Quasiregular mappings and the p-Laplace operator

Ilkka Holopainen

ABSTRACT. We describe the role of p-harmonic functions in the theory of quasiregular mappings.

Contents

1. Introduction

In this survey we discuss the importance of p-harmonic functions in the theory of quasiregular mappings as tools to obtain basic properties of these maps and, on the other hand, some Liouville-type results on the existence of non-constant quasiregular mappings between given Riemannian manifolds. Quasiregular mappings or, as they are also called, mappings of bounded distortion were introduced by Reshetnyak in the mid sixties in a series of papers; see e.g. [**27**], [**28**], and [**29**]. An interest in studying these mappings arises from a question about the existence of a geometric function theory in *real* dimensions $n \geq 3$ generalizing that of holomorphic functions $\mathbb{C} \to \mathbb{C}$. To motivate the definition of quasiregular mappings, let us write a holomorphic mapping $f\colon U \to \mathbb{C}$, where $U \subset \mathbb{C}$ is open, as a map $f = (u,v)\colon U \to \mathbb{R}^2,\ U \subset \mathbb{R}^2$,

$$f(x,y) = \big(u(x,y), v(x,y)\big).$$

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Then u and v are harmonic real-valued functions in U and they satisfy the Cauchy-Riemann system of equations

$$\begin{cases} D_1 u = D_2 v \\ D_2 u = D_1 v, \end{cases}$$

where $D_1 = \partial/\partial x,\ D_2 = \partial/\partial y$. For every $(x,y) \in U$, the differential $f'(x,y)\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map whose matrix (with respect to the standard basis of the plane) is

$$\begin{pmatrix} D_1 u & D_2 u \\ D_1 v & D_2 v \end{pmatrix} = \begin{pmatrix} D_1 u & D_2 u \\ -D_2 u & D_1 u \end{pmatrix}.$$

Hence

$$|f'(x,y)|^2 = \det f'(x,y), \tag{1.1}$$

where $|f'(x,y)| = \sup_{|h|=1} |f'(x,y)h|$ is the operator norm of the linear map $f'(x,y)$.

If we are looking for a class of mappings $f\colon U \to \mathbb{R}^n$, where $U \subset \mathbb{R}^n$ is open, sharing some geometric and topological properties of holomorphic functions, the first problem is to find an appropriate definition for such maps. The first trial definition could be maps satisfying a condition

$$|f'(x)|^n = J_f(x), \quad x \in U, \tag{1.2}$$

where $J_f(x) = \det f'(x)$. However, it has turned out that, for dimensions $n \geq 3$, a map $f\colon U \to \mathbb{R}^n$ belonging to the Sobolev space $W^{1,n}_{\mathrm{loc}}(U;\mathbb{R}^n)$ and satisfying (1.2) for a.e. $x \in U$ is either constant or a restriction of a Möbius map. This is the so-called generalized Liouville theorem due to Gehring [**10**] and Reshetn'yak [**29**]; see also the thorough discussion in [**23**].

Next candidate is obtained by replacing the equality (1.2) by a weaker condition

$$|f'(x)|^n \leq K J_f(x) \quad \text{a.e. } x \in U, \tag{1.3}$$

where $K \geq 1$ is a constant. Now there remains a question on the regularity assumption of such mapping f. Again there is some rigidity in dimensions $n \geq 3$. Indeed, if a mapping f satisfying (1.3) is nonconstant and smooth enough (more precisely, if $f \in C^k$, with $k = 2$ for $n \geq 4$ and $k = 3$ for $n = 3$), then f is a local homeomorphism. Furthermore, it then follows from a theorem of Zorich that such a map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is necessarily a homeomorphism, for $n \geq 3$; see [**36**]. We would also like a class of maps satisfying (1.3), with fixed K, to be closed under local uniform convergence. In order to obtain a rich enough class of mappings, it is thus necessary to weaken the regularity assumption from C^k-smoothness. See [**13**] and [**3**] for recent developments regarding smoothness and branching of quasiregular mappings. After this short motivation we are ready to give the following definition.

