Proceedings of the International Workshop on Quasiconformal
Mappings and their Applications (IWQCMA05)

p-Laplace operator, quasiregular mappings, and Picard-type theorems

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Abstract. We describe the role of p-harmonic functions and forms in the theory of quasiregular mappings.

Keywords. Quasiregular mapping, p-harmonic function, p-harmonic form, conformal capacity.

2000 MSC. Primary 58J60, 30C65; Secondary 53C20, 31C12, 35J60.

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1. Introduction

In this survey we emphasize the importance of the p-Laplace operator as a tool to prove basic properties of quasiregular mappings, as well as Liouville- and Picard-type results for quasiregular mappings between given Riemannian manifolds. Quasiregular mappings were introduced by Reshetnyak in the mid sixties in a series of papers; see e.g. [36], [37], and [38]. An interest in studying these mappings arises from a question about the existence of a geometric function theory in *real* dimensions $n \geq 3$ generalizing the theory of holomorphic functions $\mathbb{C} \to \mathbb{C}$.

Definition 1.1. A continuous mapping $f\colon U \to \mathbb{R}^n$ of a domain $U \subset \mathbb{R}^n$ is called *quasiregular* (or a *mapping of bounded distortion*) if

(1) $f \in W^{1,n}_{\mathrm{loc}}(U;\mathbb{R}^n)$, and
(2) there exists a constant $K \geq 1$ such that

$$|f'(x)|^n \leq K\, J_f(x) \quad \text{for a.e. } x \in U.$$

The condition (1) means that the coordinate functions of f belong to the local Sobolev space $W^{1,n}_{\mathrm{loc}}(U)$ consisting of locally n-integrable functions whose distributional (first) partial derivatives are also locally n-integrable. In Condition (2) $f'(x)$ denotes the formal derivative of f at x, i.e. the $n \times n$ matrix $\big(D_j f_i(x)\big)$ defined by the partial derivatives of the coordinate functions f_i of f. Furthermore,

$$|f'(x)| = \max_{|h|=1} |f'(x)h|$$

is the operator norm of $f'(x)$ and $J_f(x) = \det f'(x)$ is the Jacobian determinant of f at x. They exist a.e. by (1). The smallest possible K in Condition (2) is the *outer dilatation* $K_O(f)$ of f. If f is quasiregular, then

$$J_f(x) \leq K' \ell(f'(x))^n \quad \text{a.e.}$$

for some constant $K' \geq 1$, where

$$\ell(f'(x)) = \min_{|h|=1} |f'(x)h|.$$

The smallest possible K' is the *inner dilatation* $K_I(f)$ of f. It is easy to see by linear algebra that $K_O(f) \leq K_I(f)^{n-1}$ and $K_I(f) \leq K_O(f)^{n-1}$. If $\max\{K_O(f), K_I(f)\} \leq K$, f is called K-*quasiregular*.

To motivate the above definition, let us consider a holomorphic function $f\colon U \to \mathbb{C}$, where $U \subset \mathbb{C}$ is an open set. We write f as a mapping $f = (u,v)\colon U \to \mathbb{R}^2,\ U \subset \mathbb{R}^2$,

$$f(x,y) = \big(u(x,y), v(x,y)\big).$$

Then u and v are harmonic real-valued functions in U and they satisfy the Cauchy-Riemann system of equations

$$\begin{cases} D_1 u = D_2 v \\ D_2 u = -D_1 v, \end{cases}$$

where $D_1 = \partial/\partial x,\ D_2 = \partial/\partial y$. For every $(x,y) \in U$, the differential $f'(x,y)\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map whose matrix (with respect to the standard basis of the plane) is

$$\begin{pmatrix} D_1 u & D_2 u \\ D_1 v & D_2 v \end{pmatrix} = \begin{pmatrix} D_1 u & D_2 u \\ -D_2 u & D_1 u \end{pmatrix}.$$

Hence

$$|f'(x,y)|^2 = \det f'(x,y). \tag{1.1}$$

The first trial definition for mappings $f\colon U \to \mathbb{R}^n$ of a domain $U \subset \mathbb{R}^n$, sharing some geometric and topological properties of holomorphic functions, could be mappings satisfying a condition

$$|f'(x)|^n = J_f(x), \quad x \in U. \tag{1.2}$$

However, it has turned out that, in dimensions $n \geq 3$, a mapping $f\colon U \to \mathbb{R}^n$ belonging to the Sobolev space $W^{1,n}_{\mathrm{loc}}(U;\mathbb{R}^n)$ and satisfying (1.2) for a.e. $x \in U$ is either constant or a restriction of a Möbius map. This is the so-called generalized Liouville theorem due to Gehring [12] and Reshetnyak [38]; see also the thorough discussion in [29].

Next candidate for the definition is obtained by replacing the equality (1.2) by a weaker condition

$$|f'(x)|^n \leq K\, J_f(x) \quad \text{a.e. } x \in U, \tag{1.3}$$

where $K \geq 1$ is a constant. Note that $J_f(x) \leq |f'(x)|^n$ holds for a.e. $x \in U$. Now there remains the question on the regularity assumption of such a mapping f. Again there is some rigidity in dimensions $n \geq 3$. Indeed, if a mapping f satisfying (1.3) is non-constant and smooth enough (more precisely, if $f \in C^k$, with $k = 2$ for $n \geq 4$ and $k = 3$ for $n = 3$), then f is a local homeomorphism. Furthermore, it then follows from a theorem of Zorich that such mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is necessarily a homeomorphism, for $n \geq 3$; see [47]. We would also like a class of maps satisfying (1.3), with fixed K, to be closed under local uniform convergence. In order to obtain a rich enough class of mappings, it is thus necessary to weaken the regularity assumption from C^k-smoothness. See [15], [5], and [32] for recent developments regarding smoothness and branching of quasiregular mappings.

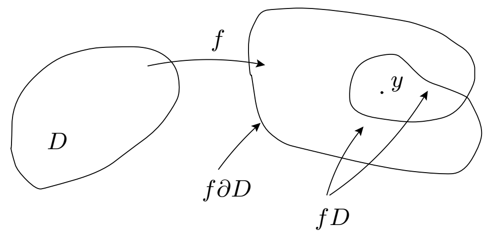
The basic analytic and topological properties of quasiregular mappings are listed in the following theorem by Reshetnyak; see [39], [41].

Theorem 1.2 (Reshetnyak's theorem)**.** *Let $U \subset \mathbb{R}^n$ be a domain and let $f\colon U \to \mathbb{R}^n$ be quasiregular. Then*

(1) *f is differentiable a.e. and*
(2) *f is either constant or it is discrete, open, and sense-preserving.*

Recall that a map $g\colon X \to Y$ between topological spaces X and Y is *discrete* if the preimage $g^{-1}(y)$ of every $y \in Y$ is a discrete subset of X and that g is *open* if gU is open for every open $U \subset X$. We also remark that a continuous discrete and open map $g\colon X \to Y$ is called a *branched covering.*

To say that $f\colon U \to \mathbb{R}^n$ is sense-preserving means that the local degree $\mu(y, f, D)$ is positive for all domains $D \Subset U$ and for all $y \in fD \setminus f\partial D$. The local degree is an integer that tells, roughly speaking, how many times f wraps D around y. It can be defined, for example, by using cohomology groups with compact support. For the basic properties of the local degree, we refer to [41, Proposition I.4.4]; see also [11], [35], and [45]. For example, if f is differentiable at x_0 with $J_f(x_0) \neq 0$, then $\mu(f(x_0), f, D) = \operatorname{sign} J_f(x_0)$ for sufficiently small connected neighborhoods D of x_0. Another useful property is the following homotopy invariance: If f and g are homotopic via a homotopy h_t, $h_0 = f$, $h_1 = g$, such that $y \in h_t D \setminus h_t \partial D$ for every $t \in [0,1]$, then $\mu(y, f, D) = \mu(y, g, D)$.

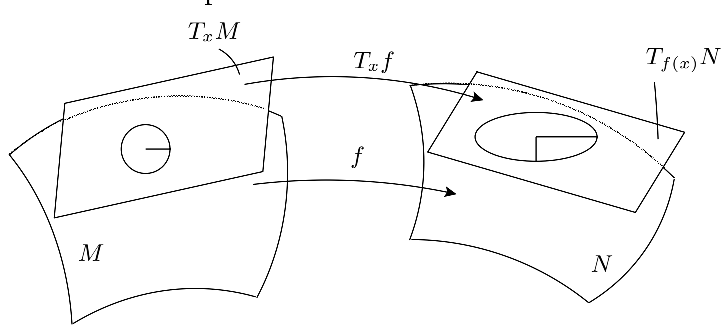


The definition of quasiregular mappings extends easily to the case of continuous mappings $f\colon M \to N$, where M and N are connected oriented Riemannian n-manifolds.

Definition 1.3. A continuous mapping $f\colon M \to N$ is quasiregular (or a mapping of bounded distortion) if it belongs to the Sobolev space $W^{1,n}_{\mathrm{loc}}(M; N)$ and there exists a constant $K \geq 1$ such that

$$|T_x f|^n \leq K J_f(x) \quad \text{for a.e. } x \in M. \tag{1.4}$$

Here again $T_x f\colon T_x M \to T_{f(x)} N$ is the formal differential (or the tangent map) of f at x, $|T_x f|$ is the operator norm of $T_x f$, and $J_f(x)$ is the Jacobian determinant of f at x uniquely defined by $(f^* \mathrm{vol}_N)_x = J(x, f)(\mathrm{vol}_M)_x$ almost everywhere. Note that $T_x f$ can be defined for a.e. x by using partial derivatives of local expressions of f at x. The geometric interpretation of (1.4) is that $T_x f$ maps balls of $T_x M$ either to ellipsoids with controlled ratios of the semi-axes or $T_x f$ is the constant linear map.



We assume from now on that M and N are connected oriented Riemannian n-manifolds.

2. $\mathcal{A}$-harmonic functions

It is well-known that the composition $u \circ f$ of a holomorphic function $f\colon U \to \mathbb{C}$ and a harmonic function $u\colon fU \to \mathbb{R}$ is a harmonic function in U. In other words, holomorphic functions are harmonic morphisms. Quasiregular mappings have a somewhat similar morphism property: If $f\colon U \to \mathbb{R}^n$ is quasiregular and u is an n-harmonic function in a neighborhood of fU, then $u \circ f$ is a so-called $\mathcal{A}$-harmonic function in U. In this section we introduce the notion of $\mathcal{A}$-harmonic functions and recall some of their basic properties that are relevant for this survey.

We denote by $\langle \cdot, \cdot \rangle$ the Riemannian metric of M. Recall that the gradient of a smooth function $u\colon M \to \mathbb{R}$ is the vector field ∇u such that

$$\langle \nabla u(x), h \rangle = du(x)h$$

for every $x \in M$ and $h \in T_x M$.

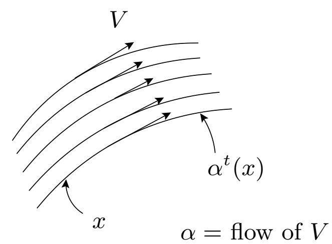
The divergence of a smooth vector field V can be defined as a function $\operatorname{div} V\colon M \to \mathbb{R}$ satisfying

$$\mathcal{L}_V \omega = (\operatorname{div} V)\, \omega,$$

where $\omega = \mathrm{vol}_M$ is the (Riemannian) volume form and

$$\mathcal{L}_V \omega = \lim_{t \to 0} \frac{(\alpha^t)^* \omega - \omega}{t}$$

is the Lie derivative of ω with respect to V, and α is the flow of V.



We say that a vector field $\nabla u \in L^1_{\mathrm{loc}}(M)$ is a *weak gradient* of $u \in L^1_{\mathrm{loc}}(M)$ if

$$\int_M \langle \nabla u, V \rangle = -\int_M u \operatorname{div} V \tag{2.1}$$

for all vector fields $V \in C_0^\infty(M)$. Conversely, a function $\operatorname{div} V \in L^1_{\mathrm{loc}}(M)$ is a weak divergence of a (locally integrable) vector field V if (2.1) holds for all $u \in C_0^\infty(M)$. Note that $\int_M \operatorname{div} Y = 0$ if Y is a smooth vector field in M with compact support.

