2-capacity of the corresponding annuli in the respective comparison spaces.

1. Introduction

In [16] the first named author solved the asymptotic Dirichlet problem at infinity for the p-Laplacian in Cartan–Hadamard manifolds of pinched negative sectional curvature. As a consequence, such a manifold admits a wealth of non-constant bounded p-harmonic functions. On the other hand, there are no non-constant positive p-harmonic functions on a complete Riemannian manifold with non-negative Ricci curvature; see e.g. [2]. The purpose of the present paper is to initiate the study of the p-Laplacian and the existence of p-harmonic functions of various types on submanifolds. In this paper we concentrate on p-hyperbolicity of submanifolds.

To describe the problem we are dealing with, suppose that S is a Riemannian submanifold of an ambient Riemannian manifold N. We look for the most general intrinsic geometric condition on N and the most general extrinsic geometric condition on S which together will assure that S is p-hyperbolic. Recall that a Riemannian manifold M is called *p-hyperbolic*, with $1 < p < \infty$, if there exists a compact set $K \subset M$ of positive p-capacity $\mathrm{Cap}_p(K, M)$ relative to M. Here the p-capacity of K is defined by

$$\mathrm{Cap}_p(K, M) = \inf_u \int_M \|\nabla u\|^p \, d\mu,$$

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where the infimum is taken over all real-valued functions $u \in C_0^\infty(M)$, with $u \geq 1$ in K. In case $p = 2$, the p-hyperbolicity of M is equivalent both to the existence of a positive Green's kernel for the Laplace-Beltrami operator and to the transience of M, (see the works [20] and [10]). Using the particular 2-capacity condition alluded to above, the two last named authors have obtained geometric criteria for 2-hyperbolicity of minimal - or close to minimal - submanifolds in manifolds with sectional curvatures bounded from above, (see [23] and [24]).

In the general case of $1 < p < \infty$, the p-hyperbolicity of M is known to be equivalent to the existence of a (positive) Green's function $g = g(\cdot, y)$ for the p-Laplace equation, i.e. a certain positive solution (in the sense of distributions) of

$$-\operatorname{div}\left(\|\nabla g\|^{p-2}\nabla g\right) = \delta_y, \quad y \in M.$$

A third equivalent criterion for the p-hyperbolicity of M is the existence of a non-constant positive p-supersolution of the p-Laplace equation; see [12] and [13]. We refer to [2], [14], and [15] for further studies on p-hyperbolicity and various Liouville-type results and to [24] for a study of the geometric conditions which have been previously applied to extend the intrinsic analysis of hyperbolicity to the extrinsic analysis which is the main concern of the present paper.

To introduce the main results of the paper requires a number of concepts and definitions and therefore we refer to Section 4. Here in the introduction we just single out one consequence of the main result (Theorem 4.1):

Corollary 4.4. *Let (M^m, g) denote a complete manifold with intrinsic concentric metric balls $B_r(o)$ centered at $o \in M$. Suppose that for some $p \geq 2$ and for some $\rho > 0$ we have*

$$\int_\rho^\infty \frac{1}{\operatorname{Vol}(\partial B_r(o))^{\frac{1}{p-1}}}\, dr = \infty,$$

and suppose that there are constants $\lambda_0 > 0$ and $b < 0$ so that

$$(p-2)\lambda_0 < (m-1)\sqrt{-b}.$$

Then (M, g) does not admit a minimal isometric immersion with bounded second fundamental form $\|\alpha\| \leq \lambda_0$ into any Cartan–Hadamard manifold N^n, $n \geq m$, with sectional curvatures bounded from above by b.

1.1. **Outline of the paper.** In Section 2 we describe some of the basic properties of the p-Laplacian and present the corresponding maximum principle, which will be fundamental for the comparison technique applied in this paper. Section 3 is devoted to set up a so-called comparison constellation, which is essentially molded from curvature restrictions and a model space construction. In Section 4 we formulate our main result together with three of its corollaries. They are proved in Sections 7, 8, and 9. As an application to the main theorem (Theorem 4.1) we study p-hyperbolicity of some surfaces of revolution in 3-dimensional hyperbolic space in Section 5. A technical tool, the drifted 2-capacity of model spaces is defined and analyzed in Section 6. Finally, in Section 10 we present an alternative proof of the main theorem based directly on finite capacity comparison results.

2. The p-Laplacian

Let M be a non-compact Riemannian manifold, with the Riemannian metric $\langle\cdot,\cdot\rangle$ and the Riemannian volume form $d\mu$. We say that a vector field $\nabla u \in L^1_{\mathrm{loc}}(M)$ is a *distributional gradient* of a function $u \in L^1_{\mathrm{loc}}(M)$ if

$$\int_M \langle \nabla u, V\rangle \, d\mu = -\int_M u \operatorname{div} V \, d\mu$$

for all compactly supported vector fields $V \in C^1_0(M)$. Let $W^{1,p}(M)$, $1 \leq p < \infty$, be the Sobolev space of all functions $u \in L^p(M)$ whose distributional gradient ∇u belongs to $L^p(M)$. We equip $W^{1,p}(M)$ with the norm $\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p$. The corresponding local space $W^{1,p}_{\mathrm{loc}}(M)$ is defined in an obvious manner. The space $W^{1,p}_0(M)$ is the closure of $C^\infty_0(M)$ in $W^{1,p}(M)$.

Let $1 < p < \infty$. A function $u \in W^{1,p}_{\mathrm{loc}}(M)$ is a (weak) solution to the p-Laplace equation

$$-\operatorname{div}\left(\|\nabla u\|^{p-2}\nabla u\right) = 0 \tag{2.1}$$

in M if

$$\int_M \langle \|\nabla u\|^{p-2}\nabla u, \nabla\phi\rangle \, d\mu = 0 \tag{2.2}$$

for all $\phi \in C^\infty_0(M)$. If, moreover, $\|\nabla u\| \in L^p(M)$, it is equivalent to require (2.2) for all $\phi \in W^{1,p}_0(M)$. Continuous solutions of (2.1) are called *p-harmonic*. Here the continuity assumption makes no restriction since every solution of (2.1) has a continuous representative by the fundamental work of Serrin [29]. In fact, p-harmonic functions have locally Hölder-continuous first order derivatives by regularity results due to Ural'tseva [33] and Lewis [18]; see also DiBenedetto [3], Evans [6], Tolksdorf [30], and Uhlenbeck [32]. Furthermore, if $D \subset M$ is a precompact open set with $C^{1,\alpha}$ boundary ($\alpha \leq 1$), $h \in C^{1,\alpha}(\partial D)$, and u is p-harmonic in D with boundary values h, then $u \in C^{1,\beta}(\bar{D})$, with $\beta = \beta(\alpha, p, \dim M)$, by Lieberman [19]. See Remark 10.2 for a discussion why these regularity results, originally proven in the Euclidean setting, apply to the Riemannian setting as well.

A function $u \in W^{1,p}_{\mathrm{loc}}(M)$ is called a *p-supersolution* in M if

$$\int_M \langle \|\nabla u\|^{p-2}\nabla u, \nabla\phi\rangle \, d\mu \geq 0$$

for all non-negative $\phi \in C^\infty_0(M)$. Similarly, a function $v \in W^{1,p}_{\mathrm{loc}}(M)$ is called a *p-subsolution* in M if

$$\int_M \langle \|\nabla v\|^{p-2}\nabla v, \nabla\phi\rangle \, d\mu \leq 0$$

for all non-negative $\phi \in C^\infty_0(M)$. A fundamental feature of solutions of (2.1) is the following well-known *maximum (or comparison) principle* which will be instrumental for the comparison technique presented below in Sections 4 and 6: If $u \in W^{1,p}(M)$ is a p-supersolution, $v \in W^{1,p}(M)$ is a p-subsolution, and $\max(v-u, 0) \in W^{1,p}_0(M)$, then $u \geq v$ a.e. in M. In particular, if $D \subset M$ is a precompact open set, $u \in C(\bar{D})$ is a p-supersolution, $v \in C(\bar{D})$ is a p-subsolution, and $u \geq v$ in ∂D, then $u \geq v$ in D. We refer to [11, 3.18] for a short proof of the comparison principle.

3. Comparison Constellations

We assume throughout the paper that S^m is a non-compact connected complete Riemannian submanifold of a complete Riemannian manifold N^n. Furthermore, we assume that N^n possesses at least one pole. Recall that a pole is a point o such that the exponential map $\exp_o\colon T_oN^n \to N^n$ is a diffeomorphism. For example, a Cartan–Hadamard manifold has everywhere non-positive sectional curvatures and since it is also by definition simply connected, *every* point is a pole. The rôle of the pole o is precisely to serve as the origin of a smooth distance function r from o: For every $x \in N^n \setminus \{o\}$ we define $r(x) = \operatorname{dist}_N(o,x)$, and this distance is realized by the length of a unique geodesic from o to x, which is the *radial geodesic from* o. We also denote by r the restriction $r|_S : S \to \mathbb{R}_+ \cup \{0\}$. This restriction is called the *extrinsic distance function* from o in S^m. The gradients of r in N and S are denoted by $\nabla^N r$ and $\nabla^S r$, respectively. Let us remark that $\nabla^S r(x)$ is just the tangential component in S of $\nabla^N r(x)$, for all $x \in S$. Then we have the following basic relation:

$$\nabla^N r = \nabla^S r + (\nabla^N r)^\perp,$$

where $(\nabla^N r)^\perp(x)$ is perpendicular to T_xS for all $x \in S$.