Definition 1.1. Let $U \subset \mathbb{R}^n$ be a domain. We say that a continuous mapping $f\colon U \to \mathbb{R}^n$ is *quasiregular* (or a *mapping of bounded distortion*) if

(1) $f \in W^{1,n}_{\mathrm{loc}}(U;\mathbb{R}^n)$, and
(2) there exists a constant $K \geq 1$ such that

$$|f'(x)|^n \leq K J_f(x) \quad \text{a.e. } x \in U. \tag{1.4}$$

Here $f'(x) = \big(D_i f_j(x)\big)$ is a linear map $\mathbb{R}^n \to \mathbb{R}^n$ (the formal derivative of f at x) and $J_f(x) = \det f'(x)$. They exist a.e. by (1). We want to emphasize that f is assumed to be continuous. We collect the basic analytic and topological properties of a quasiregular map into the following theorem by Reshetnyak; see [**30**], [**31**].

THEOREM 1.2 (Reshetnyak's theorem). *Let $f\colon U \to \mathbb{R}^n$ be quasiregular. Then*

(1) *f is differentiable a.e. and*
(2) *f is either constant or it is discrete, open, and sense-preserving.*

Recall that a map $g\colon X \to Y$ between topological spaces X and Y is *discrete* if the preimage $g^{-1}(y)$ of every $y \in Y$ is a discrete subset of X and that g is *open* if gU is open for every open $U \subset X$. We also remark that a continuous discrete and open map $g\colon X \to Y$ is called a *branched covering.*

To say that $f\colon U \to \mathbb{R}^n$ is sense-preserving means that the local degree $\mu(y, f, D)$ is positive for all domains $D \Subset U$ and for all $y \in fD \setminus f\partial D$. The local degree is an integer that tells, roughly speaking, how many times f wraps D around y. It can be defined, for example, by using cohomology groups with compact support. For the basic properties of the local degree, we refer to [**31**, Proposition I.4.4]; see also [**9**], [**26**], and [**34**]. For example, if f is differentiable at x_0 with $J_f(x_0) \neq 0$, then $\mu(f(x_0), f, D) = \operatorname{sign} J_f(x_0)$ for sufficiently small connected neighborhoods D of x_0. Another useful property is the following homotopy invariance: If f and g are homotopic via a homotopy $h_t,\ h_0 = f,\ h_1 = g$, such that $y \in h_tD \setminus h_t\partial D$ for every $t \in [0, 1]$, then $\mu(y, f, D) = \mu(y, g, D)$.

We can now easily extend the definition for mappings $f\colon M \to N$, where M and N are (oriented) Riemannian n-manifolds.

DEFINITION 1.3. A continuous mapping $f\colon M \to N$ is quasiregular (or a mapping of bounded distortion) if

(1) for every $x \in M$ there exist charts (U, φ) at x and (V, ψ) at $f(x)$, respectively, such that $fU \subset V$ and
$$\psi \circ f \circ \varphi^{-1}\colon \varphi U \to \mathbb{R}^n$$
is quasiregular, and
(2) there exists a constant $K \geq 1$ such that

$$|T_xf|^n \leq K J_f(x) \quad \text{for a.e. } x \in M. \tag{1.5}$$

Here $T_xf\colon T_xM \to T_{f(x)}N$ is the differential (or the tangent map) of f at x. It exists for a.e. x by Theorem 1.2 and Condition (1).

2. $\mathcal{A}$-harmonic functions

The very first step in developing the theory of quasiregular mappings is to prove, by direct computation, that quasiregular mappings have the following morphism property: If $f\colon U \to \mathbb{R}^n$ is quasiregular and $u \in C^2(U)$ is an n-harmonic function in a neighborhood of fU, then $u \circ f$ is a so-called $\mathcal{A}$-harmonic function in U. In this section we introduce the notion of $\mathcal{A}$-harmonic functions and recall some of their basic properties that are relevant for this survey.

Let M be a Riemannian n-manifold, with the Riemannian metric $\langle\cdot,\cdot\rangle$. Recall that the gradient of a smooth function $u\colon M \to \mathbb{R}$ is the vector field ∇u such that
$$\langle \nabla u(x), h\rangle = du(x)h$$
for every $x \in M$ and $h \in T_xM$.

The divergence of a smooth vector field V can be defined as a function $\operatorname{div} V\colon M \to \mathbb{R}$ satisfying
$$\mathcal{L}_V\omega = (\operatorname{div} V)\omega,$$

where ω is the (Riemannian) volume form and

$$\mathcal{L}_V\omega = \lim_{t\to 0}\frac{(\alpha^t)^*\omega-\omega}{t}$$

is the Lie derivative of ω with respect to V, and α is the flow of V. We say that a vector field $\nabla u \in L^1_{\mathrm{loc}}(M)$ is a *weak gradient* of $u \in L^1_{\mathrm{loc}}(M)$ if

$$\int_M \langle \nabla u, V\rangle = -\int_M u \operatorname{div} V \tag{2.1}$$

for all vector fields $V \in C_0^\infty(M)$. Conversely, a function $\operatorname{div} V \in L^1_{\mathrm{loc}}(M)$ is a weak divergence of a (locally integrable) vector field V if (2.1) holds for all $u \in C_0^\infty(M)$. Note that $\int_M \operatorname{div} Y = 0$ if Y is a smooth vector field in M with compact support.