We define the Sobolev space $W^{1,p}(M)$ and its norm as

$$W^{1,p}(M) = \{u \in L^p(M)\colon \text{the weak gradient } \nabla u \in L^p(M)\},\ 1 \le p < \infty,$$
$$\|u\|_{1,p} = \|u\|_p + \||\nabla u|\|_p.$$

Let $G \subset M$ be open. Suppose that for a.e. $x \in G$ we are given a continuous map

$$\mathcal{A}_x\colon T_x M \to T_x M$$

such that the map $x \mapsto \mathcal{A}_x(X)$ is a measurable vector field whenever X is. Suppose that there are constants $1 < p < \infty$ and $0 < \alpha \le \beta < \infty$ such that

$$\langle \mathcal{A}_x(h), h \rangle \ge \alpha |h|^p$$

and

$$|\mathcal{A}_x(h)| \le \beta |h|^{p-1}$$

for a.e. $x \in G$ and for all $h \in T_xM$. In addition, we assume that for a.e. $x \in G$

$$\langle \mathcal{A}_x(h) - \mathcal{A}_x(k), h - k\rangle > 0$$

whenever $h \neq k$, and

$$\mathcal{A}_x(\lambda h) = \lambda|\lambda|^{p-2}\mathcal{A}_x(h)$$

whenever $\lambda \in \mathbb{R} \setminus \{0\}$.

A function $u \in W^{1,p}_{\text{loc}}(G)$ is called a (weak) solution of the equation

$$-\operatorname{div} \mathcal{A}_x(\nabla u) = 0 \tag{2.2}$$

in G if

$$\int_G \langle \mathcal{A}_x(\nabla u), \nabla\varphi\rangle = 0$$

for all $\varphi \in C_0^\infty(G)$. Continuous solutions of (2.2) are called $\mathcal{A}$-*harmonic functions* (of type p). By the fundamental work of Serrin [43], every solution of (2.2) has a continuous representative. In the special case $\mathcal{A}_x(h) = |h|^{p-2}h$, $\mathcal{A}$-harmonic functions are called p-*harmonic* and, in particular, if $p = 2$, we obtain the usual harmonic functions. The conformally invariant case $p = n =$ the dimension of M is important in the sequel. In this case p-harmonic functions are called, of course, n-harmonic functions.

A function $u \in W^{1,p}_{\text{loc}}(G)$ is a *subsolution* of (2.2) in G if

$$-\operatorname{div} \mathcal{A}_x(\nabla u) \le 0$$

weakly in G, that is

$$\int_G \langle \mathcal{A}_x(\nabla u), \nabla\varphi\rangle \le 0$$

for all non-negative $\varphi \in C_0^\infty(G)$. A function v is called *supersolution* of (2.2) if $-v$ is a subsolution. The proofs of the following two basic estimates are straightforward once the appropriate test function is found. Therefore we just give the test function and leave the details to readers.

Lemma 2.1 (Caccioppoli inequality)**.** *Let u be a positive solution of* (2.2) *(for a given fixed p) in G and let $v = u^{q/p}$, where $q \in \mathbb{R} \setminus \{0, p-1\}$. Then*

$$\int_G \eta^p|\nabla v|^p \le \left(\frac{\beta|q|}{\alpha|q-p+1|}\right)^p \int_G v^p|\nabla\eta|^p \tag{2.3}$$

for every non-negative $\eta \in C_0^\infty(G)$.

Proof. Write $\kappa = q - p + 1$ and use $\varphi = u^\kappa\eta^p$ as a test function. ∎

Remark 2.2. In fact, the estimate (2.3) holds for positive supersolutions if $q < p-1,\ q \neq 0$, and for positive subsolutions if $q > p-1$.

The excluded case $q = 0$ above corresponds to the following logarithmic Caccioppoli inequality.

Lemma 2.3 (Logarithmic Caccioppoli inequality). *Let u be a positive supersolution of* (2.2) *(for a given fixed p) in G and let $C \subset G$ be compact. Then*

$$\int_C |\nabla \log u|^p \leq c \int_G |\nabla \eta|^p \tag{2.4}$$

for all $\eta \in C_0^\infty(G)$, with $\eta|C \geq 1$, where $c = c(p, \beta/\alpha)$.

Proof. Choose $\varphi = \eta^p u^{1-p}$ as a test function. ■

These two lemmas together with the Sobolev and Poincaré inequalities are used in proving Harnack's inequality for non-negative $\mathcal{A}$-harmonic functions by the familiar Moser iteration scheme. In the following $|A|$ denotes the volume of a measurable set $A \subset M$.

Theorem 2.4 (Harnack's inequality). *Let M be a complete Riemannian manifold and suppose that there are positive constants R_0, C, and $\tau \geq 1$ such that a volume doubling property*

$$|B(x, 2r)| \leq C\, |B(x, r)| \tag{2.5}$$

holds for all $x \in M$ and $0 < r \leq R_0$, and that M admits a weak $(1, p)$-Poincaré inequality

$$⨍_B |v - v_B| \leq C\, r \left(⨍_{\tau B} |\nabla v|^p \right)^{1/p} \tag{2.6}$$

for all balls $B = B(x, r) \subset M$, with $\tau B = B(x, \tau r)$ and $0 < r \leq R_0$, and for all functions $v \in C^\infty(B)$. Then there is a constant c such that

$$\sup_{B(x,r)} u \leq c \inf_{B(x,r)} u \tag{2.7}$$

whenever u is a non-negative $\mathcal{A}$-harmonic function in a ball $B(x, 2r)$, with $0 < r \leq R_0$.

In particular, if the volume doubling condition (2.5) and the Poincaré inequality (2.6) hold globally, that is, without any bound on the radius r, we obtain a global Harnack inequality. We refer to [18], [9], and [16] for proofs of the Harnack inequality.

3. Morphism property and its consequences

The very first step in developing the theory of quasiregular mappings is to prove, by direct computation, that quasiregular mappings have the morphism property in a special case where the n-harmonic function is smooth enough.

Theorem 3.1. *Let $f\colon M \to N$ be a quasiregular mapping (with a constant K) and let $u \in C^2(N)$ be n-harmonic. Then $v = u \circ f$ is $\mathcal{A}$-harmonic (of type n) in M, with*

$$\mathcal{A}_x(h) = \langle G_x h, h \rangle^{\frac{n}{2}-1} G_x h, \tag{3.1}$$

where $G_x\colon T_xM \to T_xM$ *is given by*

$$G_xh = \begin{cases} J_f(x)^{2/n} T_xf^{-1}(T_xf^{-1})^T h, & \textit{if } J_f(x) \textit{ exists and is positive,} \\ h, & \textit{otherwise.} \end{cases}$$

The constants α *and* β *for* $\mathcal{A}$ *depend only on* n *and* K.

Proof. Let us first write the proof formally and then discuss the steps in more detail. In the sequel ω stands for the volume forms in M and N. Let $V \in C^1(M)$ be the vector field $V = |\nabla u|^{n-2}\nabla u$. Since u is n-harmonic and C^2-smooth, we have $\operatorname{div} V = 0$. By Cartan's formula we obtain

$$d(V \lrcorner\, \omega) = d(V \lrcorner\, \omega) + V \lrcorner\, (d\omega) = \mathcal{L}_V \omega = (\operatorname{div} V)\, \omega = 0$$

since $d\omega = 0$. Here $X \lrcorner\, \eta$ is the contraction of a differential form η by a vector field X. Thus, for instance, $V \lrcorner\, \omega$ is the $(n-1)$-form

$$V \lrcorner\, \omega(\underbrace{\cdot, \ldots, \cdot}_{n-1}) = \omega(V, \underbrace{\cdot, \ldots, \cdot}_{n-1}).$$

Hence

$$df^*(V \lrcorner\, \omega) = f^* d(V \lrcorner\, \omega) \overset{\text{a.e.}}{=} 0. \tag{3.2}$$

On the other hand, we have a.e. in M

$$f^*(V \lrcorner\, \omega) = W \lrcorner\, f^*\omega = W \lrcorner\, (J_f \omega) = J_f W \lrcorner\, \omega, \tag{3.3}$$

where W is a vector field that will be specified later (roughly speaking, $f_* W = V$). We obtain

$$d(J_f W \lrcorner\, \omega) = 0, \tag{3.4}$$

or equivalently

$$\operatorname{div}(J_f W) = 0 \tag{3.5}$$

which can be written as

$$\operatorname{div} \mathcal{A}_x(\nabla v) = 0, \tag{3.6}$$

where $\mathcal{A}$ is as in the claim.

Some explanations are in order. When writing

$$f^* d(V \lrcorner\, \omega) \overset{\text{a.e.}}{=} 0,$$

we mean that for a.e. $x \in U$ and for all vectors $v_1, v_2, \ldots, v_n \in T_xM$

$$f^* d(V \lrcorner\, \omega)(v_1, v_2, \ldots, v_n) = d(V \lrcorner\, \omega)(f_* v_1, f_* v_2, \ldots, f_* v_n) = 0,$$

where $f_* = f_{*,x} = T_xf$ is the tangent mapping of f at x. The equality on the left-hand side of (3.2) holds in a weak sense since $f \in W^{1,n}_{\mathrm{loc}}(M)$; see [39, p. 136]. This means that, for all n-forms $\eta \in C_0^\infty(M)$,

$$\int_M \langle f^* d(V \lrcorner\, \omega), \eta \rangle = \int_M \langle f^*(V \lrcorner\, \omega), \delta\eta \rangle, \tag{3.7}$$

where δ is the codifferential. Consequently, equations (3.4)–(3.6) are to be interpreted in weak sense. In particular, combining (3.2), (3.3), and (3.7) we get

$$\int_M \langle J_f W \lrcorner\, \omega, \delta\eta\rangle = \int_M \langle f^*(V \lrcorner\, \omega), \delta\eta\rangle = \int_M \langle f^* d(V \lrcorner\, \omega), \eta\rangle = 0$$

for all n-forms $\eta \in C_0^\infty(M)$, and so (3.4) holds in weak sense.

Let us next specify the vector field W. Let $A = \{x \in M\colon J_f(x) = \det f_{*,x} \neq 0\}$. Hence $f_{*,x}$ is invertible for all $x \in A$, and $W = f_*^{-1}V$ in A. In $M \setminus A$, either $J_f(x)$ does not exist, which can happen only in a set of measure zero, or $J_f(x) \le 0$. Quasiregularity of f, more precisely the distortion condition (1.4), implies that $f_{*,x} = T_x f = 0$ for almost every such x. Hence $f_{*,x} = 0$ for a.e. $x \in M \setminus A$. Setting $W = 0$ in $M \setminus A$, we obtain

$$f^*(V \lrcorner\, \omega) = 0 = W \lrcorner\, f^*\omega$$

a.e. in $M \setminus A$. Hence $f^*(V \lrcorner\, \omega) = W \lrcorner\, f^*\omega$ a.e. in M, and so (3.3) holds. ∎

3.1. Sketch of the proof of Reshetnyak's theorem. We shall use Theorem 3.1 to sketch the proof of Reshetnyak's theorem in a way that uses analysis, in particular, $\mathcal{A}$-harmonic functions. First we recall some definitions concerning p-capacity. If $\Omega \subset M$ is an open set and $C \subset \Omega$ is compact, then the *p-capacity* of the pair (Ω, C) is

$$\operatorname{cap}_p(\Omega, C) = \inf_\varphi \int_\Omega |\nabla\varphi|^p, \tag{3.8}$$

where the infimum is taken over all functions $\varphi \in C_0^\infty(\Omega)$, with $\varphi|C \ge 1$. A compact set $C \subset M$ is of p-capacity zero, denoted by $\operatorname{cap}_p C = 0$, if $\operatorname{cap}_p(\Omega, C) = 0$ for all open sets $\Omega \supset C$. Finally, a closed set F is of p-capacity zero, denoted by $\operatorname{cap}_p F = 0$, if $\operatorname{cap}_p C = 0$ for all compact sets $C \subset F$. It is a well-known fact that a closed set $F \subset \mathbb{R}^n$ containing a continuum C cannot be of n-capacity zero. This can be seen by taking an open ball B containing C and any test function $\varphi \in C_0^\infty(B)$, with $\varphi|C = 1$, and using a potential estimate

$$|\varphi(x) - \varphi(y)| \le c\left(\int_B \frac{|\nabla\varphi|}{|x-z|^{n-1}}dz + \int_B \frac{|\nabla\varphi|}{|y-z|^{n-1}}dz\right),\ x, y \in B,$$

combined with a maximal function and covering arguments. Similarly, if C is a continuum in a domain Ω and B is an open ball, with $\bar B \subset \Omega \setminus C$, then $\operatorname{cap}_n(C, \bar B; \Omega) > 0$, where

$$\operatorname{cap}_n(C, \bar B; \Omega) = \inf_\varphi \int_\Omega |\nabla\varphi|^p > 0,$$

the infimum being taken over all functions $\varphi \in C^\infty(\Omega)$, with $\varphi|C = 1$ and $\varphi|\bar B = 0$.

f is light. Suppose that $U \subset \mathbb{R}^n$ is a domain and that $f\colon U \to \mathbb{R}^n$ is a non-constant quasiregular mapping. We will show first that f is *light* which means that, for all $y \in \mathbb{R}^n$, the preimage $f^{-1}(y)$ is totally disconnected, i.e. each component of $f^{-1}(y)$ is a point.