3.1. **Curvature restrictions.** The sectional curvatures of N *along* the radial geodesics from o are called the o-radial sectional curvatures of N.

Definition 3.1. Let o be a point in a Riemannian manifold M and let $x \in M \setminus \{o\}$. The sectional curvature $K_M(\sigma_x)$ of the two-plane $\sigma_x \in T_xM$ is then called an *o-radial sectional curvature* of M at x if σ_x contains the tangent vector to a minimal geodesic from o to x. We denote these curvatures by $K_{o,M}(\sigma_x)$.

The o-radial sectional curvatures of N control the second order behavior of $r(x)$ in N via the classical Jacobi field index theory. Indeed, a bound on the o-radial sectional curvatures gives a bound on the Hessian of radial functions, $\operatorname{Hess}^N(f(r))$, as proved by Greene and Wu [9, Theorem A]; see Theorem 3.15 below. The submanifold S and the restricted radial functions $f(r)|_S$ inherit this second order bound to the S-intrinsic Hessian, $\operatorname{Hess}^S f(r)$, and therefore also to the Laplacian $\Delta^S f(r)$ of such modified distance functions.

The mean curvatures H_S of S also appear in the Laplacian $\Delta^S f(r)$ via its radially weighted component, which we define as follows:

Definition 3.2. The *o-radial mean convexity* $\mathcal{C}(x)$ of S in N, is defined in terms of the inner product of H_S with the N-gradient of the distance function $r(x)$ as follows:

$$\mathcal{C}(x) \,=\, -\langle \nabla^N r(x), H_S(x)\rangle, \quad x \in S,$$

where $H_S(x)$ denotes the mean curvature vector of S in N, i.e. the mean trace of the second fundamental form α_x. With respect to an orthonormal basis $\{X_1, ..., X_m\}$ of T_xS at $x \in S$ we have

$$H_S(x) = \frac{1}{m}\sum_{i=1}^{m} \alpha_x\left(X_i, X_i\right).$$

Notice that H_S takes values in the normal bundle of S. We will assume, that $\mathcal{C}(x)$ is bounded from above by a function $h(r(x))$ which only depends on the distance r from o:

$$\mathcal{C}(x) \leq h(r(x)), \quad x \in S.$$

Moreover, for $p > 2$ we shall also need a particular inequality for the second fundamental form of S in N in the direction of the gradient $\nabla^N r(x)$. This gives rise to the following definition:

Definition 3.3. The *o-radial component* $\mathcal{B}(x)$ *of the second fundamental form* of S in N, is defined in terms of the following inner product:

$$\mathcal{B}(x) = -\langle \nabla^N r(x), \alpha_x(U_r, U_r) \rangle,$$

where

$$U_r = \nabla^S (r(x)) / \| \nabla^S r(x) \| \in T_x S \subset T_x N$$

is the unit tangent vector to S in the direction of $\nabla^S r(x)$ (resp. tacitly assumed to be 0 in case $\nabla^S r(x) = 0$).

We assume that $\mathcal{B}(x)$ is bounded from above by a function $\lambda(r(x))$ which only depends on the distance r from o:

$$\mathcal{B}(x) \leq \lambda(r(x)).$$

Finally, we also impose an upper control on the 'radiality' of the submanifold, i.e. a local measure of how much the submanifold is extending away from the pole o:

Definition 3.4. The *o-radial tangency* $\mathcal{T}(x)$ of S in N is defined as follows:

$$\mathcal{T}(x) = \| \nabla^S r(x) \|$$

for all $x \in S$.

We assume that this S-gradient of the restricted distance function $r|_S$ has an upper radial support function $g(r) \leq 1$:

$$\mathcal{T}(x) \leq g(r(x)).$$

Definition 3.5. Given a connected and complete m-dimensional submanifold S^m in a complete Riemannian manifold N^n with a pole o, we denote the *extrinsic metric balls* of (sufficiently large) radius R and center o by $D_R(o)$. They are defined as any connected component of the intersection

$$B_R(o) \cap S = \{x \in S \colon r(x) < R\},$$

where $B_R(o)$ denotes the open geodesic ball of radius R centered at the pole o in N^n. Using these extrinsic balls we define the o-centered extrinsic annuli

$$A_{\rho,R}(o) = D_R(o) \setminus \bar{D}_\rho(o)$$

in S^m for $\rho < R$, where $D_R(o)$ is the component of $B_R(o) \cap S$ containing $D_\rho(o)$.

The upper bounding functions $h(r)$, $g(r)$, and $\lambda(r)$ together with a suitable control on the o-radial sectional curvatures of the ambient space will eventually control the p-Laplacian of restricted radial functions on S. In particular, we consider potential functions stemming from capacity calculations of radially symmetric comparison spaces and transplant them to S

via the distance function r in N. Such transplantations are then compared with the 'correct' potentials on extrinsic metric balls of S. The maximum principle for the p-Laplacian Δ_p^S then finally gives the comparison result for capacities in S. Concerning the general strategy and types of results (in the case of $p = 2$) we refer to [21], [26], and [22].

3.2. **Warped products and model spaces.** Warped products are generalized manifolds of revolution, see e.g. [25]. Let (B^k, g_B) and (F^l, g_F) denote two Riemannian manifolds and let $w\colon B \to \mathbb{R}_+$ be a positive real function on B. We assume throughout that w is at least C^2. We consider the product manifold $M^{k+l} = B \times F$ and denote the projections onto the factors by $\pi\colon M \to B$ and $\sigma\colon M \to F$, respectively. The metric g on M is then defined by the following w-modified (warped) product metric

$$g = \pi^*(g_B) + (w \circ \pi)^2 \sigma^*(g_F).$$

Definition 3.6. The Riemannian manifold $(M, g) = (B^k \times F^l, g)$ is called a *warped product* with *warping function* w, base manifold B and fiber F. We write as follows: $M_w^m = B^k \times_w F^l$.

Definition 3.7 (See [10], [9])**.** A $w-$model M_w^m is a smooth warped product with base $B^1 = [0, \Lambda[\subset \mathbb{R}$ (where $0 < \Lambda \leq \infty$), fiber $F^{m-1} = \mathbb{S}_1^{m-1}$ (i.e. the unit $(m-1)$-sphere with standard metric), and warping function $w\colon [0, \Lambda[\to \mathbb{R}_+ \cup \{0\}$, with $w(0) = 0$, $w'(0) = 1$, and $w(r) > 0$ for all $r > 0$. The point $o_w = \pi^{-1}(0)$, where π denotes the projection onto B^1, is called the *center point* of the model space. If $\Lambda = \infty$, then o_w is a pole of M_w^m.

Proposition 3.8. *The simply connected space forms $\mathbb{K}^m(b)$ of constant curvature b are $w-$models with warping functions*

$$w(r) = Q_b(r) = \begin{cases} \frac{1}{\sqrt{b}} \sin(\sqrt{b}\, r) & \text{if } b > 0 \\ \quad r & \text{if } b = 0 \\ \frac{1}{\sqrt{-b}} \sinh(\sqrt{-b}\, r) & \text{if } b < 0. \end{cases}$$

Note that for $b > 0$ the function $Q_b(r)$ admits a smooth extension to $r = \pi/\sqrt{b}$.

Proposition 3.9 (See e.g. [25])**.** *Let $M_w^m = B^1 \times_w \mathbb{S}_1^{m-1}$ be a $w-$model. Let r_0 and r denote two points in B^1. Then the geodesic distance from every $x \in \pi^{-1}(r)$ to $\pi^{-1}(r_0)$ is $|r - r_0|$.*

Proposition 3.10 (See [25] p. 206)**.** *Let M_w^m be a $w-$model with warping function $w(r)$ and center o_w. The distance sphere of radius r and center o_w in M_w^m is the fiber $\pi^{-1}(r)$. This distance sphere has the following constant mean curvature vector in M_w^m*

$$H_{\pi^{-1}(r)} = -\eta_w(r)\, \nabla^M \pi = -\eta_w(r)\, \nabla^M r,$$

where the mean curvature function $\eta_w(r)$ is defined by

$$\eta_w(r) = \frac{w'(r)}{w(r)} = \frac{d}{dr} \log(w(r)).$$

In particular, we have for the constant curvature space forms $\mathbb{K}^m(b)$:

$$\eta_{Q_b}(r) = \begin{cases} \sqrt{b}\cot(\sqrt{b}\,r) & \text{if } b > 0 \\ \quad 1/r & \text{if } b = 0 \\ \sqrt{-b}\coth(\sqrt{-b}\,r) & \text{if } b < 0\,. \end{cases}$$

The radial curvature in model spaces is given by the following result

Proposition 3.11 (See [9] and [10])**.** *Let M_w^m be a $w-$model with center point o_w. Then the o_w-radial sectional curvatures of M_w^m at every $x \in \pi^{-1}(r)$ (for $r > 0$) are all identical and determined by*

$$K_{o_w, M_w}(\sigma_x) = -\frac{w''(r)}{w(r)}.$$

3.3. **Comparison constellation.** We now collect the previous ingredients and formulate the general framework for our p-hyperbolicity comparison result:

Definition 3.12. Let N^n denote a Riemannian manifold with a pole o and distance function $r = r(x) = \operatorname{dist}_N(o, x)$. Let S^m denote a connected complete submanifold in N^n and assume that there is an extrinsic ball $D_\rho(o)$ which is precompact with smooth boundary $\partial D_\rho(o)$ in S^m. Let M_w^m denote a w-model with warping function $w : \pi(M_w^m) \to \mathbb{R}_+$ and center o_w; see Definition 3.7. Then the triple $\{N^n, S^m, M_w^m\}$ is called a *comparison constellation* on the interval $[0, R]$ if the o-radial sectional curvatures of N are bounded from above by the o_w-radial sectional curvatures of M_w^m:

$$K_{o,N}(\sigma_x) \leq -\frac{w''(r)}{w(r)} \tag{3.1}$$

for all x with $r = r(x) \in [0, R]$ and, moreover, the radial tangency $\mathcal{T}$ and the radial convexity functions $\mathcal{B}$ and $\mathcal{C}$ of the submanifold S^m are all bounded from above by smooth radial functions $g(r)$, $\lambda(r)$, and $h(r)$, respectively:

$$\begin{aligned} &\mathcal{T}(x) \leq g(r(x)), \\ &\mathcal{B}(x) \leq \lambda(r(x)), \;\text{ and} \\ &\mathcal{C}(x) \leq h(r(x)) \;\text{ for all }\; x \in S^m \;\text{ with }\; r(x) \in [0, R]. \end{aligned} \tag{3.2}$$

Remark 3.13. We want to point out that the assumption on the smoothness of $\partial D_\rho(o)$ makes no restriction. Indeed, the distance function r is smooth in $N^n \setminus \{o\}$ since N^n is assumed to possess a pole $o \in N^n$. Hence the restriction $r|_S$ is smooth in S and consequently the radii ρ that produce smooth boundaries $\partial D_\rho(o)$ are dense in $\mathbb{R}$ by Sard's theorem and the Regular Level Set Theorem.

Remark 3.14. The definition of comparison constellation above extends a previous definition considered in [24]. In that paper, the triple $\{N^n, S^m, M_w^m\}$ is called a *comparison constellation* if inequality (3.1) holds and if in addition only the following condition holds in replacement of inequalities (3.2) for some bounding radial function $h(r)$:

$$\mathcal{C}(x) \leq h(r(x)) \leq \frac{w'(r(x))}{w(r(x))} \;\text{ for all }\; x \in S^m.$$

It is proved in [24] that under these conditions S^m is 2-hyperbolic if

$$\int_\rho^\infty \frac{\mathcal{G}^m(r)}{w^{m-1}(r)}dr < \infty,$$

where

$$\mathcal{G}(r) = \exp(\int_\rho^r h(t)dt).$$

3.4. **Hessian and Laplacian comparison analysis.** Concerning the second order analysis of the distance function r we need firstly and foremost the Hessian comparison theorem for manifolds with a pole:

Theorem 3.15 (See [9], Theorem A)**.** *Let $N = N^n$ be a manifold with a pole o, let $M = M^m_w$ denote a $w-$model with center o_w, and $m \leq n$. Suppose that every o-radial sectional curvature at $x \in N \setminus \{o\}$ is bounded from above by the o_w-radial sectional curvatures in M^m_w as follows:*

$$K_{o,N}(\sigma_x) \leq -\frac{w''(r)}{w(r)}$$

for every radial two-plane $\sigma_x \in T_xN$ at distance $r = r(x) = \mathrm{dist}_N(o,x)$ from o in N. Then the Hessian of the distance function in N satisfies

$$\begin{aligned} \mathrm{Hess}^N(r(x))(X,X) &\geq \mathrm{Hess}^M(r(y))(Y,Y) \\ &= \eta_w(r)\left(1 - \langle\nabla^M r(y), Y\rangle^2_M\right) \\ &= \eta_w(r)\left(1 - \langle\nabla^N r(x), X\rangle^2_N\right) \end{aligned} \tag{3.3}$$

for every unit vector X in T_xN and for every unit vector Y in T_yM with $r(y) = r(x) = r$ and $\langle\nabla^M r(y), Y\rangle_M = \langle\nabla^N r(x), X\rangle_N$.

Remark 3.16. In [9, Theorem A, p. 19], the Hessian of r_M is less or equal to the Hessian of r_N provided that the radial curvatures of N are bounded from above by the radial curvatures of M and provided that $\dim M \geq \dim N$. This latter dimension condition is *not* satisfied in our setting. However, since (M^m, g) is a $w-$model space it has an $n-$dimensional $w-$model space companion with the same radial curvatures and the same Hessian of radial functions as (M^m, g). In effect, therefore, applying [9, Theorem A, p. 19] to the high-dimensional comparison space gives the low-dimensional comparison inequality as stated.

If $\mu \colon N \to \mathbb{R}$ denotes a smooth function on the ambient space N, then the restriction $\tilde{\mu} = \mu_{|S}$ is a smooth function on the submanifold S and the respective Hessian tensors, $\mathrm{Hess}^N(\mu)$ and $\mathrm{Hess}^S(\tilde{\mu})$, are related as follows:

Proposition 3.17 ([17])**.**

$$\mathrm{Hess}^S(\tilde{\mu})(X,Y) = \mathrm{Hess}^N(\mu)(X,Y) + \langle\nabla^N(\mu), \alpha_x(X,Y)\rangle \tag{3.4}$$

for all tangent vectors X, $Y \in T_xS^m \subset T_xN^n$, where α_x is the second fundamental form of S at x in N.

If we compose μ with a smooth function $f \colon \mathbb{R} \to \mathbb{R}$ we then get:

Corollary 3.18 ([17]).

$$\begin{aligned}\mathrm{Hess}^S(f\circ\tilde{\mu})(X,X) &= f''(\mu)\langle\nabla^N(\mu),X\,\rangle^2\\ &+f'(\mu)\big(\mathrm{Hess}^N(\mu)(X,X)+\langle\nabla^N(\mu),\alpha_x(X,X)\rangle\big)\end{aligned}$$

for all $X\in T_xS^m$.

Combining the estimate (3.3) with Corollary 3.18 and tracing the resulting Hessian comparison statement in an orthonormal basis of T_xS^m, we obtain the following instrumental inequality for the Laplacian of (extrinsic) radial functions restricted to the submanifold S:

Proposition 3.19. *Suppose that the assumptions of Theorem 3.15 are satisfied. Then we have for every smooth real-valued function* $f\circ r$ *with* $f'\geq 0$ *the following inequality for the standard Laplacian:*

$$\Delta^S(f\circ r)\geq\big(f''(r)-f'(r)\eta_w(r)\big)\,\|\nabla^S r\|^2+mf'(r)\,\big(\eta_w(r)+\langle\nabla^N r,H_S\rangle\big)\,,$$

where H_S *denoted the mean curvature vector of* S *in* N.

4. Main results

Applying the notion of a comparison constellation as defined in the previous section, we now formulate our main p-hyperbolicity result. The proofs are developed through the following sections.

Theorem 4.1. *Consider a comparison constellation* $\{N^n,S^m,M^m_w\}$ *on the interval* $[\,0,\infty[\,$. *Assume further that the functions* $h(r)$ *and* $\lambda(r)$ *are* balanced *with respect to the warping function* $w(r)$ *by the following inequality:*

$$\mathcal{M}(r):=(m+p-2)\,\eta_w(r)-m\,h(r)-(p-2)\lambda(r)\geq 0. \tag{4.1}$$

Let $\Lambda(r)$ *denote the function*

$$\Lambda(r)=w(r)\exp\left(-\int_\rho^r\frac{\mathcal{M}(t)}{(p-1)g^2(t)}\,dt\right).$$

Suppose finally that $p\geq 2$ *and that*

$$\int_\rho^\infty\Lambda(t)\,dt<\infty. \tag{4.2}$$

Then S^m *is p-hyperbolic.*

We observe the following corollaries; the first two will be proved in Section 9.

Corollary 4.2. *Suppose (in Theorem 4.1) that we can choose* $w(r)=Q_b(r)=\sinh(\sqrt{-b}\,r)/\sqrt{-b}$ *for some* $b<0$*, i.e. we apply the negatively curved space form* $\mathbb{K}^m(b)$ *to play the role of a model space in the comparison constellation. Suppose that there exist constants* λ_0 *and* h_0 *such that*

$$\begin{aligned}&\mathcal{B}(x)\leq\lambda_0\ \text{ and}\\ &\mathcal{C}(x)\leq h_0\ \text{ for all } x\in S^m.\end{aligned}$$

Suppose further that for some $\tilde{p}\geq 2$ *we have*

$$m\,h_0+(\tilde{p}-2)\lambda_0<(m-1)\sqrt{-b}. \tag{4.3}$$

Then S^m *is p-hyperbolic for all* p *in the range* $2\leq p\leq\tilde{p}$.

Corollary 4.3. *Consider a purely intrinsic setting and comparison constellation: $S^n = N^n = M^n_w$. Then S^n is p-hyperbolic if and only if*

$$\int_\rho^\infty \frac{1}{w(t)^{\frac{n-1}{p-1}}}\, dr < \infty.$$

This observation is originally due to M. Troyanov, see [31, Corollary 5.2]. We want to point out that the result holds for values $1 < p < 2$, too.