We define the Sobolev space $W^{1,p}(M)$ and its norm as

$$\begin{aligned} &W^{1,p}(M) = \{u \in L^p(M)\colon \text{weak gradient } \nabla u \in L^p(M)\},\ 1 \le p < \infty,\\ &\quad \|u\|_{1,p} = \|u\|_p + \||\nabla u|\|_p. \end{aligned}$$

Let $G \subset M$ be open. Suppose that for a.e. $x \in G$ we are given a continuous map

$$\mathcal{A}_x\colon T_xM \to T_xM$$

such that the map $x \mapsto \mathcal{A}_x(X)$ is a measurable vector field whenever X is. Suppose that there are constants $1 < p < \infty$ and $0 < \alpha \le \beta < \infty$ such that

$$\langle \mathcal{A}_x(h), h\rangle \ge \alpha|h|^p$$

and

$$|\mathcal{A}_x(h)| \le \beta|h|^{p-1}$$

for a.e. $x \in G$ and for all $h \in T_xM$. In addition, we assume that for a.e. $x \in G$

$$\langle \mathcal{A}_x(h) - \mathcal{A}_x(k), h-k\rangle > 0$$

whenever $h \ne k$, and

$$\mathcal{A}_x(\lambda h) = \lambda|\lambda|^{p-2}\mathcal{A}_x(h)$$

whenever $\lambda \in \mathbb{R}\setminus\{0\}$.

A function $u \in W^{1,p}_{\mathrm{loc}}(G)$ is called a (weak) solution of the equation

$$-\operatorname{div}\mathcal{A}_x(\nabla u) = 0 \tag{2.2}$$

in G if

$$\int_G \langle \mathcal{A}_x(\nabla u), \nabla\varphi\rangle = 0$$

for all $\varphi \in C_0^\infty(G)$. Continuous solutions of (2.2) are called *$\mathcal{A}$-harmonic functions* (of type p). By the fundamental work of Serrin [**32**], every solution of (2.2) has a continuous representative. In the special case $\mathcal{A}_x(h) = |h|^{p-2}h$, $\mathcal{A}$-harmonic functions are called *p-harmonic* and, in particular, if $p = 2$, we obtain the usual harmonic functions.

A function $u \in W^{1,p}_{\mathrm{loc}}(G)$ is a *subsolution* of (2.2) in G if

$$-\operatorname{div}\mathcal{A}_x(\nabla u) \le 0$$

weakly in G, that is

$$\int_G \langle \mathcal{A}_x(\nabla u), \nabla\varphi\rangle \le 0$$

for all nonnegative $\varphi \in C_0^\infty(G)$. A function v is called *supersolution* of (2.2) if $-v$ is a subsolution. The proofs of the following two basic estimates are straightforward once the appropriate test function is found. Therefore we just give the test function and omit the details.

2.1. Caccioppoli and logarithmic Caccioppoli inequality.

LEMMA 2.1 (Caccioppoli inequality). *Let u be a positive solution of* (2.2) *(for a given fixed p) in G and let $v = u^{q/p}$, where $q \in \mathbb{R} \setminus \{0, p-1\}$. Then*

$$\int_G \eta^p |\nabla v|^p \le \left(\frac{\beta |q|}{\alpha |q-p+1|} \right)^p \int_G v^p |\nabla \eta|^p \tag{2.3}$$

for every nonnegative $\eta \in C_0^\infty(G)$.

PROOF. Write $\kappa = q - p + 1$ and use $\varphi = u^\kappa \eta^p$ as a test function. □

REMARK 2.2. In fact, the estimate (2.3) holds for positive supersolutions if $q < p-1,\ q \ne 0$, and for positive subsolutions if $q > p-1$.

The excluded case $q = 0$ above corresponds to the following logarithmic Caccioppoli's inequality.