Fix $y \in \mathbb{R}^n$ and define $u\colon \mathbb{R}^n \setminus \{y\} \to \mathbb{R}$ by

$$u(x) = \log \frac{1}{|x-y|}.$$

Then u is C^∞ and n-harmonic in $\mathbb{R}^n \setminus \{y\}$ by a direct computation. By Theorem 3.1, $v = u \circ f$,

$$v(x) = \log \frac{1}{|f(x)-y|},$$

is $\mathcal{A}$-harmonic in an open non-empty set $U \setminus f^{-1}(y)$ and $v(x) \to +\infty$ as $x \to z \in f^{-1}(y)$. We set $v(z) = +\infty$ for $z \in f^{-1}(y)$.

To show that f is light we use the logarithmic Caccioppoli inequality (2.4). Suppose that $C \subset f^{-1}(y) \cap U$ is a continuum. Since f is non-constant and continuous, there exists $m > 1$ such that the set $\Omega = \{x \in U\colon v(x) > m\}$ is an open neighborhood of C and $\bar{\Omega} \subset U$. We choose another neighborhood D of C such that $\bar{D} \subset \Omega$ is compact. Now we observe that $v_i = \min\{v, i\}$ is a positive supersolution for all $i > m$. The logarithmic Caccioppoli inequality (2.4) then implies that

$$\int_D |\nabla \log v_i|^n \le c \operatorname{cap}_n(\Omega, \bar{D}) \le c < \infty$$

uniformly in i. Hence $|\nabla \log v| \in L^n(D)$. Choose an open ball B such that $\bar{B} \subset D \setminus f^{-1}(y)$. We observed earlier that

$$\operatorname{cap}_n(C, \bar{B}; D) > 0$$

since C is a continuum. Let

$$M_B = \max_{\bar{B}} \log v.$$

Now the idea is to use

$$\min\{1, \max\{0, \frac{1}{k} \log \frac{v}{M_B}\}\}$$

as a test function for $\operatorname{cap}_n(C, \bar{B}; D)$ for every $k \in \mathbb{N}$. We get a contradiction since

$$0 < \operatorname{cap}_n(C, \bar{B}; D) \le k^{-n} \|\nabla \log v\|_{L^p(D)} \to 0$$

as $k \to \infty$. Thus $f^{-1}(y)$ can not contain a continuum.

Differentiability a.e. Assume that $f = (f_1, \ldots, f_n)\colon U \to \mathbb{R}^n$, $U \subset \mathbb{R}^n$, is quasiregular. Then the coordinate functions f_j are $\mathcal{A}$-harmonic again by Theorem 3.1, since the functions $x = (x_1, \ldots, x_n) \mapsto x_j$ are n-harmonic. Now there are at least two ways to prove that f is differentiable almost everywhere. For instance, since each f_j is $\mathcal{A}$-harmonic, one can show by employing reverse Hölder inequality techniques that, in fact, $f \in W^{1,p}_{\mathrm{loc}}(U)$, with some $p > n$. This then implies that f is differentiable a.e. in U; see e.g. [3]. Another way is to conclude that f is monotone, i.e. each coordinate function f_j is monotone, and therefore differentiable a.e. since $f \in W^{1,n}_{\mathrm{loc}}(U)$; see [41]. The monotonicity of f_j holds since $\mathcal{A}$-harmonic functions obey the maximum principle.

f **is sense-preserving.** Here one first shows that conditions $f \in W^{1,n}_{\mathrm{loc}}(U)$ and $J_f(x) \geq 0$ a.e. imply that f is weakly sense-preserving, i.e. $\mu(y, f, D) \geq 0$ for all domains $D \Subset U$ and for all $y \in fD \setminus f\partial D$. This step employs approximation of f by smooth mappings. Pick then a domain $D \Subset U$ and $y \in fD \setminus f\partial D$. Denote by Y the y-component of $\mathbb{R}^n \setminus f\partial D$ and write $V = D \cap f^{-1}Y$. Since f is light, $D \setminus f^{-1}(y)$ is non-empty. Thus we can find a point $x_0 \in f^{-1}(y) \cap V$. Next we conclude that the set $\{x \in V : J_f(x) > 0\}$ has positive measure. Otherwise, since f is ACL and $|f'(x)| = 0$ a.e. in V, f would be constant in a ball centered at x_0 contradicting the fact that f is light. Thus there is a point x in V where f is differentiable and $J_f(x) > 0$. Now a homotopy argument, using the differential of f at z, and $\mu(y, f, D) \geq 0$ imply that f is sense-preserving.

f **is discrete and open.** This part of the proof is purely topological. A sense-preserving light mapping is discrete and open by Titus and Young; see e.g. [41].

Further properties of f. Once Reshetnyak's theorem is established it is possible to prove further properties for quasiregular mappings. We collect these properties to the following theorems and refer to the books [39] and [41] for the proofs.

Theorem 3.2. *Let $f\colon M \to N$ be a non-constant quasiregular map. Then*

1. $|fE| = 0$ *if and only if* $|E| = 0$.
2. $|B_f| = 0$, *where B_f is the branch set of f, i.e. the set of all $x \in M$ where f does not define a local homeomorphism.*
3. $J_f(x) > 0$ *a.e.*
4. *The integral transformation formula*
$$\int_A (h \circ f)(x) J_f(x) dm(x) = \int_N h(y) N(y, f, A) dm(y)$$
holds for every measurable $h\colon N \to [0, +\infty]$ and for every measurable $A \subset M$, where $N(y, f, A) = \operatorname{card} f^{-1}(y) \cap A$.
5. *If $u \in W^{1,n}_{loc}(N, \mathbb{R})$, then $v = u \circ f \in W^{1,n}_{loc}(M, \mathbb{R})$ and*
$$\nabla v(x) = T_x f^T \nabla u(f(x)) \ \textit{a.e.}$$

Furthermore, we have a generalization of the morphism property.

Theorem 3.3. *Let $f\colon M \to N$ be quasiregular and let $u\colon N \to \mathbb{R}$ be an $\mathcal{A}$-harmonic function (or a subsolution or a supersolution, respectively) of type n. Then $v = u \circ f$ is $f^{\#}\mathcal{A}$-harmonic (a subsolution or a supersolution, respectively), where*
$$f^{\#}\mathcal{A}_x(h) = \begin{cases} J_f(x) T_x f^{-1} \mathcal{A}_{f(x)}(T_x f^{-1})^T h), & \textit{if } J_f(x) \textit{ exists and is positive,} \\ |h|^{n-2} h, & \textit{otherwise.} \end{cases}$$

The ingredients of the proof of Theorem 3.3 include, for instance, the locality of $\mathcal{A}$-harmonicity, Theorem 3.2, and a method to "push-forward" (test) functions; see e.g. [16] and [41].

4. Modulus and capacity inequalities

Although the main emphasis of this survey is on the relation between quasiregular mappings and p-harmonic functions, we want to introduce also the other main tool in the theory of quasiregular mappings. Let $1 \leq p < \infty$ and let Γ be a family of paths in M. We denote by $\mathcal{F}(\Gamma)$ the set of all Borel functions $\varrho\colon M \to [0, +\infty]$ such that

$$\int_\gamma \varrho ds \geq 1$$

for all locally rectifiable path $\gamma \in \Gamma$. We call the functions in $\mathcal{F}(\Gamma)$ *admissible* for Γ. The *p-modulus of* Γ is defined by

$$M_p(\Gamma) = \inf_{\varrho \in \mathcal{F}(\Gamma)} \int_M \varrho^p dm.$$

There is a close connection between p-modulus and p-capacity. Indeed, suppose that $\Omega \subset M$ is open and $C \subset \Omega$ is compact. Let Γ be the family of all paths in $\Omega \setminus C$ connecting C and $\partial\Omega$. Then

$$\operatorname{cap}_p(\Omega, C) = M_p(\Gamma). \tag{4.1}$$

The inequality $\operatorname{cap}_p(\Omega, C) \geq M_p(\Gamma)$ follows easily since $\varrho = |\nabla\varphi|$ is admissible for Γ for each function φ as in (3.8). The other direction is harder and requires an approximation argument; see [41, Proposition II.10.2].

If $p = n =$ the dimension of M, we call $M_n(\Gamma)$ the *conformal modulus* of Γ, or simply the modulus of Γ. In that case $M_n(\Gamma)$ is invariant under conformal changes of the metric. In fact, $M_n(\Gamma)$ can be interpreted as follows: Define a new measurable Riemannian metric

$$\langle\langle \cdot, \cdot \rangle\rangle = \varrho^2 \langle \cdot, \cdot \rangle.$$

Then, with respect to $\langle\langle \cdot, \cdot \rangle\rangle$, the length of γ has a lower bound

$$\ell_{\langle\langle \cdot, \cdot \rangle\rangle}(\gamma) = \int_\gamma \varrho ds \geq 1$$

and the volume of M is given by

$$\operatorname{Vol}_{\langle\langle \cdot, \cdot \rangle\rangle}(M) = \int_M \varrho^n dm.$$

Thus we are minimizing the volume of M under the constraint that paths in Γ have length at least 1.

The importance of the conformal modulus for quasiregular mappings lies in the following invariance properties; see [41, II.2.4, II.8.1]

Theorem 4.1. *Let $f\colon M \to N$ be a non-constant quasiregular mapping. Let $A \subset M$ be a Borel set with $N(f,A) := \sup_y N(y,f,A) < \infty$, and let Γ be a family of paths in A. Then*

$$M_n(\Gamma) \le K_O(f)\, N(f,A)\, M_n(f\Gamma). \tag{4.2}$$

Theorem 4.2 (Poletsky's inequality)**.** *Let $f\colon M \to N$ be a non-constant quasiregular mapping and let Γ be a family of paths in M. Then*

$$M_n(f\Gamma) \le K_I(f)\, M_n(\Gamma). \tag{4.3}$$

The proof of (4.2) is based on the change of variable formula for integrals (Theorem 3.2 4.) and on Fuglede's theorem. The estimate (4.3) in the converse direction is more useful than (4.2) but also much harder to prove; see [41, p. 39–50].

As an application of the use of p-modulus and p-capacity, we prove a Harnack's inequality for positive $\mathcal{A}$-harmonic functions of type $p > n-1$. Assume that $\Omega \subset M$ is a domain, $D \Subset \Omega$ another domain, and $C \subset D$ is compact. For $p > n-1$, we set

$$\lambda_p(C,D) = \inf_{E,F} M_p(\Gamma(E,F;D)),$$

where E and F are continua joining C and $\Omega \setminus D$, and $\Gamma(E,F;D)$ is the family of all paths joining E and F in D.

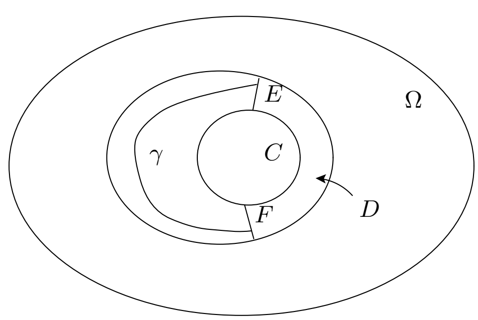
Theorem 4.3 (Harnack's inequality, $p > n-1$)**.** *Let Ω, D, and C be as above. Let u be a positive $\mathcal{A}$-harmonic function in Ω of type $p > n-1$. Then*

$$\log \frac{M_C}{m_C} \le c_0 \left(\frac{\operatorname{cap}_p(\Omega, \bar{D})}{\lambda_p(C,D)} \right)^{1/p}, \tag{4.4}$$

where

$$M_C = \max_{x\in C} u(x), \quad m_C = \min_{x \in C} u(x),$$

and $c_0 = c_0(p, \beta/\alpha)$.