Corollary 4.4. *Let (M^m, g) denote a complete manifold with intrinsic concentric metric balls $B_r(o)$ centered at $o \in M$. Suppose that for some $p \geq 2$ and for some $\rho > 0$ we have*

$$\int_\rho^\infty \frac{1}{\mathrm{Vol}(\partial B_r(o))^{\frac{1}{p-1}}}\, dr = \infty, \tag{4.4}$$

and suppose that there are constants $\lambda_0 > 0$ and $b < 0$ so that

$$(p-2)\lambda_0 < (m-1)\sqrt{-b}. \tag{4.5}$$

Then (M, g) does not admit a minimal isometric immersion with bounded second fundamental form $\|\alpha\| \leq \lambda_0$ into any Cartan–Hadamard manifold N^n, $n \geq m$, with sectional curvatures bounded from above by b.

Proof. Condition (4.4) implies that the manifold (M^m, g) is p-parabolic according to [31, Corollary 5.4], whereas the condition (4.5) implies p-hyperbolicity of (M^m, g) according to Corollary 4.2 of the present work - upon observing that $\mathcal{C}(x) \equiv 0$ by the minimality assumption and that $\|\alpha_x\| \leq \lambda_0$ implies $\mathcal{B}(x) \leq \lambda_0$. □

5. p-Hyperbolic surfaces of revolution in $\mathbb{K}^3(-1)$

We consider a specific family of surfaces of revolution in 3-dimensional hyperbolic space and show the p-hyperbolicity of these surfaces - first as an application of Theorem 4.1 (for $p \geq 2$) and then by applying Corollary 4.3 (for all $1 < p < \infty$). We model the ambient hyperbolic space on $\mathbb{R}^2 \times \mathbb{R}_+ = \{(x, y, z) \in \mathbb{R}^3 \colon z > 0\}$ with the conformal factor $1/z^2$ and consider the curves:

$$\gamma_k(s) = \big(x(s),\ 0,\ z(s)\big) = \left(k \sin\left(\frac{s}{k}\right),\ 0,\ k\cos\left(\frac{s}{k}\right) + 1 - k\right), \tag{5.1}$$

where k is a (family)-parameter in the range $k \in]0, \infty[$. We also need that $s \in [0, k \arccos((k-1)/k)[$ for $k \geq 1/2$ to guarantee $z(s) > 0$ and that $s \in [0, k\pi]$ for $0 < k < 1/2$. The curve $\gamma_k(s)$ is a segment of the circle of (Euclidean) radius k in the quarter-plane $x \geq 0,\ y = 0,\ z > 0$ which goes through the point $(0, 0, 1)$, where it hits the z-axis orthogonally. The circle is unit length parameterized with respect to the Euclidean metric in the quarter-plane. Following the setting and calculations of [27, Appendix p. 2759] and [4] we rotate $\gamma_k(s)$ around the z-axis and obtain a complete surface of revolution S^2_k in $\mathbb{K}^3(-1)$. We can find more information about these surfaces in [5, pp. 177-184].

Proposition 5.1. *For each $k \in]0,\infty[$ the surface S^2_k is totally umbilical with principal curvatures*

$$\kappa_1(s) = \kappa_2(s) = \frac{k-1}{k}. \tag{5.2}$$

In particular, S^2_k is of constant sectional curvature

$$K^S = -1 + \frac{(k-1)^2}{k^2}. \tag{5.3}$$

Proof. Let us denote $S = S^2_k$ and $N = \mathbb{K}^3(-1)$. According to [27, Appendix p. 2759] the principal curvatures of S are for any general generating curve $\big(x(s), 0, z(s)\big)$:

$$\begin{aligned}\kappa_1(s) &= z(s)\alpha'(s) + \cos(\alpha(s))\\ \kappa_2(s) &= \cos(\alpha(s)) + \left(\frac{z(s)}{x(s)}\right)\sin(\alpha(s)),\end{aligned}$$

where $\alpha(s)$ is the angle that the curve makes with the x-axis. When inserting our specific choices for $x(s)$ and $z(s)$ the result in equation (5.2) follows easily. Sectional curvature on S can be calculated by using the Gauss formula (see e.g. [5, Theorem 2.5, p. 130]). First we observe that $|\alpha_x(X,X)| \equiv (k-1)/k$ for all unit vectors $X \in TS$ since both principal curvatures are equal to $(k-1)/k$ at every point in S. Furthermore, all vectors in T_xS, $x \in S$, are eigenvectors (principal directions) of the second fundamental form. Hence $\alpha_x(X_1, X_2) = 0$ whenever X_1 and X_2 are orthogonal and thus, if moreover they are unit vectors

$$\begin{aligned}K^S(X_1,X_2) &= K^N(X_1,X_2) + \langle \alpha_x(X_1,X_1), \alpha_x(X_2,X_2)\rangle - |\alpha_x(X_1,X_2)|\\ &= -1 + \frac{(k-1)^2}{k^2}.\end{aligned}$$

□

Proposition 5.2. *The surface S^2_k is p-hyperbolic for every $1 < p < \infty$ whenever $1/2 < k < \infty$. Furthermore, $S^2_{1/2}$ is p-hyperbolic if and only if $1 < p < 2$ and S^2_k is p-parabolic for every $p > 1$ if $0 < k < 1/2$.*

Although the above statement follows from Proposition 5.1 and from well-known results on p-hyperbolicity of spaces of constant curvature, (see [31], and [10]), we will give below two proofs of the result to illustrate the use of Theorem 4.1 (for $p \geq 2$) and Corollary 4.3 (for all $1 < p < \infty$).

Indeed, we recognize S^2_1 as the totally geodesic surface which goes through the point $(0,0,1)$ orthogonally to the z-axis. The surface S^2_1 is isometric to the hyperbolic plane $\mathbb{K}^2(-1)$ and is therefore p-hyperbolic for every $1 < p < \infty$. Similarly, S^2_k is isometric to the simply connected space form $\mathbb{K}^m(b)$ of constant curvature $b = -1 + \big((k-1)/k\big)^2$. Hence S^2_k is of constant negative curvature if $1/2 < k < \infty$, and therefore p-hyperbolic for every $1 < p < \infty$. On the other hand, $S^2_{1/2}$ is the horosphere at $(0,0,0)$ passing through the point $(0,0,1)$. The surface $S^2_{1/2}$ is totally umbilical with constant mean curvature 1 and it is well-known to be isometric to the flat Euclidean two-plane. Thus $S^2_{1/2}$ is p-hyperbolic if and only if $1 < p < 2$. At the other end, when k tends to ∞ we recognize another horosphere S^2_∞ as the plane $z = 1$ in $\mathbb{K}^3(-1)$. Similarly, S^2_∞ is totally umbilical with constant mean curvature 1 and isometric to the flat Euclidean two-plane. Hence S^2_∞ is p-hyperbolic if and only if $1 < p < 2$. Finally, for $0 < k < 1/2$, the surface S^2_k is compact and therefore p-parabolic for every $p > 1$.

Proof of Proposition 5.2 using Theorem 4.1: Case $p \geq 2,\ k > 1/2$. We will use $o = (0, 0, 1)$ as the pole of $N = \mathbb{K}^m(b)$. Since the principal curvatures are the same and the surface $S = S^2_k$ is of codimension 1, we have

$$H_S(x) = \alpha_x(X, X) = \frac{k-1}{k}\nu \tag{5.4}$$

for all unit vectors $X \in T_xS$, where $\nu \in (T_xS)^\perp$ is the unit vector normal to S pointing towards the z-axis. In particular, (5.4) holds for the unit vector U_r. It follows that $\mathcal{C}(x) = \mathcal{B}(x)$ and we may choose $h(r(x)) = \lambda(r(x)) = \mathcal{C}(x)$. Next we define $\beta(r)$ by setting $\beta(r(x))$ to be the angle between $\nabla^N r(x)$ and $\nabla^S r(x)$ (i.e. the angle between $\nabla^N r(x)$ and the surface S). Then $\beta(r)$ increases from 0 to $\beta(\infty) = |\arcsin((k-1)/k)|$. Now

$$\begin{aligned} h\big(r(x)\big) = \lambda\big(r(x)\big) &= -|H_S(x)|\cos(\beta(r(x)) + \pi/2) \\ &= \frac{|k-1|}{k}\sin\beta(r(x)) \end{aligned} \tag{5.5}$$

which increases from 0 to $(k-1)^2/k^2$. Similarly,

$$\mathcal{T}(x) = \cos\beta(r(x)) \tag{5.6}$$

and hence $g(r(x)) = \mathcal{T}(x)$ decreases from 1 to $\cos\beta(\infty) = \sqrt{1-(k-1)^2/k^2}$. Furthermore, we may choose $\mathbb{K}^2(-1)$ as the w-model M^2_w in the comparison constellation. The balance condition (4.1) holds since

$$\begin{aligned} \mathcal{M}(r) &= (m + p - 2)\eta_w(r) - m\,h(r) - (p-2)\lambda(r) \\ &= p\left(\coth r - \frac{|k-1|}{k}\sin\beta(r)\right) \\ &\geq p\left(\coth r - \frac{(k-1)^2}{k^2}\right) > 0. \end{aligned} \tag{5.7}$$