LEMMA 2.3 (logarithmic Caccioppoli inequality). *Let u be a positive supersolution of* (2.2) *(for a given fixed p) in G and let $C \subset G$ be compact. Then*

$$\int_C |\nabla \log u|^p \le c \int_G |\nabla \eta|^p \tag{2.4}$$

for all $\eta \in C_0^\infty(G)$, with $\eta \mid C \ge 1$, where $c = c(p, \beta/\alpha)$.

PROOF. Take $\varphi = \eta^p u^{1-p}$ as a test function. □

These two lemmas together with the Sobolev and Poincaré inequalities are used in proving Harnack's inequality for nonnegative $\mathcal{A}$-harmonic functions by the familiar Moser iteration scheme. In the following $|A|$ denotes the volume of a measurable set $A \subset M$.

THEOREM 2.4 (Harnack's inequality). *Let M be a complete Riemannian manifold and suppose that there are positive constants R_0, C, and $\tau \ge 1$ such that a volume doubling property*

$$|B(x, 2r)| \le C\, |B(x, r)| \tag{2.5}$$

holds for all $x \in M$ and $0 < r \le R_0$, and that M admits a weak $(1,p)$-Poincaré inequality

$$\fint_B |v - v_B| \le C\, r \left(\fint_{\tau B} |\nabla v|^p \right)^{1/p} \tag{2.6}$$

for all balls $B = B(x, r) \subset M$, with $\tau B = B(x, \tau r)$ and $0 < r \le R_0$, and for all functions $v \in C^\infty(B)$. Then there is a constant c such that

$$\sup_{B(x,r)} u \le c \inf_{B(x,r)} u \tag{2.7}$$

whenever u is a nonnegative $\mathcal{A}$-harmonic function in a ball $B(x, 2r)$, with $0 < r \le R_0$.

In particular, if the volume doubling condition (2.5) and the Poincaré inequality (2.6) hold globally, that is, without any bound on the radius r, we obtain a global Harnack inequality. We refer to [**16**], [**7**], and [**14**] for proofs of the Harnack inequality.

3. Morphism property and its consequences

Let us now prove the morphism property for quasiregular mappings that were mentioned at the beginning of Section 2. In the following "n-harmonic" means, of course, "p-harmonic" with $p = n$, the dimension of M.

THEOREM 3.1. *Let $f\colon M \to N$ be a quasiregular mapping (with a constant K) and let $u \in C^2(N)$ be n-harmonic. Then $v = u \circ f$ is $\mathcal{A}$-harmonic (of type n) in M, with*

$$\mathcal{A}_x(h) = \langle G_x h, h\rangle^{\frac{n}{2}-1} G_x h, \tag{3.1}$$

where $G_x\colon T_xM \to T_xM$ is given by

$$G_x h = \begin{cases} J_f(x)^{2/n} T_x f^{-1}(T_x f^{-1})^T h, & \text{if } J_f(x) \text{ exists and is positive,} \\ h, & \text{otherwise.} \end{cases}$$

The constants α and β for $\mathcal{A}$ depend only on n and K.

PROOF. Let us first write the proof formally and then discuss the steps in more detail. In the sequel ω stands for the volume forms in M and N. Let $V \in C^1(M)$ be the vector field $V = |\nabla u|^{n-2}\nabla u$. Since u is n-harmonic and C^2-smooth, we have $\operatorname{div} V = 0$. By Cartan's formula we obtain

$$d(V \lrcorner\, \omega) = d(V \lrcorner\, \omega) + V \lrcorner\, (d\omega) = \mathcal{L}_V \omega = (\operatorname{div} V)\omega = 0$$

since $d\omega = 0$. Here $X \lrcorner\, \eta$ is the contraction of a differential form η by a vector field X. Thus, for instance, $V \lrcorner\, \omega$ is the $(n-1)$-form

$$V \lrcorner\, \omega(\underbrace{\cdot, \dots, \cdot}_{n-1}) = \omega(V, \underbrace{\cdot, \dots, \cdot}_{n-1}).$$

Hence

$$df^*(V \lrcorner\, \omega) = f^* d(V \lrcorner\, \omega) \overset{\text{a.e.}}{=} 0. \tag{3.2}$$

On the other hand, we have a.e. in M

$$f^*(V \lrcorner\, \omega) = W \lrcorner\, f^*\omega = W \lrcorner\, (J_f \omega) = J_f W \lrcorner\, \omega, \tag{3.3}$$

where W is a vector field that will be specified later (roughly speaking, $f_* W = V$). We obtain

$$d(J_f W \lrcorner\, \omega) = 0, \tag{3.4}$$

or equivalently

$$\operatorname{div}(J_f W) = 0 \tag{3.5}$$

which can be written as

$$\operatorname{div} \mathcal{A}_x(\nabla v) = 0, \tag{3.6}$$

where $\mathcal{A}$ is as in the claim.