Proof. We may assume that $M_C > m_C$. Let $\varepsilon > 0$ be so small that $M_C - \varepsilon > m_C + \varepsilon$. Then the sets $\{x\colon u(x) \ge M_C - \varepsilon\}$ and $\{x\colon u(x) \le m_C + \varepsilon\}$ contain continua E and F, respectively, that join C and $\Omega \setminus D$. Write

$$w = \frac{\log u - \log(m_C + \varepsilon)}{\log(M_C - \varepsilon) - \log(m_C + \varepsilon)}$$

and observe that $w \geq 1$ in E and $w \leq 0$ in F. Therefore $|\nabla w|$ is admissible for $\Gamma(E, F; D)$ and hence

$$\int_D |\nabla w|^p \geq M_p(\Gamma(E, F; D)) \geq \lambda_p(C, D).$$

On the other hand,

$$\int_D |\nabla \log u|^p dm \leq c(p, \beta/\alpha) \operatorname{cap}_p(\Omega, \bar{D})$$

by the logarithmic Caccioppoli inequality (2.4), and

$$\nabla \log u = \left(\log \frac{M_C - \varepsilon}{m_C + \varepsilon} \right) \nabla w.$$

Hence

$$\log \frac{M_C - \varepsilon}{m_C + \varepsilon} \leq c_0 \left(\frac{\operatorname{cap}_p(\Omega, \bar{D})}{\lambda_p(C, D)} \right)^{1/p}$$

and (4.4) follows by letting $\varepsilon \to 0$. ■

We can define $\lambda_p(C, D)$ analogously for $p \leq n-1$, too. However, $\lambda_p(C, D)$ vanishes for $p \leq n-1$. Consequently, Theorem 4.3 is useful only for $p > n-1$. The idea of the proof is basically due to Granlund [13]. In the above form, (4.4) appeared first time in [17]. In general, it is difficult to obtain an effective lower bound for $\lambda_p(C, D)$ together with an upper bound for $\operatorname{cap}_p(\Omega, \bar{D})$. However, if $M = \mathbb{R}^n$ and $p = n$, one obtains a global Harnack inequality by choosing C, D, and Ω as concentric balls.

5. Liouville-type results for $\mathcal{A}$-harmonic functions

We say that M is *strong p-Liouville* if M does not support non-constant positive $\mathcal{A}$-harmonic functions for any $\mathcal{A}$ of type p. We have already mentioned that a global Harnack inequality

$$\max_{B(x,r)} u \leq c \min_{B(x,r)} u$$

holds for non-negative $\mathcal{A}$-harmonic functions on $B(x, 2r)$ with a (Harnack-)constant c independent of x, r, and u if M is complete and admits a global volume doubling condition and a $(1, p)$-Poincaré inequality. It follows from the global Harnack inequality that such manifold M is strong p-Liouville.

Example 5.1. 1. Let M be complete with non-negative Ricci curvature. Then it is well-known that M admits a global volume doubling property by the Bishop-Gromov comparison theorem (see [2], [8]). Furthermore, Buser's isoperimetric inequality [6] implies that M also admits a $(1, p)$-Poincaré inequality for every $p \geq 1$. Hence M is strong p-Liouville.

2. Let $\mathbb{H}_n$ be the Heisenberg group. We write elements of $\mathbb{H}_n$ as (z,t), where $z=(z_1,\dots,z_n)\in\mathbb{C}^n$ and $t\in\mathbb{R}$. Furthermore, we assume that $\mathbb{H}_n$ is equipped with a left-invariant *Riemannian* metric in which the vector fields
$$X_j=\frac{\partial}{\partial x_j}+2y_j\frac{\partial}{\partial t},$$
$$Y_j=\frac{\partial}{\partial y_j}-2x_j\frac{\partial}{\partial t},$$
$$T=\frac{\partial}{\partial t},$$
$j=1,\dots,n$, form an orthonormal frame. Harnack's inequality for non-negative $\mathcal{A}$-harmonic functions on $\mathbb{H}_n$ was proved in [18] by using Jerison's version of Poincaré's inequality. Jerison proved in [31] that (1,1)-Poincaré's inequality holds for the horizontal gradient
$$\nabla_0 u=\sum_{j=1}^{n}\left((X_ju)X_j+(Y_ju)Y_j\right)$$
and for balls in the so-called Carnot-Carathéodory metric. Since the L^p-norm of the Riemannian gradient is larger than that of the horizonal gradient, we have (1,1)-Poincaré's inequality for the Riemannian gradient as well if geodesic balls are replaced by Carnot-Carathéodory balls or Heisenberg balls $B_H(r)=\{(z,t)\in\mathbb{H}_n\colon(|z|^4+t^2)^{1/4}<r\}$ and their left-translations.

Classically, a Riemannian manifold M is called parabolic if it does not support a positive Green's function for the Laplace equation.

Definition 5.2. We say that a Riemannian manifold M is p-parabolic, with $1<p<\infty$, if
$$\operatorname{cap}_p(M,C)=0$$
for all compact sets $C\subset M$. Otherwise, we say that M is p-hyperbolic.

Example 5.3. 1. A compact Riemannian manifold is p-parabolic for all $p\geq 1$.

2. In the Euclidean space $\mathbb{R}^n$ we have precise formulas for p-capacities of balls:
$$\operatorname{cap}_p(\mathbb{R}^n,\bar{B}(r))=\begin{cases}c\,r^{n-p}, & \text{if } 1\leq p<n,\\ 0, & \text{otherwise.}\end{cases}$$
Hence $\mathbb{R}^n$ is p-parabolic if and only if $p\geq n$.
3. If the Heisenberg group $\mathbb{H}_n$ is equipped with the left-invariant Riemannian metric, we do not have precise formulas for capacities of balls. However, for $r\geq 1$,
$$\operatorname{cap}_p(\mathbb{H}_n,\bar{B}_H(r))\approx r^{2n+2-p}$$
if $1\leq p<2n+2$, and $\operatorname{cap}_p(\mathbb{H}_n,\bar{B}_H(r))=0$ if $p\geq 2n+2$. Hence $\mathbb{H}_n$ is p-parabolic if and only if $p\geq 2n+2$.

4. Any complete Riemannian manifold M with finite volume $\mathrm{Vol}(M) < \infty$ is p-parabolic for all $p \geq 1$. This is easily seen by fixing a point $o \in M$ and taking a function $\varphi \in C_0^\infty\big(B(o,R)\big)$, with $\varphi|\bar{B}(o,r) = 1$ and $|\nabla\varphi| \leq c/(R-r)$. We obtain an estimate
$$\mathrm{cap}_p\big(B(o,R), \bar{B}(o,r)\big) \leq c\,\mathrm{Vol}(M)/(R-r)^p \to 0$$
as $R \to \infty$.
5. Let M^n be a Cartan-Hadamard n-manifold, i.e. a complete, simply connected Riemannian manifold of non-positive sectional curvatures and dimension n. If sectional curvatures have a negative upper bound $K_M \leq -a^2 < 0$, then M is p-hyperbolic for all $p \geq 1$. This follows since M^n satisfies an isoperimetric inequality
$$\mathrm{Vol}(D) \leq \frac{a}{n-1}\,\mathrm{Area}(\partial D)$$
for all domains $D \Subset M$, with smooth boundary; see [46], [7]. Another proof uses the Laplace comparison and Green's formula. If $p > 1$, then $v(x) = \exp(-\delta d(x,o))$ is a positive supersolution of the p-Laplace equation for some $\delta = \delta(n,p) > 0$ (see [20]). Hence the p-hyperbolicity of M also follows from the theorem below for $p > 1$.

Theorem 5.4. *Let M be a Riemannian manifold and $1 < p < \infty$. Then the following conditions are equivalent:*

1. *M is p-parabolic.*
2. *$M_p(\Gamma_\infty) = 0$, where Γ_∞ is the family of all paths $\gamma\colon [0,\infty) \to M$ such that $\gamma(t) \to \infty$ as $t \to \infty$.*
3. *Every non-negative supersolution of*
$$-\operatorname{div}\mathcal{A}_x(\nabla u) = 0 \qquad (5.1)$$
on M is constant for all $\mathcal{A}$ of type p.
4. *M does not support a positive Green's function $g(\cdot,y)$ for (5.1) for any $\mathcal{A}$ of type p and $y \in M$.*

Here $\gamma(t) \to \infty$ means that $\gamma(t)$ eventually leaves any compact set. For the proof of Theorem 5.4 as well as for the discussion below we refer to [17].

Let us explain what a Green's function for (5.1) is. We define it first in a "regular" domain $\Omega \Subset M$, where regular means that the Dirichlet problem for the $\mathcal{A}$-harmonic equation is solvable with continuous boundary data. For this notion, see [16]. We need a concept of $\mathcal{A}$-capacity. Let $C \subset \Omega$ be compact, and assume for simplicity that $\Omega \setminus C$ is regular. Thus there exists a unique $\mathcal{A}$-harmonic function in $\Omega \setminus C$ with continuous boundary values $u = 0$ on $\partial\Omega$ and $u = 1$ in C. Call u the $\mathcal{A}$-potential of (Ω, C). We define
$$\mathrm{cap}_{\mathcal{A}}(\Omega, C) = \int_\Omega \langle \mathcal{A}_x(\nabla u), \nabla u\rangle.$$
Then
$$\mathrm{cap}_{\mathcal{A}}(\Omega, C) \approx \mathrm{cap}_p(\Omega, C)$$

and furthermore,

$$\operatorname{cap}_{\mathcal{A}}(\Omega_1, C_1) \geq \operatorname{cap}_{\mathcal{A}}(\Omega_2, C_2) \tag{5.2}$$

if $C_2 \subset C_1$ and/or $\Omega_1 \subset \Omega_2$. Note that this property is obvious for variational capacities but $\operatorname{cap}_{\mathcal{A}}$ is not necessary a variational capacity.

The definition of Green's function, and in particular its uniqueness when $p = n$, relies on the following observation.

Lemma 5.5. *Let $\Omega \Subset M$ be a domain and let $C \subset \Omega$ be compact such that $\Omega \setminus C$ is regular. Let u be the $\mathcal{A}$-potential of (Ω, C). Then, for every $0 \leq a < b \leq 1$,*

$$\operatorname{cap}_{\mathcal{A}}(\{u > a\}, \{u \geq b\}) = \frac{\operatorname{cap}_{\mathcal{A}}(\Omega, C)}{(b-a)^{p-1}}.$$

Definition 5.6. Suppose that $\Omega \Subset M$ is a regular domain and let $y \in \Omega$. A function $g = g(\cdot, y)$ is called a *Green's function for (5.1) in* Ω if

1. g is positive and $\mathcal{A}$-harmonic in $\Omega \setminus \{y\}$,
2. $\lim_{x \to z} g(x) = 0$ for all $z \in \partial\Omega$,
3. $$\lim_{x \to y} g(x) = \operatorname{cap}_{\mathcal{A}}(\Omega, \{y\})^{1/(1-p)},$$
   which we interpret to mean $\lim_{x \to y} g(x) = \infty$ if $p \leq n$,
4. for all $0 \leq a < b < \operatorname{cap}_{\mathcal{A}}(\Omega, \{y\})^{1/(1-p)}$,
   $$\operatorname{cap}_{\mathcal{A}}(\{g > a\}, \{g \geq b\}) = (b-a)^{1/(1-p)}.$$

Theorem 5.7. *Let $\Omega \Subset M$ be a regular domain and $y \in \Omega$. Then there exists a Green's function for (5.1) in Ω. Furthermore, it is unique at least if $p \geq n$.*

Monotonicity properties (5.2) of $\mathcal{A}$-capacity and the so-called Loewner property, i.e. $\operatorname{cap}_n C > 0$ if C is a continuum, are crucial in proving the uniqueness when $p = n$. Indeed, we can show that on sufficiently small spheres $S(y, r)$

$$|g(x, y) - \operatorname{cap}_{\mathcal{A}}(\Omega, \bar{B}(y, r))^{1/(1-n)}| \leq c, \quad x \in S(y, r),$$

which then easily implies the uniqueness.

Next take an exhaustion of M by regular domains $\Omega_i \subset \Omega_{i+1} \Subset M$, $M = \cup_i \Omega_i$. We can construct an increasing sequence of Green's functions $g_i(\cdot, y)$ on Ω_i. Then the limit is either identically $+\infty$ or

$$g(\cdot, y) := \lim_{i \to \infty} g_i(\cdot, y)$$

is a positive $\mathcal{A}$-harmonic function on $M \setminus \{y\}$. In the latter case we call the limit function $g(\cdot, y)$ a Green's function for (5.1) on M.