Fix $\varepsilon > 0$ to be specified later and choose a sufficiently large (inner) radius ρ such that $g^2(r) \leq 1 - (k-1)^2/k^2 + \varepsilon$ for all $r \geq \rho$. Then

$$\begin{aligned} \Lambda(r) &= (\sinh r)\exp\left(-\int_\rho^r \frac{\mathcal{M}(t)}{(p-1)g^2(t)}\,dt\right) \\ &\leq (\sinh r)\exp\left(-\int_\rho^r \frac{p\left(\coth t - (k-1)^2/k^2\right)}{(p-1)(1-(k-1)^2/k^2+\varepsilon)}\,dt\right) \\ &\leq c(\rho)\exp\left(r\big(1 - C + C(k-1)^2/k^2\big)\right), \end{aligned}$$

where $c(\rho)$ is a constant depending only on ρ and

$$C = \frac{p}{(p-1)(1-(k-1)^2/k^2+\varepsilon)}.$$

If we choose

$$\varepsilon < \frac{2k-1}{(p-1)k^2},$$

$1 - C + C(k-1)^2/k^2 < 0$ and consequently the integral

$$\int_\rho^\infty \Lambda(t)\, dt$$

is finite. Hence S_k^2 is p-hyperbolic for every $2 \le p < \infty$ if $k > 1/2$. $\square$

Proof of Proposition 5.2 using Corollary 4.3. Since the surfaces S_k^2 are models, Corollary 4.3 gives the precise criteria for p-hyperbolicity. Thus it suffices to study whether the integrals

$$I_k(p) := \int_c^\infty \frac{dt}{A_k(t)^{1/(p-1)}}$$

are finite. We show that $I_k(p) < \infty$ for every $1 < p < \infty$ and $1/2 < k < \infty$ and that $I_{1/2}(p) < \infty$ if and only if $1 < p < 2$. Here c is a positive constant which will change its actual value in what follows and $A_k(t)$ is the intrinsic (i.e. hyperbolic) length of the circle on S_k^2 of intrinsic radius t centered at $(0,0,1)$. This circle is the intersection of the surface S_k^2 with an appropriate plane $z =$ constant. Let us denote by $c(s)$ the intersection circle of S_k^2 and the plane $z = z(s)$. The Euclidean radius of $c(s)$ is $x(s)$ and hence the intrinsic length of $c(s)$ is

$$L\big(\delta(s)\big) := 2\pi x(s)/z(s).$$

Denote by $\delta(s)$ the intrinsic distance (i.e. the intrinsic radius) of $c(s)$ from the point $(0,0,1)$. Thus

$$\delta(s) = \int_0^s 1/z(t)\, dt$$

and hence

$$I_k(p) = \int_c^\infty \frac{d\big(\delta(s)\big)}{L\big(\delta(s)\big)^{1/(p-1)}}.$$

After change of variables and forgetting irrelevant multiplicative factors we obtain an integral

$$\int_c^{K(k)} \left(\frac{z(s)^{2-p}}{x(s)}\right)^{1/(p-1)} ds,$$

where $K(k) = k \arccos\big(\frac{k-1}{k}\big)$ is the value for the parameter s such that $z(K(k)) = 0$. Performing another change of variables

$$z(s) = kz, \quad x(s) = k\sqrt{1-(z+1-1/k)^2}, \quad ds = \frac{-k\, dz}{\sqrt{1-(z+1-1/k)^2}}$$

and again forgetting irrelevant multiplicative factors we get

$$\int_0^c \Big(\sqrt{1-(z+1-1/k)^2}\Big)^{\frac{p}{1-p}} z^{\frac{2-p}{p-1}}\, dz.$$

Hence $I_{1/2}(p) < \infty$ if and only if $1 < p < 2$ whereas $I_k(p) < \infty$ for every $1 < p < \infty$ and $1/2 < k < \infty$. $\square$

Remark 5.3. In this remark we discuss how Theorem 4.1 applies to the study of p-hyperbolicity of a horosphere in a Cartan–Hadamard manifold of negative curvature. Let us first consider the case $N = \mathbb{K}^{m+1}(-1)$, with $m \ge 3$. Exactly as in the case $m = 2$, all horospheres are of constant mean curvature 1. Of course, it is well-known that horospheres are isometric to $\mathbb{R}^m$ and hence p-hyperbolic if and only if $1 < p < m$, but we want to deduce the p-hyperbolicity of a horosphere for $2 \le p < m$ from Theorem 4.1. It is convenient to use the unit ball model for $\mathbb{K}^{m+1}(-1)$ equipped with the Riemannian metric

$$ds^2 = \frac{4|dz|^2}{(1+|x|^2)^2}, \tag{5.8}$$

and choose the origin as a pole o. Let S be the horosphere at $-e_{m+1} = (0, 0, \dots, 0, -1) \in \partial\mathbb{B}^{m+1}$ passing through the origin. Thus S is the intersection of the unit ball $\mathbb{B}^{m+1}$ and the m-dimensional sphere $S^m(-e_{m+1}/2, 1/2)$ of (Euclidean) radius $1/2$ centered at $-e_{m+1}/2$. The hyperbolic distance between a point $x \in \mathbb{K}^{m+1}(-1)$ and the pole o is given by the formula

$$r(x) = \log \frac{1+|x|}{1-|x|}.$$

As in (5.5) and (5.6), we have

$$h\big(r(x)\big) = \lambda\big(r(x)\big) = \sin\beta\big(r(x)\big)$$

and

$$g\big(r(x)\big) = \cos\beta\big(r(x)\big),$$

where $\beta\big(r(x)\big)$ is the angle between $\nabla^N r(x)$ and $\nabla^S r(x)$. In order to apply Theorem 4.1 we need a sharp estimate, or rather a formula, for $\beta(r)$. By elementary (Euclidean) trigonometry, it is easy to see that $\beta\big(r(x)\big) = \alpha_x/2$, where α_x is the angle at $-e_{m+1}/2$ between (Euclidean) line segments from $-e_{m+1}/2$ to o and to x, respectively. Note that the conformal change of the metric (5.8) keeps angles invariant. Furthermore, $\sin\beta\big(r(x)\big) = |x|$, and therefore

$$\sin\beta(r) = \frac{e^r - 1}{e^r + 1}.$$

Thus

$$\begin{aligned}\frac{\mathcal{M}(t)}{(p-1)g^2(t)} &= \frac{(m+p-2)\big(\coth t - \sin\beta(t)\big)}{2(p-1)\big(1-\sin^2\beta(t)\big)} \\ &= \frac{(m+p-2)}{2(p-1)}\left(1 + O(e^{-t})\right)\end{aligned}$$

as $t \to \infty$, and consequently

$$\Lambda(r) \le c(\rho)(\sinh r)\exp\left(-\frac{m+p-2}{2(p-1)}\left(r + O(e^{-r})\right)\right)$$

as $r \to \infty$. If $p < m$, we have

$$1 - \frac{m+p-2}{2(p-1)} < 0,$$

and therefore

$$\int_\rho^\infty \Lambda(t)\,dt < \infty$$

showing the p-hyperbolicity of S.

It turns out that Theorem 4.1 can not be applied to studying p-hyperbolicity of horospheres on a general Cartan–Hadamard manifold since the balance condition (4.1) need not be valid. For example, the complex hyperbolic space $\mathbb{CH}^2$ equipped with a (Riemannian) metric of constant holomorphic sectional curvature -4 is a Cartan–Hadamard manifold of pinched (Riemannian) sectional curvature $-4 \leq K \leq -1$. Furthermore, for every $x \in \mathbb{CH}^2$ and every $K \in [-4,-1]$ there exists a 2-plane $P \subset T_x\mathbb{CH}^2$ such that the sectional curvature of P at x takes the value K. Thus in order to apply Theorem 4.1 the best possible model would be $\mathbb{K}^3(-1)$. On the other hand, the principal curvatures of a horosphere S in $\mathbb{CH}^2$ are either 1 or 2 (with multiplicities 2 and 1, respectively), and S has constant mean curvature 4/3. Since, moreover,

$$-\langle \nabla^N r(x), H_S(x)\rangle \to \frac{4}{3}$$

as $r(x) \to \infty$, we must have

$$\liminf_{r\to\infty} h(r) \geq \frac{4}{3} \quad \text{and} \quad \liminf_{r\to\infty} \lambda(r) \geq 1,$$

and thus the balance condition (4.1) does not hold for large r. We refer to [1] and [8] for the curvature results above. It is worth pointing out that every horosphere in $\mathbb{CH}^2$ is isometric to the first Heisenberg group equipped with a left-invariant Riemannian metric (see e.g. [8]) and hence is p-hyperbolic if and only if $1 < p < 4$ by [13].

6. Drifted 2-capacity of model spaces

Definition 6.1. Let (M, g) denote a Riemannian manifold with Laplace operator Δ^M, and let V denote a continuous vector field on M. The drifted Brownian motion on M with the drift vector field V is then generated by the modified Laplacian L

$$\mathrm{L}\, f = \Delta^M f + \langle \nabla^M f, V\rangle$$

for every smooth function f on M.