Some explanations are in order. When writing

$$f^* d(V \lrcorner\, \omega) \overset{\text{a.e.}}{=} 0,$$

we mean that for a.e. $x \in U$ and for all vectors $v_1, v_2, \ldots, v_n \in T_x M$

$$f^* d(V \lrcorner\, \omega)(v_1, v_2, \ldots, v_n) = d(V \lrcorner\, \omega)(f_* v_1, f_* v_2, \ldots, f_* v_n) = 0,$$

where $f_* = f_{*,x} = T_x f$ is the tangent mapping of f at x. The left-hand side of (3.2) holds in a weak sense since $f \in W^{1,n}_{\mathrm{loc}}(M)$; see [**30**, p. 136]. This means that, for all n-forms $\eta \in C_0^\infty(M)$,

$$\int_M \langle f^* d(V \lrcorner\, \omega), \eta \rangle = \int_M \langle f^*(V \lrcorner\, \omega), \delta\eta \rangle, \tag{3.7}$$

where δ is the codifferential. Consequently, equations (3.4)–(3.6) are to be interpreted in weak sense. In particular, combining (3.2), (3.3), and (3.7) we get

$$\int_M \langle J_f W \lrcorner\, \omega, \delta\eta \rangle = \int_M \langle f^*(V \lrcorner\, \omega), \delta\eta \rangle = \int_M \langle f^* d(V \lrcorner\, \omega), \eta \rangle = 0$$

for all n-forms $\eta \in C_0^\infty(M)$, and so (3.4) holds in weak sense.

Let us next specify the vector field W. Let $A = \{x \in M \colon J_f(x) = \det f_{*,x} \neq 0\}$. Hence $f_{*,x}$ is invertible for all $x \in A$, and $W = f_*^{-1} V$ in A. In $M \setminus A$, either $J_f(x)$ does not exist, which can happen only in a set of measure zero, or $J_f(x) \le 0$. Quasiregularity of f, more precisely the distortion condition (1.5), implies that $f_{*,x} = T_x f = 0$ for almost every such x. Hence $f_{*,x} = 0$ for a.e. $x \in M \setminus A$. Setting $W = 0$ in $M \setminus A$, we obtain

$$f^*(V \lrcorner\, \omega) = 0 = W \lrcorner\, f^* \omega$$

a.e. in $M \setminus A$. Hence $f^*(V \lrcorner\, \omega) = W \lrcorner\, f^*\omega$ a.e. in M, and so (3.3) holds. □

3.1. Sketch of the proof of Reshetnyak's theorem. We shall use Theorem 3.1 to sketch the proof of Reshetnyak's theorem in a way that uses analysis, in particular, $\mathcal{A}$-harmonic functions. First we recall some definitions concerning p-capacity. If $\Omega \subset M$ is an open set and $C \subset \Omega$ is compact, then the *p-capacity* of the pair (Ω, C) is

$$\operatorname{cap}_p(\Omega, C) = \inf_{\varphi} \int_\Omega |\nabla\varphi|^p, \tag{3.8}$$

where the infimum is taken over all functions $\varphi \in C_0^\infty(\Omega)$, with $\varphi \mid C \ge 1$. A compact set $C \subset M$ is of p-capacity zero, denoted by $\operatorname{cap}_p C = 0$, if $\operatorname{cap}_p(\Omega, C) = 0$ for all open sets $\Omega \supset C$. Finally, a closed set F is of p-capacity zero, denoted by $\operatorname{cap}_p F = 0$, if $\operatorname{cap}_p C = 0$ for all compact sets $C \subset F$. It is a well-known fact that a closed set $F \subset \mathbb{R}^n$ containing a continuum C cannot be of p-capacity zero. This can be seen by taking an open ball B containing C and any test function $\varphi \in C_0^\infty(B)$, with $\varphi \mid C = 1$, and using a potential estimate

$$|\varphi(x) - \varphi(y)| \le c \left(\int_B \frac{|\nabla\varphi|}{|x - z|^{n-1}} dz + \int_B \frac{|\nabla\varphi|}{|y - z|^{n-1}} dz \right), \ x, y \in B,$$

combined with a maximal function and covering arguments. Similarly, if C is a continuum in a domain Ω and B is an open ball, with $\bar{B} \subset \Omega \setminus C$, then $\operatorname{cap}_n(C, \bar{B}; \Omega) > 0$, where

$$\operatorname{cap}_n(C, \bar{B}; \Omega) = \inf_{\varphi} \int_\Omega |\nabla\varphi|^p > 0,$$

the infimum being taken over all functions $\varphi \in C^\infty(\Omega)$, with $\varphi \mid C = 1$ and $\varphi \mid \bar{B} = 0$.