We have the following list of Liouville-type properties of M (which may or may not hold for M):

(1) M is p-parabolic.
(2) Every non-negative $\mathcal{A}$-harmonic function on M is constant for every $\mathcal{A}$ of type p. (*Strong p-Liouville.*)

(3) Every bounded $\mathcal{A}$-harmonic function on M is constant for every $\mathcal{A}$ of type p. (*p-Liouville.*)
(4) Every $\mathcal{A}$-harmonic function u on M with $\nabla u \in L^p(M)$ is constant for every $\mathcal{A}$ of type p. (*D_p-Liouville.*)

We refer to [17] for the proof of the following general result, and to [18] and [25] for studies concerning the converse directions.

Theorem 5.8.

$$(1) \;\Rightarrow\; (2) \;\Rightarrow\; (3) \;\Rightarrow\; (4).$$

Next we discuss the close connection between the volume growth and p-parabolicity. Suppose that M is complete. Fix a point $o \in M$ and write $V(t) = \operatorname{Vol}\bigl(B(o,t)\bigr)$.

Theorem 5.9. *Let $1 < p < \infty$ and suppose that*

$$\int^{\infty} \left(\frac{t}{V(t)}\right)^{1/(p-1)} dt = \infty,$$

or

$$\int^{\infty} \frac{dt}{V'(t)^{1/(p-1)}} = \infty.$$

Then M is p-parabolic.

Proof. One can either construct a test function involving the integrals above, or use a p-modulus estimate for separating (spherical) rings. More precisely, write $B(t) = B(o,t)$ and $S(t) = S(o,t) = \partial B(o,t)$. For $R > r > 0$ and integers $k \geq 1$, we write $t_i = r + i(R-r)/k,\ i = 0, 1, \ldots, k$. Then, by a well-known property of the modulus,

$$M_p\left(\Gamma\bigl(S(r), S(R); \bar{B}(R)\bigr)\right)^{1/(1-p)} \geq \sum_{i=0}^{k-1} M_p\left(\Gamma\left(S(t_i), S(t_{i+1}); \bar{B}(t_{i+1})\right)\right)^{1/(1-p)};$$

see e.g. [41, II.1.5]. Here $\Gamma\bigl(S(r), S(R); \bar{B}(R)\bigr)$ is the family of all paths joining $S(r)$ and $S(R)$ in $\bar{B}(R)$. For each $i = 0, \ldots, k-1$ we have an estimate

$$M_p\left(\Gamma\left(S(t_i), S(t_{i+1}); \bar{B}(t_{i+1})\right)\right) \leq \left(V(t_{i+1}) - V(t_i)\right)\left(t_{i+1} - t_i\right)^{-p}.$$

Hence
(5.3)

$$M_p\bigl(\Gamma\bigl(S(r), S(R); \bar{B}(R)\bigr)\bigr)^{1/(1-p)} \geq \sum_{i=0}^{k-1} \left(\frac{V(t_{i+1}) - V(t_i)}{t_{i+1} - t_i}\right)^{1/(1-p)} (t_{i+1} - t_i).$$

Thus the right-hand side of (5.3) tends to the integral

$$\int_r^R \frac{dt}{V'(t)^{1/(p-1)}}$$

as $k \to \infty$. We obtain an estimate

$$M_p\left(\Gamma\left(S(r), S(R); \bar{B}(R)\right)\right) \leq \left(\int_r^R \frac{dt}{V'(t)^{1/(p-1)}}\right)^{1-p}.$$

In particular, if

$$\int_r^\infty \frac{dt}{V'(t)^{1/(p-1)}} = \infty$$

for some $r > 0$, then M is p-parabolic. ■

The converse is not true in general. That is, M can be p-parabolic even if

$$\int^\infty \left(\frac{t}{V(t)}\right)^{1/(p-1)} dt < \infty$$

or

$$\int^\infty \frac{dt}{V'(t)^{1/(p-1)}} < \infty;$$

see [44].

It is interesting to study under which conditions the converse is true. We refer to [19] for the proofs of the following two theorems.

Theorem 5.10. *Suppose that M is complete and admits a global doubling property and global $(1, p)$-Poincaré inequality for $1 < p < \infty$. Then*

$$\text{(5.4)} \qquad M \textit{ is } p\textit{-hyperbolic if and only if } \int^\infty \left(\frac{t}{V(t)}\right)^{1/(p-1)} dt < \infty.$$

In some cases, we can estimate Green's functions:

Theorem 5.11. *Suppose that M is complete and has non-negative Ricci curvature everywhere. Let $1 < p < \infty$. Then*

$$M \textit{ is } p\textit{-hyperbolic if and only if } \int^\infty \left(\frac{t}{V(t)}\right)^{1/(p-1)} dt < \infty.$$

Furthermore, we have estimates for Green's functions for (5.1)

$$c^{-1} \int_{2r}^\infty \left(\frac{t}{V(t)}\right)^{1/(p-1)} dt \le g(x, o) \le c \int_{2r}^\infty \left(\frac{t}{V(t)}\right)^{1/(p-1)} dt$$

for every $x \in \partial M(r)$, where $M(r)$ is the union of all unbounded components of $M \setminus \bar{B}(o, r)$. The constant c depends only on n, p, α, and β.

Theorem 5.10 follows also from the following sharper result; see [21].

Theorem 5.12. *Suppose that M is complete and that there exists a geodesic ray $\gamma\colon [0, \infty) \to M$ such that for all $t > 0$,*

$$|B(\gamma(t), 2s)| \le c|B(\gamma(t), s)|,$$

whenever $0 < s \le t/4$, and that

$$\fint_{B_{\gamma(t)}} |u - u_{B_{\gamma(t)}}| dm \le c \left(\fint_{2B_{\gamma(t)}} |\nabla u|^p dm \right)^{1/p}$$

for all $u \in C^\infty\big(2B_\gamma(t)\big)$, *where* $B_\gamma(t) = B(\gamma(t), t/8)$. *Then* M *is* p*-hyperbolic if*

$$\int^\infty \left(\frac{t}{|B(\gamma(t), t/4)|}\right)^{1/(p-1)} dt < \infty.$$

Theorem 5.12 can be applied to obtain the following.

Theorem 5.13. *Let* M *be a complete Riemannian* n*-manifold whose Ricci curvature is non-negative outside a compact set. Suppose that* M *has maximal volume growth* ($V(t) \approx r^n$). *Then* M *is* p*-parabolic if and only if* $p \geq n$.

To our knowledge it is an open problem whether the equivalence (5.4) holds for a complete Riemannian n-manifold whose Ricci curvature is non-negative outside a compact set.

6. Liouville-type results for quasiregular mappings

Here we give applications of the above results on n-parabolicity and various Liouville properties to the existence of non-constant quasiregular mappings between given Riemannian manifolds.

Let us start with the Gromov-Zorich "global homeomorphism theorem" that is a generalization of Zorich's theorem, which we mentioned in the introduction; see [14], [48].

Theorem 6.1. *Suppose that* M *is* n*-parabolic,* $n = \dim M \geq 3$, *and that* N *is simply connected. Let* $f\colon M \to N$ *be a locally homeomorphic quasiregular map. Then* f *is injective and* fM *is* n*-parabolic.*

Proof. We give here a very rough idea of the proof. First one observes that fM is n-parabolic (see Theorem 6.2 below), and so $N \setminus fM$ is of n-capacity zero. Then one shows, again by using the n-parabolicity of M, that the set E of all asymptotic limits of f is of zero capacity. Consequently, E is of Hausdorff dimension zero. Recall that an asymptotic limit of f is a point $y \in N$ such that $f\big(\gamma(t)\big) \to y$ as $t \to \infty$ for some path $\gamma \in \Gamma_\infty$ in M. Removing $E \cup (N \setminus FM)$ from N has no effect on the simply connectivity for dimensions $n \geq 3$. That is, $fM \setminus E$ remains simply connected. Thus one can extend uniquely any branch of local inverses of f and obtain a homeomorphism $g\colon fM \setminus E \to g(fM \setminus E)$ such that $f \circ g = \mathrm{id}\,|(fM \setminus E)$. Finally, g can be extended to E to obtain the inverse of f. ∎

In [22] we generalized the global homeomorphism theorem for mappings of finite distortion under mild conditions on the distortion. See also [49] for a related result for locally quasiconformal mappings.

Theorem 6.2. *If* N *is* n*-hyperbolic and* M *is* n*-parabolic, then every quasiregular mapping* $f\colon M \to N$ *is constant.*

Proof. Suppose that $f\colon M \to N$ is a non-constant quasiregular mapping. Then $fM \subset N$ is open. If $fM \neq N$, pick a point $y \in \partial(N \setminus fM)$ and let $g = g(\cdot, y)$ be the Green's function on N for the n-Laplacian. Then $g \circ f$ is a non-constant positive $\mathcal{A}$-harmonic function on M which gives a contradiction with the n-parabolicity of M and Theorem 5.8. If $fM = N$, let u be a non-constant positive supersolution on N for the n-Laplacian. Then $u \circ f$ is a non-constant supersolution on M for some $\mathcal{A}$ of type n which is again a contradiction. ∎

Example 6.3.
1. If N is a Cartan-Hadamard manifold, with $K_N \leq -a^2 < 0$, then every quasiregular mapping $f\colon \mathbb{R}^n \to N$ is constant.
2. Let $\mathbb{H}_n$ be the Heisenberg group with a left-invariant Riemannian metric, then every quasiregular mapping $f\colon \mathbb{R}^{2n+1} \to \mathbb{H}_n$ is constant.

Theorem 6.4. *Suppose that M is strong n-Liouville while N is not. Then every quasiregular map $f\colon M \to N$ is constant.*

Proof. If N is not strong n-Liouville, then it is n-hyperbolic by Theorem 5.8. Suppose that $f\colon M \to N$ is a non-constant quasiregular mapping. Then $fM \subset N$ is open. If $fM \neq N$, choose a point $y \in \partial(N \setminus fM)$ and let $g = g(\cdot, y)$ be the Green's function for the n-Laplacian on N. Then $g \circ f$ is a non-constant positive $\mathcal{A}$-harmonic function, with $\mathcal{A}$ of type n. This is a contradiction. If $fM = N$, we choose a non-constant positive n-harmonic function u on N and get a contradiction as above. ∎

Theorem 6.5. *Let N be a Cartan-Hadamard n-manifold, with $-b^2 \leq K \leq -a^2 < 0$, and let M be a complete Riemannian n-manifold admitting a global doubling property and a global $(1, n)$-Poincaré inequality. Then every quasiregular mapping $f\colon M \to N$ is constant.*

Proof. By [20], N admits non-constant positive n-harmonic functions. Hence N is not strong n-Liouville. On the other hand, the assumptions on M imply that a global Harnack's inequality for positive $\mathcal{A}$-harmonic functions of type n holds on M. Thus M is strong n-Liouville, and the claim follows from Theorem 6.4. ∎

Theorem 6.6 ("One-point Picard")**.** *Suppose that N is n-hyperbolic and M is strong n-Liouville. Then every quasiregular mapping $f\colon M \to N \setminus \{y\}$, with $y \in N$, is constant.*

Proof. Suppose that $f\colon M \to N \setminus \{y\}$ is a non-constant quasiregular mapping. Then $\partial(N \setminus fM) \neq \emptyset$. Choose a point $z \in \partial(N \setminus fM)$, and let $g = g(\cdot, z)$ be the Green's function on N for the n-Laplacian. Then $g \circ f$ is a non-constant positive $\mathcal{A}$-harmonic function for some $\mathcal{A}$ of type n leading to a contradiction. ∎

7. Picard-type theorems

The classical big Picard theorem states that a holomorphic mapping of the punctured unit disc $\{z \in \mathbb{C}\colon 0 < |z| < 1\}$ into the complex plane omitting two values has a meromorphic extension to the whole disc; see e.g. [1, Theorem 1-14]. In [40] Rickman proved a counterpart of Picard's theorem for quasiregular mappings (Theorem 7.1) and its local version (Theorem 7.2) corresponding to the big Picard theorem.