We consider, in particular, the drift vector field

$$V = \mathcal{V}(r)\nabla^M r$$

with

$$\mathcal{V}(r) = \frac{\mathcal{M}(r)}{(p-1)g^2(r)} - m\,\eta_w(r)$$

on model spaces $M = M_w^m$, so that the modified Laplacian then reads as

$$\mathrm{L}\,\psi(x) = \Delta^M\psi(x) + \psi'(r(x))\mathcal{V}(r(x))$$

for smooth functions ψ on M_w^m. For purely *radial* functions $\psi(r)$ we get

Lemma 6.2. *Let $\psi = \psi(r)$ denote a function on the w-model space $M = M_w^m$ which only depends on the radial distance r to the center o_w. Then*

$$\mathrm{L}\,\psi(r) = \psi''(r) + \psi'(r)\left(\frac{\mathcal{M}(r)}{(p-1)\,g^2(r)} - \eta_w(r)\right).$$

The Dirichlet problem associated to L defined on so-called *extrinsic annuli* is defined as follows:

First, the annular domains in the model space are denoted by

$$A^w_{\rho,R} = \{x \in M^n_w \colon \pi(x) \in [\rho, R]\} = \pi^{-1}([\rho, R]),$$

and the corresponding boundaries are denoted by $\partial D^w_\rho = \pi^{-1}(\rho)$ and $\partial D^w_R = \pi^{-1}(R)$, respectively. We consider the unique radial function $\psi_{\rho,R}(r)$ which solves the one-dimensional Laplace-Dirichlet problem on the model space annulus $A^w_{\rho,R}$:

$$\begin{cases} \mathrm{L}\,\psi &= 0 \ \text{ on } A^w_{\rho,R} \\ \quad \psi &= 0 \ \text{ on } \partial D^w_\rho \\ \quad \psi &= 1 \ \text{ on } \partial D^w_R. \end{cases} \tag{6.1}$$

The explicit solution to the Dirichlet problem (6.1) is given in the following Proposition, with a focus towards the corresponding expression for the drifted annular capacity in the model space; see [24], [23], and Section 10 below.

Proposition 6.3. *The solution to the Dirichlet problem (6.1) only depends on r and is given explicitly - via the function $\Lambda(r)$ introduced in Theorem 4.1, by:*

$$\psi_{\rho,R}(r) = \frac{\int_\rho^r \Lambda(t)\, dt}{\int_\rho^R \Lambda(t)\, dt}. \tag{6.2}$$

The corresponding 'drifted' 2-capacity is

$$\begin{aligned} \mathrm{Cap}_{\mathrm{L}}(A^w_{\rho,R}) &= \int_{\partial D^w_\rho} \langle \nabla^M \psi_{\rho,R}, \nu\rangle \, dA \\ &= \mathrm{Vol}(\partial D^w_\rho)\Lambda(\rho) \left(\int_\rho^R \Lambda(t)\, dt \right)^{-1} . \end{aligned} \tag{6.3}$$

7. p-Laplacian comparison

Let us consider comparison constellations $\{N^n, S^m, M^m_w\}$ on intervals $[0, R]$ for $R > 0$. Since the o-radial mean convexity of S has an upper bound

$$\mathcal{C}(x) = -\langle \nabla^N r(x), H_S(x)\rangle \le h(r(x)),$$

we obtain the following estimate using Proposition 3.19

$$\Delta^S (f \circ r) \ge \big(f''(r) - f'(r)\eta_w(r)\big)\, \|\nabla^S r\|^2 + m\, f'(r)\, (\eta_w(r) - h(r))\, . \tag{7.1}$$

In what follows we use shorthand

$$F(x) = f'(r(x))\|\nabla^S r(x)\| \tag{7.2}$$

for all $x \in S$ to simplify the notation. To get estimates for the p-Laplacian of $f \circ r$ we first compute

$$
\begin{aligned}
&\Delta_p^S f(r(x)) = \operatorname{div}^S \left(\|\nabla^S f(r(x))\|^{p-2}\, \nabla^S f(r(x))\right)\\
&= \|\nabla^S f(r(x))\|^{p-2}\, \Delta^S f(r(x)) + \left\langle \nabla^S \|\nabla^S f(r(x))\|^{p-2}, \nabla^S f(r(x))\right\rangle\\
&= F^{p-2}(x)\Delta^S f(r(x)) + \left\langle \nabla^S F^{p-2}(x), f'(r(x))\nabla^S r(x)\right\rangle\\
&= F^{p-2}(x)\Delta^S f(r(x))\\
&\qquad + \Big\langle (p-2)F^{p-3}(x)\left(f''(r(x))\|\nabla^S r(x)\|\nabla^S r(x) + f'(r(x))\nabla^S\|\nabla^S r(x)\|\right),\\
&\qquad\quad f'(r(x))\nabla^S r(x)\Big\rangle\\
&= F^{p-2}(x)\Big((p-2)\Big(f''(r(x))\|\nabla^S r(x)\|^2 + f'(r(x))\frac{\left\langle \nabla^S r(x), \nabla^S\|\nabla^S r(x)\|\right\rangle}{\|\nabla^S r(x)\|}\Big)\\
&\qquad + \Delta^S f(r(x))\Big).
\end{aligned}
$$

This partial 'isolation' of the factor $(p-2)$ is the reason behind the general assumption $p \geq 2$ in this work. The factor on $(p-2)$ is controlled via the following observation, which introduces the bound $\lambda(r)$ into this setting:

Lemma 7.1. *Let $\{N^n, S^m, M_w^m\}$ be a comparison constellation on $[0, R]$ for $R > 0$. Suppose that the o-radial component of the second fundamental form of S (see Definition 3.3) has an upper bound*

$$\mathcal{B}(x) \leq \lambda(r(x)).$$

Then

$$
\begin{aligned}
&\frac{\left\langle \nabla^S r(x), \nabla^S\|\nabla^S r(x)\|\right\rangle}{\|\nabla^S r(x)\|}\\
&\qquad = \operatorname{Hess}^S(r(x))\,(U_r, U_r)\\
&\qquad = \operatorname{Hess}^N(r(x))\,(U_r, U_r) + \left\langle \nabla^N r(x), \alpha_x\,(U_r, U_r)\right\rangle\\
&\qquad \geq \eta_w(r(x))\left(1 - \|\nabla^S r(x)\|^2\right) - \lambda(r(x)).
\end{aligned}
\tag{7.3}
$$

Proof. By definition of the Hessian via the induced connection D^S in S we have directly for the first equality in (7.3):

$$
\begin{aligned}
\operatorname{Hess}^S(r)\left(\nabla^S r, \nabla^S r\right) &= \left\langle \mathrm{D}^S_{\nabla^S r}\, \nabla^S r, \nabla^S r\right\rangle\\
&= \tfrac{1}{2}\, \mathrm{D}^S_{\nabla^S r} \left\langle \nabla^S r, \nabla^S r\right\rangle\\
&= \tfrac{1}{2}\nabla^S r \left\langle \nabla^S r, \nabla^S r\right\rangle\\
&= \tfrac{1}{2}\left\langle \nabla^S\|\nabla^S r\|^2, \nabla^S r\right\rangle\\
&= \|\nabla^S r\|\left\langle \nabla^S\|\nabla^S r\|, \nabla^S r\right\rangle,
\end{aligned}
$$

so that

$$
\begin{aligned}
\operatorname{Hess}^S(r(x))\,(U_r, U_r) &= \frac{\operatorname{Hess}^S(r)\left(\nabla^S r, \nabla^S r\right)}{\|\nabla^S r\|^2}\\
&= \frac{\left\langle \nabla^S r(x), \nabla^S\|\nabla^S r(x)\|\right\rangle}{\|\nabla^S r(x)\|}.
\end{aligned}
$$

The other (in)equalities in (7.3) follow from (3.4) and (3.3), respectively. $\square$

The following result relates the p-Laplacian of a radial function $f(r)$ with its 2-drifted Laplacian, as defined in Section 6.

Lemma 7.2. *Let $\{N^n, S^m, M_w^m\}$ be a comparison constellation on $[0,R]$ for $R > 0$. Let $f \circ r$ be a smooth real-valued function with $f' \geq 0$, and suppose now that $f(r)$ satisfies the following condition (to be molded shortly from the balance condition (4.1)):*

$$f''(r) - f'(r)\eta_w(r) \leq 0. \tag{7.4}$$

Then, for all $x \in S$,

$$\Delta_p^S f(r(x)) \geq (p-1)F^{p-2}(x)g^2(r(x))\,\mathrm{L}(f(r(x))),$$

where L is the modified 2-Laplacian defined in Lemma 6.2 and F is given by (7.2).