f is light. Suppose then that $U \subset \mathbb{R}^n$ is a domain and that $f\colon U \to \mathbb{R}^n$ is a nonconstant quasiregular mapping. We will show first that f is *light* which means that, for all $y \in \mathbb{R}^n$, the preimage $f^{-1}(y)$ is totally disconnected, i.e. each component of $f^{-1}(y)$ is a point.

Fix $y \in \mathbb{R}^n$ and define $u\colon \mathbb{R}^n \setminus \{y\} \to \mathbb{R}$ by

$$u(x) = \log \frac{1}{|x-y|}.$$

Then u is C^∞ and n-harmonic in $\mathbb{R}^n \setminus \{y\}$ by a direct computation. By Theorem 3.1, $v = u \circ f$,

$$v(x) = \log \frac{1}{|f(x)-y|},$$

is $\mathcal{A}$-harmonic in an open non-empty set $U \setminus f^{-1}(y)$ and $v(x) \to +\infty$ as $x \to z \in f^{-1}(y)$. We set $v(z) = +\infty$ for $z \in f^{-1}(y)$.

To show that f is light we use the logarithmic Caccioppoli inequality (2.4). Suppose that $C \subset f^{-1}(y) \cap U$ is a continuum. Since f is nonconstant and continuous, there exists $m > 1$ such that the set $\Omega = \{x \in U \colon v(x) > m\}$ is an open neighborhood of C and $\bar{\Omega} \subset U$. We choose another neighborhood D of C such that $\bar{D} \subset \Omega$ is compact. Now we observe that $v_i = \min\{v, i\}$ is a positive supersolution for all $i > m$. The logarithmic Caccioppoli inequality (2.4) then implies that

$$\int_D |\nabla \log v_i|^n \le c \operatorname{cap}_n(\Omega, \bar{D}) \le c < \infty$$

uniformly in i. Hence $|\nabla \log v| \in L^n(D)$. Choose an open ball B such that $\bar{B} \subset D \setminus f^{-1}(y)$. We observed earlier that

$$\operatorname{cap}_n(C, \bar{B}; D) > 0$$

since C is a continuum. Let

$$M_B = \max_{\bar{B}} \log v.$$

Now the idea is to use

$$\min\{1, \max\{0, \frac{1}{k} \log \frac{v}{M_B}\}\}$$

as a test function for $\operatorname{cap}_n(C, \bar{B}; D)$ for every $k \in \mathbb{N}$. We get a contradiction since

$$0 < \operatorname{cap}_n(C, \bar{B}; D) \le k^{-n} \|\nabla \log v\|_{L^p(D)} \to 0$$

as $k \to \infty$. Thus $f^{-1}(y)$ can not contain a continuum.

Differentiability a.e. Assume that $f = (f_1, \dots, f_n)\colon U \to \mathbb{R}^n$, $U \subset \mathbb{R}^n$, is quasiregular. Then coordinate functions f_j are $\mathcal{A}$-harmonic again by Theorem 3.1, since functions $x = (x_1, \dots, x_n) \mapsto x_j$ are n-harmonic. Now there are at least two ways to prove that f is differentiable almost everywhere. For instance, since each f_j is $\mathcal{A}$-harmonic, one can show by employing reverse Hölder inequality techniques that, in fact, $f \in W^{1,p}_{\text{loc}}(U)$, with some $p > n$. This then implies that f is differentiable a.e. in U; see e.g. [**2**]. Another way is to conclude that f is monotone, i.e. each coordinate function f_j is monotone, and therefore differentiable a.e. since $f \in W^{1,n}_{\text{loc}}(U)$; see [**31**]. The monotonicity of f_j holds since $\mathcal{A}$-harmonic functions obey the maximum principle.