Theorem 7.1 ([40]). *For each integer $n \geq 2$ and each $K \geq 1$ there exists a positive integer $q = q(n, K)$ such that if $f\colon \mathbb{R}^n \to \mathbb{R}^n \setminus \{a_1, \ldots, a_q\}$ is K-quasiregular and $a_1, \ldots, a_q$ are distinct points in $\mathbb{R}^n$, then f is constant.*

Theorem 7.2 ([40]). *Let $G = \{x \in \mathbb{R}^n \colon |x| > s\}$ and let $f\colon G \to \mathbb{R}^n \setminus \{a_1, \ldots, a_q\}$ be a K-quasiregular mapping, where $a_1, \ldots, a_q$ are distinct points in $\mathbb{R}^n$ and $q = q(n, K)$ is the integer in Theorem 7.1. Then the limit $\lim_{|x|\to\infty} f(x)$ exists.*

In this section we consider corollaries and extensions of the big Picard theorem for quasiregular mappings.

Although the short argument yielding Theorem 7.1 from Theorem 7.2 is well-known, it seems that the following corollary employing the same argument has gone unnoticed in the literature.

Corollary 7.3. *Let $K \geq 1$ and $R > 0$. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping omitting at least $q = q(n, K)$ points, where $q(n, K)$ is as in Theorem 7.1. Then at least one of the following conditions fails:*

(i) *$f|\mathbb{R}^n \setminus \bar{B}^n(R)$ is K-quasiregular,*
(ii) *$fB^n(r)$ is open for some $r > R$.*

Proof. Suppose towards a contradiction that both conditions (i) and (ii) hold. By Theorem 7.2, the mapping f has a limit at the infinity. Hence we may extend f to a continuous mapping $\bar{\mathbb{R}}^n \to \bar{\mathbb{R}}^n$. Moreover, f is K-quasiregular in $\bar{\mathbb{R}}^n \setminus \bar{B}^n(R)$. By composing f with a Möbius mapping if necessary, we may assume that $f(\infty) = \infty$. Since f is a non-constant quasiregular mapping on $\bar{\mathbb{R}}^n \setminus \bar{B}^n(R)$, $f|\bar{\mathbb{R}}^n \setminus \bar{B}^n(R)$ is an open mapping. Hence $f\bar{\mathbb{R}}^n$ is open in $\bar{\mathbb{R}}^n$, by (ii). Since $f\bar{\mathbb{R}}^n$ is both open and closed, $f\bar{\mathbb{R}}^n = \bar{\mathbb{R}}^n$ and $f\mathbb{R}^n = \mathbb{R}^n$. This contradicts the assumption that f omits q points. The claim follows. ■

In [23] the authors consider quasiregular mappings of the punctured unit ball into a Riemannian manifold N. We say that N has at least q ends, if there exists a compact set $C \subset N$ such that $N \setminus C$ has at least q components which are not relatively compact. Such a component of $N \setminus C$ is called an *end of M with respect to C*. Let $\mathcal{E}$ be the set of ends of N, that is, $E \in \mathcal{E}$ is an end of N with respect to some compact set $C \subset N$. We compactify N with respect to its ends as follows. There is a natural partial order in $\mathcal{E}$ induced by inclusion. We call a maximal totally ordered subset of $\mathcal{E}$ *an asymptotic end of N*. The set of asymptotic ends of N is denoted by ∂N and $\hat{N} = N \cup \partial N$. We endow $\hat{N}$ with a topology such that the inclusion $N \subset \hat{N}$ is an embedding and for every $e \in \partial N$ sets $E \in e$ form a neighborhood basis at e. The main result is the following version of the big Picard theorem.

Theorem 7.4 ([23, Theorem 1.3]). *For every $K \geq 1$ there exists $q = q(K, n)$ such that every K-quasiregular mapping $f\colon B^n \setminus \{0\} \to N$ has a limit $\lim_{x\to 0} f(x)$ in $\hat{N}$ if N has at least q ends.*

In the spirit of Corollary 7.3 we formulate the following consequence Theorem 7.4.

Corollary 7.5. *Given $n \geq 2$ and $K \geq 1$ there exists $\tilde{q} = \tilde{q}(n, K)$ such that the following holds. Suppose that M is compact, $\{z_1, \ldots, z_k\} \subset M$, where $1 \leq k < \tilde{q}$, and that N has at least $\tilde{q}$ ends. Let $f\colon M \setminus \{z_1, \ldots, z_k\} \to N$ be a continuous mapping and let Ω_i be a neighborhood of z_i for every $1 \leq i \leq k$. Then at least one of the following conditions fails:*

(i) *f is K-quasiregular in $\Omega_i \setminus \{z_i\}$ for every i,*
(ii) *there exists a neighborhood Ω of $M \setminus (\Omega_1 \cup \cdots \cup \Omega_k)$ such that $f\Omega$ is open.*

Proof. Suppose that both conditions are satisfied. For every $1 \leq i \leq k$ we fix a 2-bilipschitz chart $\varphi_i\colon U_i \to \varphi_i U_i$ at z_i. We may assume that $U_i \subset \Omega_i$. Every mapping $f \circ \varphi_i^{-1}|\varphi_i(U_i \setminus z_i)$ is $2^n K$-quasiregular, and therefore it has a limit at $\varphi_i(z_i)$ by Theorem 7.4 if N has at least $q(n, 2^n K)$ ends. Hence f has a limit at every point z_i. We extend f to a continuous mapping $\hat{f}\colon M \to \hat{N}$. Denote $M' = M \setminus \{z_1, \ldots, z_k\}$. Since f is an open mapping, $\partial f M' \cap f M' = \emptyset$. Furthermore, since M is compact,

$$\overline{fM'} \subset \overline{\hat{f}M} = \hat{f}M.$$

Hence $\partial f M' \subset \hat{f}M \setminus fM'$. Thus $\operatorname{card}(\partial f M') \leq \operatorname{card}(\hat{f}M \setminus fM') \leq k$ and

$$\hat{N} = \overline{N} = \overline{fM'} = \hat{f}M.$$

This is a contradiction, since

$$\operatorname{card}(\hat{f}M \setminus fM') \leq k < q \leq \operatorname{card}(\hat{N} \setminus fM') = \operatorname{card}(\hat{f}M \setminus fM').$$

■

In [26] Holopainen and Rickman applied a method of Lewis ([33]) that relies on Harnack's inequality to prove the following general version of Picard's theorem on the number of omitted values of a quasiregular mapping. We say that a complete Riemannian n-manifold M belongs to the class $\mathcal{M}(m, \vartheta)$, where $m\colon (0,1) \to \mathbb{N}$ and $\vartheta\colon (0, \infty) \to (0, \infty)$ are given functions, if the following two conditions hold:

(m) for each $0 < \lambda < 1$ every ball of radius r in M can contain at most $m(\lambda)$ disjoint balls of radius λr, and
(ϑ) M admits a global Harnack's inequality for non-negative $\mathcal{A}$-harmonic functions of type n with Harnack-constant $\vartheta(\beta/\alpha)$, where α and β are the constants of $\mathcal{A}$.

Theorem 7.6 ([26]). *Given $n \geq 2$, $K \geq 1$, $m\colon (0,1) \to \mathbb{N}$, and $\vartheta\colon (0, \infty) \to (0, \infty)$ there exists $q = q(n, K, m, \vartheta) \geq 2$ such that the following holds. Suppose that M belongs to the class $\mathcal{M}(m, \vartheta)$ and that N has at least q ends. Then every K-quasiregular mapping $f\colon M \to N$ is constant.*

Next we show that this theorem admits a local version. Suppose that M is complete. We say that an asymptotic end e of M is of type $\mathcal{E}(m,\vartheta)$ if there exists $E \in e$ such that

($\mathcal{E}m$) for each $0 < \lambda < 1$ every ball of radius r in E can contain at most $m(\lambda)$ disjoint balls of radius λr, and

($\mathcal{E}\vartheta$) E admits a uniform Harnack inequality for non-negative $\mathcal{A}$-harmonic functions of E of type n for balls $B \subset E$ satisfying $4B \subset E$. We also assume that the Harnack constant ϑ depends only on β/α, where α and β are the constants of $\mathcal{A}$.

We also say that an asymptotic end e of M is *p-parabolic* (with $p \geq 1$) if there exists $E \in e$ such that for every $\varepsilon > 0$ there exists $E' \in e$ such that

$$M_p(\Gamma(E', M \setminus E; M)) < \varepsilon.$$

Furthermore, we say that an asymptotic end e of M is *locally C-quasiconvex* if for every $E \in e$ there exists $E' \in e$, $E' \subset E$, such that each pair of points $x, y \in E'$ can be joint by a path in E' of length at most $Cd(x,y)$, where d is the Riemannian distance of M.

Theorem 7.7. *Let $n \geq 2$, $K \geq 1$, $m\colon (0,1) \to \mathbb{N}$, and $\vartheta\colon (0,\infty) \to (0,\infty)$. Then there exists $q = q(n,K,m,\vartheta)$ such that the following holds. Suppose that M is complete and e is an n-parabolic locally C-quasiconvex asymptotic end of M of type $\mathcal{E}(m,\vartheta)$, and that N has at least q ends. Let $E \in e$ and $f\colon E \to N$ be a K-quasiregular mapping. Then f has a limit at e.*

Corollary 7.8. *Let $n \geq 2$, $K \geq 1$, $m\colon (0,1) \to \mathbb{N}$, and $\vartheta\colon (0,\infty) \to (0,\infty)$. Then there exists $q = q(n,K,m,\vartheta)$ such that the following holds. Suppose that a complete Riemannian n-manifold M has asymptotic ends $\{e_1,\ldots,e_k\}$, $k < q$, of type $\mathcal{E}(m,\vartheta)$ which are all n-parabolic and locally C-quasiconvex, and that N has at least q ends. Let $f\colon M \to N$ be a continuous mapping and $E_i \in e_i$ for every $1 \leq i \leq k$. Then at least one of the following conditions fails:*

(i) *f is K-quasiregular in E_i for every i,*
(ii) *there exists a neighborhood Ω of $M \setminus (E_1 \cup \cdots \cup E_k)$ such that $f\Omega$ is open.*

Proof. Suppose that both conditions hold. By Theorem 7.7, we may extend f to a continuous mapping $\hat{f}\colon \hat{M} \to \hat{N}$. Since $\hat{M}$ is compact, we may follow the proof of Corollary 7.5. ∎

We need several lemmas in order to prove Theorem 7.7. Let us first recall the definition of a Harnack function. Let M be a Riemannian manifold. A continuous function $u\colon M \to \mathbb{R}$ is called *a Harnack function with constant θ* if

$$M(h,x,r) := \sup_{B(x,r)} h \leq \theta \inf_{B(x,r)} h$$

holds in each ball $B(x,r)$ whenever the function h is nonnegative in $B(x,2r)$, has the form $h = \pm u + a$ for some $a \in \mathbb{R}$, and $\bar{B}(x,2r) \subset M$ is compact. The original version of Lewis' lemma is stated for Harnack functions. It is well known (see [16, 6.2]) that $\mathcal{A}$-harmonic functions in the Euclidean setting are Harnack functions with some θ depending only on n and on the constants p, α, and β of $\mathcal{A}$. In that case θ is called the Harnack constant of $\mathcal{A}$.

Lemma 7.9. *Let e be an n-parabolic locally C-quasiconvex asymptotic end of a complete Riemannian n-manifold M. Suppose $u\colon E \to \mathbb{R}$, where $E \in e$, is a Harnack function with constant θ such that $\limsup_{x\to e} u(x) = \infty$ and $\liminf_{x\to e} u(x) < 0$. Then for every $C_0 > 0$ there exists a ball $B = B(x_0, r_0) \subset E$ such that*

(1) $B(x_0, 100Cr_0) \subset E$,
(2) $u(x_0) = 0$, *and*
(3) $\max_B u \geq C_0$.

Proof. It is sufficient to modify the proof of [23, Lemma 2.1] as follows. Let $E' \in e$ be such that $E' \subset E$ and E' is C-quasiconvex. Let $F' \subset M$ be a compact set such that E' is a component of $M \setminus F'$, fix $o \in M$, and let $R_0 > 0$ be such that $F' \subset \bar{B}(o, R_0/2)$.

Fix $k \in \mathbb{N}$ such that given $r > R_0$ and $x, y \in \partial B(o, r) \cap E'$ there exists k balls $B_i = B(x_i, r/1000)$, $1 \leq i \leq k$, in E such that

(1) $x \in B_1$,
(2) $y \in B_k$,
(3) $x_i \in E'$ for every i, and
(4) $B_i \cap B_{i+1} \neq \emptyset$ for every $i \in \{1, \ldots, k-1\}$.