Proof. By using the assumption $p \geq 2$ together with the comparison constellation assumptions (3.2) we obtain from (7.1) and (7.3) that

$$\begin{aligned}
&\Delta_p^S(f(r(x)))\\
&\geq F^{p-2}(x)(p-2)\left(f''(r)\|\nabla^S(r)\|^2 + f'(r)\,\mathrm{Hess}^S(r)\,(U_r,U_r)\right)\\
&\quad + F^{p-2}(x)\left(f''(r)\|\nabla^S(r)\|^2 - f'(r)\eta_w(r)\|\nabla^S(r)\|^2 + m\,f'(r)\,(\eta_w(r) - h(r))\right)\\
&\geq F^{p-2}(x)(p-1)\|\nabla^S(r)\|^2\left(f''(r) - f'(r)\eta_w(r)\right)\\
&\quad + F^{p-2}(x)f'(r)\,((p-2+m)\eta_w(r) - (p-2)\lambda(r) - m\,h(r))\\
&= F^{p-2}(x)\left(\left(f''(r) - f'(r)\eta_w(r)\right)(p-1)\|\nabla^S(r)\|^2 + f'(r)\mathcal{M}(r)\right).
\end{aligned}$$

Since $f(r)$ satisfies inequality (7.4), we have, via $\|\nabla^S(r)\| \leq g(r)$, that:

$$\begin{aligned}
&\Delta_p^S(f(r(x)))\\
&\geq F^{p-2}(x)\left(\left(f''(r) - f'(r)\eta_w(r)\right)(p-1)g^2(r) + f'(r)\mathcal{M}(r)\right)\\
&= (p-1)F^{p-2}(x)g^2(r)\left(f''(r) - f'(r)\eta_w(r) + f'(r)\frac{\mathcal{M}(r)}{(p-1)g^2(r)}\right)\\
&= (p-1)F^{p-2}(x)g^2(r)\left(f''(r) + f'(r)\left(\frac{\mathcal{M}(r)}{(p-1)g^2(r)} - \eta_w(r)\right)\right)\\
&= (p-1)F^{p-2}(x)g^2(r)\,\mathrm{L}(f(r)),
\end{aligned}$$

as claimed in the lemma. □

8. First proof of Theorem 4.1

Next we show that (4.2) is also a sufficient condition for p-hyperbolicity of S^m. First we transplant the model space solutions $\psi_{\rho,R}(r)$ of equation (6.1) into the extrinsic annulus $A_{\rho,R} = D_R(o) \setminus \bar{D}_\rho(o)$ in S by defining

$$\Psi_{\rho,R}\colon A_{\rho,R} \to \mathbb{R}, \quad \Psi_{\rho,R}(x) = \psi_{\rho,R}(r(x)).$$

Here the extrinsic ball $D_\rho(o)$ is as in Definition 3.12 and $D_R(o)$ is that component of $B_R(o) \cap S$ which contains $D_\rho(o)$. Next we extend $\Psi_{\rho,R}$ to $S \cap \bar{B}_\rho(o)$ by setting $\Psi_{\rho,R}(x) = 0$ for $x \in S \cap \bar{B}_\rho(o)$.

Using $w'(r) = \eta_w(r)w(r)$ and the balance condition (4.1) it is straightforward to check that

$$\psi''_{\rho,R}(r) - \psi'_{\rho,R}(r)\eta_w(r) \leq 0.$$

Since $\psi'_{\rho,R}(r) \geq 0$ and $\mathrm{L}\,\psi_{\rho,R} = 0$ in $A^w_{\rho,R}$, we obtain from Lemma 7.2 that

$$\Delta^S_p \Psi_{\rho,R} \geq 0 \quad \text{in} \quad D_R(o) \setminus \bar{B}_\rho(o).$$

Thus $\Psi_{\rho,R}$ is a p-subsolution in $D_R(o) \setminus \bar{B}_\rho(o)$. In fact, $\Psi_{\rho,R}$ is a p-subsolution in the whole extrinsic ball $D_R(o)$ since $\Psi_{\rho,R}(x) = 0$ for $x \in S \cap \bar{B}_\rho(o)$; see [11, Theorem 7.25, Lemma 7.28]. Furthermore, for fixed ρ and fixed $x \in S$, $\Psi_{\rho,R}(x)$ is defined for sufficiently large R and it is decreasing as a function of R, see equation (6.2). Hence the limit function

$$\Psi_\rho := \lim_{R\to\infty} \Psi_{\rho,R}$$

exists in S and, moreover, it is positive in $S \setminus \bar{B}_\rho(o)$ by (4.2). By [11, Theorem 3.75], Ψ_ρ is a p-subsolution in S. Hence $1 - \Psi_\rho$ is a non-negative, non-constant p-supersolution in S, and therefore S is p-hyperbolic. This proves Theorem 4.1.

9. Proof of Corollaries

Proof of Corollary 4.2. The balance condition (4.1) is clearly satisfied by (4.3). Thus we only need to check the p-hyperbolicity condition (4.2). Since $g(r) \leq 1$, we have

$$\frac{\mathcal{M}(r)}{(p-1)g^2(r)} > (1+c)\sqrt{-b}$$

for some positive constant c by (4.3). Hence

$$\Lambda(r) \leq \frac{\sinh(\sqrt{-b}\,r)}{\sqrt{-b}} \exp\left(-\int_1^r (1+c)\sqrt{-b}\,dt\right)$$

and therefore it is straightforward to check that

$$\int_\rho^\infty \Lambda(t)\,dt < \infty,$$

which concludes the proof. □

Proof of Corollary 4.3. The assumptions amount to $g(r) \equiv 1$, $h(r) \equiv 0$, and $\lambda(r) \equiv 0$ and the only 'free' function is $w(r)$. In this intrinsic setting we therefore have

$$\mathcal{M}(r) = (m+p-2)\eta_w(r),$$

so that with $g(r) = 1$ we get

$$\begin{aligned}
\int_\rho^r \frac{\mathcal{M}(t)}{(p-1)g^2(t)}\,dt &= \tfrac{m+p-2}{p-1} \int_\rho^r \frac{w'(t)\,dt}{w(t)} \\
&= \tfrac{m+p-2}{p-1} \log \frac{w(r)}{w(\rho)},
\end{aligned}$$

and hence

$$\begin{aligned}
\Lambda(r) &= w(r) \exp\left(-\tfrac{m+p-2}{p-1} \log \frac{w(r)}{w(\rho)}\right) \\
&= w(r)^{1-\frac{m+p-2}{p-1}} w(\rho)^{-\frac{m+p-2}{p-1}} \\
&= w(r)^{-\frac{m-1}{p-1}}\, c(\rho),
\end{aligned}$$

where $c(\rho)$ is a constant depending on the fixed inner radius of the annuli used in the proof of the p-hyperbolicity. Then $\Lambda(r)$ has bounded integral precisely if

$$\int_{\rho}^{\infty} \frac{1}{w(r)^{\frac{m-1}{p-1}}}\, dt < \infty,$$

as claimed. $\square$

10. p-capacity bounds

In this section we give lower bounds on the p-capacity of closed (compact) extrinsic balls relative to S^m. Let $G \subset S^m$ be a precompact open set such that $\bar{D}_\rho(o) \subset G$. We recall from the introduction that the p-capacity of $\bar{D}_\rho(o)$ relative to G is defined by

$$\operatorname{Cap}_p(\bar{D}_\rho(o), G) = \inf_v \int_G \|\nabla^S v\|^p \, d\mu,$$

where the infimum is taken over all real-valued functions $v \in C_0^\infty(G)$, with $v \geq 1$ in $\bar{D}_\rho(o)$. If ∂G is regular for the Dirichlet problem for p-harmonic functions, then there exists a unique function $u \in C(\bar{G})$ which is p-harmonic in $G \setminus \bar{D}_\rho(o)$ such that $u = 0$ in $\bar{D}_\rho(o)$, $u = 1$ in ∂G, and that

$$\operatorname{Cap}_p(\bar{D}_\rho(o), G) = \int_G \|\nabla^S u\|^p \, d\mu.$$

We refer to [11, Chapter 6] for the boundary regularity. For our purposes it is enough to know that every open set can be exhausted by open sets with regular boundaries.

Since u is p-harmonic in $G \setminus \bar{D}_\rho(o)$, we have

$$\operatorname{Cap}_p(\bar{D}_\rho(o), G) = \int_G \langle \|\nabla^S u\|^{p-2} \nabla^S u, \nabla^S \varphi \rangle \, d\mu \tag{10.1}$$

for every function $\varphi \in W^{1,p}(G)$ which is continuous in $\bar{G}$ with values $\varphi = 0$ in $\bar{D}_\rho(o)$ and $\varphi = 1$ in ∂G. In particular, (10.1) holds for all $0 \leq t < s \leq 1$ with the function

$$\varphi(x) = \begin{cases} 0 & \text{if } u(x) \leq t \\ \frac{u(x)-t}{s-t} & \text{if } t < u(x) < s \\ 1 & \text{if } u(x) \geq s. \end{cases}$$

Applying the co-area formula ([28], [7, 3.2.12, 3.2.46], [34]) we obtain

$$\operatorname{Cap}_p(\bar{D}_\rho(o), G) = \frac{1}{s-t} \int_t^s \left(\int_{u^{-1}(\tau)} \|\nabla^S u\|^{p-1} \, d\mathcal{H}^{m-1} \right) d\tau.$$

Letting $s \to t$ we finally get

$$\operatorname{Cap}_p(\bar{D}_\rho(o), G) = \int_{u^{-1}(t)} \|\nabla^S u\|^{p-1} \, d\mathcal{H}^{m-1} \tag{10.2}$$

for a.e. $t \in [0, 1]$. We will use the equation (10.2) to get lower bounds on the p-capacity $\operatorname{Cap}_p(\bar{D}_\rho(o), D_R(o))$ in terms of the corresponding drifted 2-capacity in the model space.