f is sense-preserving. Here one first shows that conditions $f \in W^{1,n}_{\mathrm{loc}}(U)$ and $J_f(x) \geq 0$ a.e. imply that f is weakly sense-preserving, i.e. $\mu(y, f, D) \geq 0$ for all domains $D \Subset U$ and for all $y \in fD \setminus f\partial D$. This step employs approximation of f by smooth mappings. Pick then a domain $D \Subset U$ and $y \in fD \setminus f\partial D$. Denote by Y the y-component of $\mathbb{R}^n \setminus f\partial D$ and write $V = D \cap f^{-1}Y$. Since f is light, $D \setminus f^{-1}(y)$ is non-empty. Thus we can find a point $x_0 \in f^{-1}(y) \cap V$. Next we conclude that the set $\{x \in V : J_f(x) > 0\}$ has positive measure. Otherwise, since f is ACL and $|f'(x)| = 0$ a.e. in V, f would be constant in a ball centered at x_0 contradicting the fact that f is light. Thus there is a point x in V where f is differentiable and $J_f(x) > 0$. Now a homotopy argument, using the differential of f at z, and $\mu(y, f, D) \geq 0$ imply that f is sensepreserving.

f is discrete and open. This part of the proof is purely topological. A sense-preserving light mapping is discrete and open by Titus and Young; see e.g. [**31**].

Further properties of f. Once Reshetnyak's theorem is established it is possible to prove further properties for quasiregular mappings. We collect these properties to the following theorems and refer to the books [**30**] and [**31**] for the proofs.

THEOREM 3.2. *Let $f\colon M \to N$ be a nonconstant quasiregular map. Then*

(1) $|fE| = 0$ *if and only if* $|E| = 0$.
(2) $|B_f| = 0$, *where B_f is the branch set of f, i.e. the set of all $x \in M$ where f does not define a local homeomorphism.*
(3) $J_f(x) > 0$ *a.e.*
(4) *The integral transformation formula*
$$\int_A (h \circ f)(x) J_f(x)\, dm(x) = \int_N h(y) N(y, f, A)\, dm(y)$$
holds for every measurable $h\colon N \to [0, +\infty]$ and for every measurable $A \subset M$, where $N(y, f, A) = \operatorname{card} f^{-1}(y) \cap A$.
(5) *If $u \in W^{1,n}_{loc}(N, \mathbb{R})$, then $v = u \circ f \in W^{1,n}_{loc}(M, \mathbb{R})$ and*
$$\nabla v(x) = T_x f^T \nabla u(f(x)) \text{ a.e.}$$

Furthermore, we have a generalization of the morphism property.

THEOREM 3.3. *Let $f\colon M \to N$ be quasiregular and let $u\colon N \to \mathbb{R}$ be an $\mathcal{A}$-harmonic function (or a subsolution or a supersolution, respectively) of type n. Then $v = u \circ f$ is $f^{\#}\mathcal{A}$-harmonic (a subsolution or a supersolution, respectively), where*
$$f^{\#}\mathcal{A}_x(h) = \begin{cases} J_f(x) T_x f^{-1} \mathcal{A}_{f(x)}(T_x f^{-1})^T h), & \text{if } J_f(x) \text{ exists and is positive,} \\ |h|^{n-2} h, & \text{otherwise.} \end{cases}$$

The ingredients of the proof of Theorem 3.3 include, for instance, the locality of $\mathcal{A}$-harmonicity, Theorem 3.2, and a method to "push-forward" (test) functions.

To describe the latter, we say that D is a normal neighborhood of a point $x_0 \in M$ if $\partial fD = f\partial D$ and $\{x_0\} = D \cap f^{-1}(f(x_0))$. The local degree $\mu(f(x_0), f, D)$ is independent of the normal neighborhood of x_0 and we denote $\mu(f(x_0), f, D) = i(x_0, f)$. For the proof of the following lemma we refer to [**14**, 14.31] and [**31**, p. 151].

Lemma 3.4. *Let $f\colon M \to \mathbb{N}$ be a nonconstant quasiregular map, $\varphi \in C_0^\infty(M)$, and define*

$$\varphi_*(y) = \sum_{x\in f^{-1}(y)} i(x,f)\varphi(x), \quad y \in fM.$$

Then $\varphi_* \in W_0^{1,n}(fM)$.