Indeed, since $B_i \cap B(o, R_0/2) = \emptyset$, we have $B_i \subset E$, and since E' is C-quasiconvex, we may choose any $k > 2000C$. We may now apply the proof of [23, Lemma 5] almost verbatim. ∎

Lemma 7.10. *Let e be an n-parabolic locally C-quasiconvex asymptotic end of a complete Riemannian n-manifold M and $E \in e$. If $f\colon E \to N$ is a quasiregular mapping such that fE is n-hyperbolic, then f has a limit in $\hat{N}$ at e.*

Proof. Suppose that fE is n-hyperbolic. We may assume that E is C-quasiconvex. If f has no limit at e, there exists a compact set $F \subset N$ such that $fE' \cap F \neq \emptyset$ for every $E' \in e$. Hence there exists a sequence (x_k) such that $x_k \to e$ and $f(x_k) \to z \in N$ as $k \to \infty$. Let (y_k) be another sequence such that $y_k \to e$ as $k \to \infty$. We show that the hyperbolicity of fE yields $f(y_k) \to z$ as $k \to \infty$, which is a contradiction.

For every k we fix a path $\alpha_k\colon [0,1] \to E$ such that $\alpha_k(0) = x_k$, $\alpha_k(1) = y_k$, and $\ell(\alpha_k) \leq C d(x_k, y_k)$. Then

$$\operatorname{cap}_n(E, |\alpha_k|) \to 0$$

as $k \to \infty$. By Poletsky's inequality (4.3) and (4.1),

$$\operatorname{cap}_n(fE, f|\alpha_k|) \leq K_I(f) \operatorname{cap}_n(E, |\alpha_k|)$$

for every k. Suppose that $f(y_k) \not\to z$. Then, by passing to a subsequence if necessary, we may assume that $d(f(y_k), z) \geq \delta > 0$ for every k. Since

$d(f(\alpha_k(0)), f(\alpha_k(1))) \geq \delta/2$ for large k, we have, by the n-hyperbolicity of fE, that

$$\operatorname{cap}_n(fE, f|\alpha_k|) \geq \varepsilon > 0.$$

for every k. This is a contradiction. ∎

The following lemma is a reformulation of [29, Lemma 19.3.2].

Lemma 7.11. *Let $E \subset M$, let $u\colon E \to \mathbb{R}$ be a non-constant Harnack function with constant θ, and let $\alpha\colon [a,b] \to E$ be a path. If $\ell(\alpha) \leq k \operatorname{dist}(|\alpha|, u^{-1}(0) \cup M \setminus E)$, then u has a constant sign on $|\alpha|$. Furthermore,*

$$\max_{|\alpha|} u \leq \theta^k \min_{|\alpha|} u$$

if u is positive on $|\alpha|$, and

$$\max_{|\alpha|} u \leq \theta^{-k} \min_{|\alpha|} u$$

if u is negative on $|\alpha|$.

Proof. Since $|\alpha|$ is connected, every non-vanishing function on $|\alpha|$ has constant sign. We may assume without loss of generality that u is positive on $|\alpha|$. Let $a = a_0 < a_1 < \ldots < a_k = b$ be a partition of $[a,b]$ such that $\ell(\alpha|[a_i, a_{i+1}]) = \ell(\alpha)/k$ for every $i = 0, 1, \ldots, k-1$. For every i fix $x_i \in \alpha([a_i, a_{i+1}])$ such that $\ell(\alpha|[a_i, x_i]) = \ell(\alpha|[x_i, a_{i+1}])$. Then $\alpha([a_i, a_{i+1}]) \subset \bar{B}(x_i, \ell(\alpha)/(2k))$. Furthermore, $B(x_i, \ell(\alpha)/k) \subset E$ and $B(x_i, \ell(\alpha)/k) \cap u^{-1}(0) = \emptyset$. Since $\alpha(a_{i+1}) \in \bar{B}(x_i, \ell(\alpha)/(2k)) \cap \bar{B}(x_{i+1}, \ell(\alpha)/(2k))$ for every $i = 1, \ldots, k-1$, a repeated use of Harnack's inequality yields $\max_{|\alpha|} u \leq \theta^k \min_{|\alpha|} u$. ∎

Lemma 7.12 (Lewis' lemma)**.** *Let M, e, E, and u be as in Theorem 7.7. Then for every $C_0 > 0$ there exists a ball $B = B(x_0, r_0) \subset E$ such that*

(1) $6B \subset E$,
(2) $u(x_0) = 0$, *and*
(3) $C_0 \leq \max_{6B} u \leq \theta^6 \max_B u$.

Proof. Let $C_0 > 0$ and $B(x_0, R)$ be as in Lemma 7.9. Let $Z = u^{-1}(0)$ and $Z_R = Z \cap \bar{B}(x_0, 41R)$. For each $x \in Z_R$ we set $r_x = R - d(x, x_0)/41$ and $B_x = B(x, r_x)$. Then $F = \bigcup_{x \in Z_R} \bar{B}_x$ is compact and $x \mapsto \max_{\bar{B}_x} u$ is continuous. Let $a \in Z_R$ be a point of maximum for this function. Thus

$$\max_{\bar{B}(a, r_a)} u \geq \max_{\bar{B}(x_0, R)} u \geq C_0.$$

As in [29, Lemma 19.4.1], we have that

$$\operatorname{dist}(Z, \bar{B}(a, 6r_a) \setminus F) \geq \frac{5r_a}{6}.$$

Let $y_0 \in \bar{B}(a, 6r_a)$ be such that

$$u(y_0) = \max_{\bar{B}(a, 6r_a)} u \geq C_0 > 0.$$

If $y_0 \in F$, then, by the maximal property of the ball $B(a, r_a)$,

$$\max_{\bar{B}(a,6r_a)} u = u(y_0) \leq \max_F u = \max_{\bar{B}(a,r_a)} u \leq \theta^6 \max_{\bar{B}(a,r_a)} u.$$

If $y_0 \notin F$, let $y_1 \in F \cap \bar{B}(a, 6r_a)$ be nearest to y_0 in the length metric. As $\bar{B}(a, r_a) \subset F$ it follows that

$$d(y_0, y_1) \leq \operatorname{dist}(y_0, \bar{B}(a, r_a)) \leq 6r_a - r_a = 5r_a.$$

Let $\alpha\colon [0,1] \to E$ be a path of minimal length such that $\alpha(0) = y_0$ and $\alpha(1) = y_1$. Then $\alpha[0,1) \cap F = \emptyset$. Hence

$$\operatorname{dist}(Z, |\alpha|) \geq \frac{5r_a}{6}.$$

Thus $\ell(\alpha) \leq Cd(y_0, y_1) \leq 6C \operatorname{dist}(Z, |\alpha|)$. By Lemma 7.11,

$$u(y_0) \leq \max_{|\alpha|} u \leq \theta^{6C} \min_{|\alpha|} u \leq \theta^{6C} u(y_1) \leq \theta^{6C} \max_F u = \theta^{6C} \max_{\bar{B}(a,r_a)} u.$$

■

Lemma 7.13 ([24],[26])**.** *Let N be an n-parabolic Riemannian manifold. Suppose that $C \subset N$ is compact such that N has q ends $V_1, \ldots, V_q$ with respect to C. Then there exist n-harmonic functions v_j, $j = 2, \ldots, q$, and a positive constant κ such that*

(7.1) $\quad |v_j| \leq \kappa$ *in* C,

(7.2) $\quad |v_j - v_i| \leq 2\kappa$ *in* V_1,

(7.3) $\quad \sup_{V_1} v_j = \infty$,

(7.4) $\quad \inf_{V_j} v_j = -\infty$,

(7.5) $\quad v_j$ *is bounded in* V_k *for* $k \neq 1, j$,

(7.6) $\quad$ *if* $v_j(x) > \kappa$, *then* $x \in V_1$,

(7.7) $\quad$ *if* $v_j(x) < -\kappa$ *then* $x \in V_j$.

Proof of Theorem 7.7. Suppose that a K-quasiregular mapping $f\colon E \to N$ has no limit at e. By Lemma 7.10, N is n-parabolic. Let $C \subset N$ be a compact set such that N has q ends $V_1, \ldots, V_q$ with respect to C. For every $j = 2, \ldots, q$ let us fix an n-harmonic function v_j with properties (7.1) - (7.7) given in Lemma 7.13. For every $j = 2, \ldots, q$ we set $u_j = v_j \circ f$. Then functions u_j are $\mathcal{A}$-harmonic in E. Next we show that

$$\limsup_{x \to e} u_j(x) = +\infty \quad \text{and} \quad \liminf_{x \to e} u_j(x) = -\infty, \tag{7.8}$$

and hence they satisfy the assumptions of Lemma 7.9. This can be seen by observing that the sets $\{x \in N\colon v_j(x) > c\}$ and $\{x \in N\colon v_j(x) < -c\}$ are non-empty and open for every $c > 0$ and $j = 2, \ldots, q$. By Lemma 7.10, $f(E \setminus F)$ intersects these sets for every compact $F \subset M$, and therefore (7.8) follows. By

Lemma 7.12 there are sequences $x_i \in E$ and $r_i \in (0,\infty)$, $i \in \mathbb{N}$, such that $u_2(x_i) = 0$, $B(x_i, 3r_i) \subset E$,

$$M(u_2, x_i, 3r_i) \le \theta^6 M(u_2, x_i, r_i/2),$$

and $M(u_2, x_i, r_i/2) \to \infty$ as $i \to \infty$. Let us fix an index i such that $M(u_2, x_i, r_i/2) \ge 4\theta\kappa$, where $\theta > 1$ is the Harnack constant of $\mathcal{A}$ and κ is the constant in Lemma 7.13. We write $x = x_i$ and $r = r_i$. By (7.6), $f\big(B(x, r/2)\big) \cap V_1 \neq \emptyset$. Thus, by (7.2), we have

$$M(u_2, x, s) - 2\kappa \le M(u_j, x, s) \le M(u_2, x, s) + 2\kappa \tag{7.9}$$

whenever $s \ge r/2$. Next we conclude by using Harnack's inequality that

$$M(u_j, x, r) \le (\theta - 1) M(-u_j, x, 2r) \tag{7.10}$$

for all j. Let us first show that $u_j(z) = 0$ for some $z \in B(x, r)$. Suppose on the contrary, that $u_j > 0$ in $B(x, r)$. Then $u_j(y) \le \theta u_j(x)$ for all $y \in B(x, r/2)$ by Harnack's inequality. Since $M(u_2, x, r/2) \ge 4\theta\kappa$, there exists $y \in B(x, r/2)$ such that $u_j(y) > 2\theta\kappa$ by (7.9). Thus $u_j(x) > 2\kappa$, and so $x \in V_1$. By (7.2), $u_2(x) \ge u_j(x) - 2\kappa > 0$ contradicting the assumption $u_2(x) = 0$. Therefore there exists $z \in B(x, r)$ such that $u_j(z) = 0$. Thus $\inf_{B(x,r)} u_j \le 0$. Inequality (7.10) follows now from the calculation

$$\begin{aligned} M(u_j, x, r) &= \sup_{B(x,r)} u_j = \sup_{B(x,r)} \left(u_j - \inf_{B(x,2r)} u_j \right) + \inf_{B(x,2r)} u_j \\ &\le \theta \inf_{B(x,r)} \left(u_j - \inf_{B(x,2r)} u_j \right) + \inf_{B(x,2r)} u_j \\ &= \theta \inf_{B(x,r)} u_j + (1-\theta) \inf_{B(x,2r)} u_j \\ &\le -(\theta - 1) \inf_{B(x,2r)} u_j = (\theta - 1) \sup_{B(x,2r)} (-u_j) \\ &= (\theta - 1) M(-u_j, x, 2r), \end{aligned}$$

since $u_j - \inf_{B(x,2r)} u_j \ge 0$ in $B(x, 2r)$.