Our main comparison estimate for the p-capacity now reads as follows:

Theorem 10.1. *Let $\{N^n, S^m, M_w^m\}$ denote a comparison constellation on $[0,R]$, $R > \rho$, in the sense of Definition 3.12. Then*

$$\operatorname{Cap}_p\big(\bar{D}_\rho(o), D_R(o)\big) \geq \left(\frac{\operatorname{Cap}_{\mathrm{L}}(A^w_{\rho,R})}{\operatorname{Vol}(\partial D^w_\rho)}\right)^{p-1} \int_{\partial D_\rho} \|\nabla^S r\|^{p-1}\, d\mathcal{H}^{m-1}. \tag{10.3}$$

Proof. Let $G \subset D_R(o)$ be a precompact open set with regular boundary such that $\bar{D}_\rho(o) \subset G$. Let $u \in C(\bar{G})$ be p-harmonic in $G \setminus \bar{D}_\rho(o)$ with $u = 0$ in $\bar{D}_\rho(o)$ and $u = 1$ in ∂G. Furthermore, let $\Psi_{\rho,R}$ be the p-subsolution in $D_R(o)$ defined in Section 8. By the comparison principle,

$$u(x) \geq \Psi_{\rho,R}(x)$$

for all $x \in D_R(o)$. Recall that $\partial D_\rho(o)$ is assumed to be smooth; see Remark 3.13. Since $\nabla^S u$ is Hölder-continuous up to the boundary $\partial D_\rho(o)$ by [19] (see Remark 10.2 below) and $u(x) = \Psi_{\rho,R}(x) = 0$ for all $x \in \bar{D}_\rho(o)$, we obtain

$$\|\nabla^S u(x)\| \geq \|\nabla^S \Psi_{\rho,R}(x)\| \tag{10.4}$$

for all $x \in \partial D_\rho(o)$. Combining (10.2) and (10.4), we arrive at

$$\begin{aligned}
\operatorname{Cap}_p(\bar{D}_\rho(o), G) &\geq \int_{\partial D_\rho} \|\nabla^S \Psi_{\rho,R}\|^{p-1}\, d\mathcal{H}^{n-1} \\
&= \big(\psi'_{\rho,R}(\rho)\big)^{p-1} \int_{\partial D_\rho} \|\nabla^S r\|^{p-1}\, d\mathcal{H}^{m-1} \\
&= \left(\frac{\operatorname{Cap}_{\mathrm{L}}(A^w_{\rho,R})}{\operatorname{Vol}(\partial D^w_\rho)}\right)^{p-1} \int_{\partial D_\rho} \|\nabla^S r\|^{p-1}\, d\mathcal{H}^{m-1}.
\end{aligned}$$

The desired estimate (10.3) now follows since

$$\operatorname{Cap}_p\big(\bar{D}_\rho(o), D_R(o)\big) = \inf_G \operatorname{Cap}_p(\bar{D}_\rho(o), G),$$

where $G \subset D_R(o)$ is a precompact open set with regular boundary. □

Remark 10.2. The $C^{1,\beta}$-regularity results for p-harmonic functions that were mentioned in the introduction are proven in the Euclidean setting. In this remark we explain why these regularity results apply to p-harmonic functions in precompact open subsets D of a Riemannian n-manifold as well.

The p-Laplace equation is the Euler-Lagrange equation of the variational integral

$$\int_D \|\nabla u\|^p\, d\mu\,.$$

A key idea in proofs of $C^{1,\beta}$-regularity (also up to the boundary) is to consider, for a fixed $0 < \varepsilon < 1$, the minimizer u_ε of the variational integral

$$\int_D \big(\varepsilon + \|\nabla u\|^2\big)^{p/2}\, d\mu \tag{10.5}$$

with fixed boundary values and prove estimates for ∇u_ε that are independent of ε. Since u_ε is a minimizer, it is a weak solution to

$$\operatorname{div}\Big(\big(\varepsilon + \|\nabla u_\varepsilon\|^2\big)^{p/2} \nabla u_\varepsilon\Big) = 0. \tag{10.6}$$

Recall that in local coordinates the divergence of a vector field $X = X^i\partial_i$ is

$$\operatorname{div} X = \partial_i X^i + \Gamma^i_{ki} X^k,$$

where $\partial_1, \dots, \partial_n$ is the coordinate frame and Γ^j_{ki} are the corresponding Christoffel symbols. Writing (10.6) in local coordinates yields an equation

$$\partial_i \Big(\big(\varepsilon + \|\nabla u_\varepsilon\|^2\big)^{p/2-1} g^{ji}\partial_j u_\varepsilon \Big) + \Gamma^i_{ki}\big(\varepsilon + \|\nabla u_\varepsilon\|^2\big)^{p/2-1} g^{jk}\partial_j u_\varepsilon = 0\,, \tag{10.7}$$

where g^{ij} are the entries of the matrix $(g_{ij})^{-1}$, $g_{ij} = \langle \partial_i, \partial_j\rangle$.

For simplicity let us use the same notation for functions on M and their local representations. Then (10.7) is equivalent to the equation (now in $\mathbb{R}^n$)

$$\operatorname{div} A\big(x, \nabla u_\varepsilon(x)\big) + B\big(x, \nabla u_\varepsilon(x)\big) = 0,$$

where the kth component of the vector $A(x,h)$, $h = (h_1, \dots, h_n)$, is

$$A^k(x,h) = \Big(\varepsilon + g^{i\ell}(x) h_i h_\ell\Big)^{p/2-1} g^{jk}(x) h_j\,,$$

and

$$B(x,h) = \Gamma^i_{ki}\Big(\varepsilon + g^{i\ell}(x) h_i h_\ell\Big)^{p/2-1} g^{jk}(x) h_j\,.$$

As in [19] we write

$$\begin{aligned} a^{kj}(x,h) &= \frac{\partial A^k}{\partial h_j}(x,h) \\ &= \Big(\varepsilon + g^{i\ell}(x) h_i h_\ell\Big)^{p/2-1} \left(g^{jk}(x) + \frac{(p-2)\big(g^{ik}(x)h_i\big)\big(g^{ij}(x)h_i\big)}{\varepsilon + g^{i\ell}(x) h_i h_\ell} \right). \end{aligned}$$

Next recall that the components of the Riemannian metric with respect to normal coordinates at a point $y \in M$ behave like $g_{ij}(\exp_y v) = \delta_{ij} + O(|v|^2)$ as $v \to 0$. Hence $g_{ij}(y) = \delta_{ij}$, $\partial_k g_{ij}(y) = 0$, and $\Gamma^j_{ki}(y) = 0$. By smoothness of g_{ij}, $\partial_k g_{ij}$, and Γ^j_{ki} we conclude that each point $y \in \bar D$ has a neighborhood such that A, a^{jk}, and B, when written with respect to normal coordinates at y, satisfy the structure conditions

$$\begin{aligned} a^{kj}(x,h)\xi_k\xi_j &\ge \lambda\left(\varepsilon + |h|^2\right)^{p/2-1}|\xi|^2, \quad \xi \in \mathbb{R}^n, \\ \left|a^{kj}(x,h)\right| &\le \Lambda\left(\varepsilon + |h|^2\right)^{p/2-1}, \\ |B(x,h)| &\le \Lambda\,(1+|h|)^p\,, \\ |A(x,h) - A(y,h)| &\le \Lambda\,(1+|h|)^{p-1}\,|x-y|, \end{aligned}$$

where the constants λ and Λ depend only on p and the dimension n.

Suppose then that $D \subset M$ is a precompact open set with $C^{1,\alpha}$ boundary ($\alpha \le 1$), $h \in C^{1,\alpha}(\partial D)$, and that u is p-harmonic in D with boundary values h. Applying [19] and the discussion above we conclude that each point $y \in \bar D$ has a neighborhood U such that $u \in C^{1,\beta}(\bar D \cap U)$, with $\beta = \beta(\alpha, p, n)$, By compactness of $\bar D$ we finally obtain that $u \in C^{1,\beta}(\bar D)$ as stated in the introduction.

10.1. **Second Proof of Theorem 4.1.** Using the explicit capacity comparison obtained in Theorem 10.1 we finally observe the following direct proof of the main theorem.

Let $\{N^n, S^m, M_w^m\}$ denote a comparison constellation on $[0, \infty]$ in the sense of Definition 3.12. By assumption $D_\rho(o)$ is precompact with a smooth boundary (cf. Remark 3.13) and thence, in equation (10.3) we have

$$\int_{\partial D_\rho} \|\nabla^S r\|^{p-1} \, d\mathcal{H}^{m-1} \, > 0.$$

From (6.3) and the assumption (4.2) we also have

$$\lim_{R\to\infty} \operatorname{Cap}_{\mathrm{L}}(A^w_{\rho,R}) \, > \, 0,$$

so that Theorem 10.1 implies:

$$\operatorname{Cap}_p\big(\bar{D}_\rho(o), S^m\big) \, = \, \lim_{R\to\infty} \operatorname{Cap}_p\big(\bar{D}_\rho(o), D_R(o)\big) \, > \, 0.$$

Thus $\bar{D}_\rho(o)$ is a compact subset with positive p-capacity in S^m, and p-hyperbolicity of that submanifold follows again.

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P.O. Box 68 (Gustaf Hällströmin katu 2b), FIN-00014 University of Helsinki, Finland.
E-mail address: ilkka.holopainen@helsinki.fi

Department of Mathematics, Technical University of Denmark.
E-mail address: S.Markvorsen@mat.dtu.dk

Departament de Matemàtiques, Universitat Jaume I, Castellon, Spain.
E-mail address: palmer@mat.uji.es