Inequalities (7.9) and (7.10), and the assumption $M(u_2, x, r/2) \ge 4\theta\kappa$ together yield the inequality

$$M(u_2, x, r) \le \theta M(-u_j, x, 2r). \tag{7.11}$$

Indeed,

$$\begin{aligned} M(u_2, x, r) &\le M(u_j, x, r) + \theta^{-1} M(u_2, x, r) \\ &\le (\theta - 1) M(-u_j, x, 2r) + \theta^{-1} M(u_2, x, r), \end{aligned}$$

which is equivalent to (7.11). We fix $z_j \in \bar{B}(x, 2r)$ such that

$$M(-u_j, x, 2r) = -u_j(z_j). \tag{7.12}$$

The well-known oscillation estimate (see e.g. [16, 6.6])

$$\operatorname*{osc}_{B(y,\rho)} u_j \le c(\rho/r)^\gamma \operatorname*{osc}_{B(y,r)} u_j$$

together with [24, Lemma 4.2] and (7.9) imply that

$$\operatorname*{osc}_{B(z_j,\rho)} u_j \le c_1(\rho/r)^\gamma M(u_2,x,3r) \tag{7.13}$$

for $\rho \in (0,r)$. See [24, (5.5)] for details. Thus

$$\begin{aligned}\max_{\bar{B}(z_j,\rho)} u_j &= \operatorname*{osc}_{B(z_j,\rho)} u_j + \min_{\bar{B}(z_j,\rho)} u_j \\ &\le c_1(\rho/r)^\gamma M(u_2,x,3r) + u_j(z_j) \\ &\le c_1(\rho/r)^\gamma M(u_2,x,3r) - \theta^{-1}M(u_2,x,r),\end{aligned}$$

by (7.13), (7.12), and (7.11). Since $M(u_2,x,3r) \le \theta^6 M(u_2,x,r)$, we obtain

$$c_1(\rho/r)^\gamma M(u_2,x,3r) \le (2\theta)^{-1}M(u_2,x,r)$$

by choosing $\rho = (2\theta^7 c_1)^{-1/\gamma} r$. Hence

$$\max_{\bar{B}(z_j,\rho)} u_j \le -(2\theta)^{-1}M(u_2,x,r) \le -2\kappa.$$

By (7.7), we conclude that $f\big(B(z_j,\rho)\big) \subset V_j$ and hence the balls $B(z_j,\rho)$ are disjoint. Since $B(z_j,\rho) \subset B(x,3r)$, there can be at most $m(\rho/3r)$ of them. Hence q has an upper bound that depends only on n, K, ϑ, and m. ■

8. Quasiregular mappings, p-harmonic forms, and de Rham cohomology

The use of n-harmonic functions in studying Liouville-type theorems for quasiregular mappings $f\colon M \to N$ is restricted to the case, where N is non-compact. The reason for this restriction is simple: a compact Riemannian manifold does not carry non-constant p-harmonic functions. Therefore, in the case of a compact target manifold, we have to use p-harmonic forms. In this final section we discuss briefly p-harmonic and $\mathcal{A}$-harmonic forms and their connections to quasiregular mappings. For detailed discussions on $\mathcal{A}$-harmonic forms, see e.g. [27], [28], [29], [30], and [42]. For the connection of $\mathcal{A}$-harmonic forms to quasiregular mappings, see e.g. [4], [29], and [34].

The Riemannian metric of M induces an inner product to the exterior bundle $\bigwedge^\ell T^*M$ for every $\ell \in \{1,\dots,n\}$, see e.g. [29, 9.6] for details. We denote this inner product by $\langle\cdot,\cdot\rangle$ and the corresponding norm by $|\cdot|$. As usual, sections of the bundle $\bigwedge^\ell T^*M$ are called ℓ-forms. The L^p-space of measurable ℓ-forms is denoted by $L^p(\bigwedge^\ell M)$ and the L^p-norm is defined by

$$\|\xi\|_p = \left(\int_M |\xi|^p dx\right)^{1/p}.$$

The local L^p-spaces of ℓ-forms are denoted by $L^p_{\mathrm{loc}}(\bigwedge^\ell M)$. The space of C^∞-smooth ℓ-forms on M is denoted by $C^\infty(\bigwedge^\ell M)$, and the space of compactly supported C^∞-smooth ℓ-forms by $C_0^\infty(\bigwedge^\ell M)$.

Let $\ell \in \{1, \ldots, n-1\}$ and $p > 1$. Let $\mathcal{A}\colon \bigwedge^\ell T^*M \to \bigwedge^\ell T^*M$ be a measurable bundle map such that there exists positive constants a and b satisfying

$$\langle \mathcal{A}(\xi) - \mathcal{A}(\zeta), \xi - \zeta \rangle \geq a(|\xi| + |\zeta|)^{p-2}|\xi - \zeta|^2, \tag{8.1}$$

$$|\mathcal{A}(\xi) - \mathcal{A}(\zeta)| \leq b(|\xi| + |\zeta|)^{p-2}|\xi - \zeta|, \text{ and} \tag{8.2}$$

$$\mathcal{A}(t\xi) = t|t|^{p-2}\mathcal{A}(\xi) \tag{8.3}$$

for all $\xi, \zeta \in \bigwedge^\ell T^*_x M$, $t \in \mathbb{R}$, and for almost every $x \in M$. We also assume that $x \mapsto \mathcal{A}_x(\omega)$ is a measurable ℓ-form for every measurable ℓ-form $\omega\colon M \to \bigwedge^\ell T^*M$.

We say that an ℓ-form ξ is $\mathcal{A}$-harmonic (of type p) on M if ξ is a weakly closed continuous form in $W^{d,p}_{\mathrm{loc}}(\bigwedge^\ell M)$ and satisfies the equality

$$\delta(\mathcal{A}(\xi)) = 0$$

weakly, that is,

$$\int_M \langle \mathcal{A}(\xi), d\varphi \rangle = 0$$

for all $\varphi \in C^\infty_0(\bigwedge^{\ell-1} M)$. Here $W^{d,p}_{\mathrm{loc}}(\bigwedge^\ell M)$ is the partial Sobolev space of ℓ-forms. A form $\omega \in L^p_{\mathrm{loc}}(\bigwedge^\ell M)$ is in the space $W^{d,p}_{\mathrm{loc}}(\bigwedge^\ell M)$ if the distributional exterior derivative $d\omega$ exists and $d\omega \in L^p_{\mathrm{loc}}(\bigwedge^{\ell+1} M)$. The global space $W^{d,p}(\bigwedge^\ell M)$ is defined similarly. A form $\omega \in W^{d,p}_{\mathrm{loc}}(\bigwedge^\ell M)$ is *weakly closed* if $d\omega = 0$ and *weakly exact* if $\omega = d\tau$ for some $\tau \in W^{d,p}_{\mathrm{loc}}(\bigwedge^{\ell-1} M)$.

Apart from minor differences between conditions (8.1)-(8.3) and the corresponding conditions in Section 2, we can say that $\mathcal{A}$-harmonic functions correspond to $\mathcal{A}$-harmonic weakly exact 1-forms.

Let $f\colon M \to N$ be a quasiregular mapping. Since f is almost everywhere differentiable, we may define the pull-back $f^*\xi$ of the form $\xi \in L^{n/\ell}_{\mathrm{loc}}(\bigwedge^\ell N)$ by

$$(f^*\xi)_x = (T_x f)^* \xi_{f(x)}.$$

By the quasiregularity of f, $f^*\xi \in L^{n/\ell}_{\mathrm{loc}}(\bigwedge^\ell M)$. Furthermore, $d(f^*\xi) = f^*(d\xi)$ if $\xi \in W^{1,n/\ell}_{\mathrm{loc}}(\bigwedge^\ell N)$. Hence $f^*\xi \in W^{1,n/\ell}_{\mathrm{loc}}(\bigwedge^\ell M)$ for $\xi \in W^{1,n/\ell}_{\mathrm{loc}}(\bigwedge^\ell M)$. The quasiregularity of f also yields that the pull-back $f^*\xi$ of an (n/ℓ)-harmonic ℓ-form is $\mathcal{A}$-harmonic. Similarly to the case of $\mathcal{A}$-harmonic functions, $\mathcal{A}\colon \bigwedge^\ell T^*M \to \bigwedge^\ell T^*M$ is defined by

$$\mathcal{A}(\eta) = \langle G^*\eta, \eta \rangle^{(n/\ell)-2} G^*\eta,$$

where

$$G_x = J_f(x)^{2/n}(T_x f)^{-1}\left((T_x f)^{-1}\right)^T \text{ a.e. }.$$

Recently in [4] Bonk and Heinonen studied cohomology of quasiregularly elliptic manifolds using p-harmonic forms. A connected Riemannian manifold is called K-quasiregularly elliptic if it receives a non-constant K-quasiregular mapping from $\mathbb{R}^n$. The main result of [4] is the following theorem.

Theorem 8.1 ([4, Theorem 1.1])**.** *Given $n \geq 2$ and $K \geq 1$ there exists a constant $C = C(n, K) > 1$ such that $\dim H^*(N) \leq C$ for every K-quasiregularly elliptic closed n-manifold N.*

As the Picard-type theorem 7.6, also this theorem has a local counterpart.

Theorem 8.2 ([34, Theorem 2])**.** *Given $n \geq 2$ and $K \geq 1$ there exists a constant $C' = C'(n,K) > 1$ such that every K-quasiregular mapping $f\colon B^n \setminus \{0\} \to N$ has a limit at the origin if N is a closed, connected, and oriented Riemannian n-manifold with $\dim H^*(N) \geq C'$.*

We close this section with a sketch of the proof of Theorem 8.2. The following theorem on exact $\mathcal{A}$-harmonic forms is essential in the proof. For details, see [34].

Theorem 8.3. *Let $n \geq 3$ and let η be a weakly exact $\mathcal{A}$-harmonic ℓ-form, $\ell \in \{2,\dots,n-1\}$, on $\mathbb{R}^n \setminus \bar{B}^n$ such that*

$$\int_{\mathbb{R}^n \setminus \bar{B}^n(2)} |\eta|^{n/\ell} = \infty. \tag{8.4}$$

Then there exists $\gamma = \gamma(n,a,b) > 0$ such that

$$\liminf_{r\to\infty} \frac{1}{r^\gamma} \int_{B^n(r)\setminus \bar{B}^n(2)} |\eta|^{n/\ell} > 0. \tag{8.5}$$

Here a and b are as in (8.1) *and* (8.2)*.*

Sketch of the proof of Theorem 8.2. Let us first consider some exceptions. For Riemannian surfaces the result is classical and follows from the uniformization theorem and the measurable Riemann mapping theorem, see [34, Theorem 3]. For $n \geq 3$ we may give a bound for the first cohomology using a well-known result of Varopoulos on the fundamental group and n-hyperbolicity. For details, see [34, Theorem 4]. Hence we may restrict our discussion to dimensions $n \geq 3$ and to cohomology dimensions $\ell \geq 2$.

Let $n \geq 3$ and $2 \leq \ell \leq n-1$, and suppose that $f\colon B^n \setminus \{0\} \to N$ does not have a limit at the origin. Without changing the notation we precompose f with a sense-preserving Möbius mapping σ such that $\sigma(\mathbb{R}^n \setminus \bar{B}^n) = B^n \setminus \{0\}$. Let us now show that $\dim H^\ell(N)$ is bounded from above by a constant depending only on n and K. We fix p-harmonic ℓ-forms ξ_i generating $H^\ell(N)$, with $p = n/\ell$. This can be done by a result of Scott [42]. Furthermore, we may assume that forms ξ_i are uniformly separated and uniformly bounded in L^p, that is, $\|\xi_i - \xi_j\|_p \geq 1$ and $\|\xi_i\|_p = 1$ for every i and j.

A local version [34, Theorem 6] of the value distribution result of Mattila and Rickman yields that

$$\int_{B^n(r)\setminus B^n(2)} |f^*\xi|^{n/\ell} \sim \int_{B^n(r)\setminus B^n(2)} J_f \tag{8.6}$$

for large radii r. Using Theorem 8.3 and a decomposition technique due to Rickman, we find a radius R and a decomposition of the annulus $B^n(R) \setminus B^n(2)$

into domains quasiconformally equivalent to B^n in such a way that we have a quasiregular embedding $\psi\colon B^n \to \mathbb{R}^n \setminus \bar{B}^n(2)$ with properties

$$\int_{\psi B^n(1/2)} J_f \gtrsim \left(\int_{B^n(R)\setminus B^n(2)} J_f \right)^{1/4} \tag{8.7}$$

and

$$\int_{\psi B^n} J_f \lesssim \int_{B^n(R)\setminus B^n(2)} J_f . \tag{8.8}$$

Combining (8.6) with (8.7) and (8.8), we have that forms $\varphi^* f^* \xi_i$ are uniformly bounded in $L^p(B^n)$ and uniformly separated in $L^p(B^n(1/2))$. By compactness, the number of forms is bounded by a constant depending on data. ∎

Remark 8.4. The use of $\mathcal{A}$-harmonic forms in the proof of Theorem 8.2 is very similar to their use in the proof of Theorem 8.1. Also Theorem 8.3 corresponds to a theorem of Bonk and Heinonen ([4, Theorem 1.11]).

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