4. Modulus and capacity inequalities

Although the main emphasis of this survey is on the relation between quasiregular mappings and p-harmonic functions, we want to introduce also the other main tool in the theory of quasiregular mappings. Let $1 \le p < \infty$ and let Γ be a family of paths in M. We denote by $\mathcal{F}(\Gamma)$ the set of all Borel functions $\varrho\colon M \to [0,+\infty]$ such that

$$\int_\gamma \varrho ds \ge 1$$

for all locally rectifiable path $\gamma \in \Gamma$. We call the functions in $\mathcal{F}(\Gamma)$ *admissible* for Γ. The *p-modulus of* Γ is defined by

$$M_p(\Gamma) = \inf_{\varrho\in\mathcal{F}(\Gamma)} \int_M \varrho^p dm.$$

There is a close connection between p-modulus and p-capacity. Indeed, suppose that $\Omega \subset M$ is open and $C \subset \Omega$ is compact. Let Γ be the family of all paths in $\Omega \setminus C$ connecting C and $\partial\Omega$. Then

$$\operatorname{cap}_p(\Omega, C) = M_p(\Gamma).$$

The inequality $\operatorname{cap}_p(\Omega, C) \ge M_p(\Gamma)$ follows easily since $\varrho = |\nabla\varphi|$ is admissible for Γ for each function φ as in (3.8). The other direction is harder and requires an approximation argument; see [**31**, Proposition II.10.2].

If $p = n =$ the dimension of M, we call $M_n(\Gamma)$ the *conformal modulus* of Γ, or simply the modulus of Γ. In that case $M_n(\Gamma)$ is invariant under conformal changes of the metric. In fact, $M_n(\Gamma)$ can be interpreted as follows: Define a new measurable Riemannian metric

$$\langle\langle\cdot,\cdot\rangle\rangle = \varrho^2\langle\cdot,\cdot\rangle.$$

Then, with respect to $\langle\langle\cdot,\cdot\rangle\rangle$, the length of γ has a lower bound

$$\ell_{\langle\langle\cdot,\cdot\rangle\rangle}(\gamma) = \int_\gamma \varrho ds \ge 1$$

and the volume of M is given by

$$\operatorname{Vol}_{\langle\langle\cdot,\cdot\rangle\rangle}(M) = \int_M \varrho^n dm.$$

Thus we are minimizing the volume of M under the constraint that paths in Γ have length at least 1.

The importance of the conformal modulus for quasiregular mappings lies in the following invariance properties; see [**31**, II.2.4, II.8.1]

Theorem 4.1. *Let $f\colon M \to N$ be a nonconstant quasiregular mapping. Let $A \subset M$ be a Borel set with $N(f,A) := \sup_y N(y,f,A) < \infty$, and let Γ be a family of paths in A. Then*

$$M_n(\Gamma) \le K\,N(f,A)\,M_n(f\Gamma). \tag{4.1}$$

THEOREM 4.2 (Poletsky's inequality). *Let $f\colon M \to N$ be a nonconstant quasiregular mapping and let Γ be a family of paths in M. Then*

$$M_n(f\Gamma) \le K^{n-1} M_n(\Gamma). \tag{4.2}$$

The proof of (4.1) is based on the change of variable formula for integrals (Theorem 3.2 3.) and on Fuglede's theorem. The estimate (4.2) in the converse direction is more useful than (4.1) but also much harder to prove; see [**31**, p. 39–50].

Application: Harnack's inequality. As an application of the use of p-modulus and p-capacity, we prove a Harnack's inequality for positive $\mathcal{A}$-harmonic functions of type $p > n-1$. Assume that $\Omega \subset M$ is a domain, $D \Subset \Omega$ another domain, and $C \subset D$ is compact. For $p > n-1$, we set

$$\lambda_p(C, D) = \inf_{E,F} M_p(\Gamma(E, F; D)),$$

where E and F are continua joining C and $\Omega \setminus D$, and $\Gamma(E, F; D)$ is the family of all paths joining E and F in D.

THEOREM 4.3 (Harnack's inequality, $p > n-1$). *Let Ω, D, and C be as above. Let u be a positive $\mathcal{A}$-harmonic function in Ω of type $p > n-1$. Then*

$$\log \frac{M_C}{m_C} \le c_0 \left(\frac{\operatorname{cap}_p(\Omega, \bar{D})}{\lambda_p(C, D)} \right)^{1/p}, \tag{4.3}$$

where

$$M_C = \max_{x \in C} u(x), \quad m_C = \min_{x \in C} u(x),$$

and $c_0 = c_0(p, \beta/\alpha)$.

PROOF. We may assume that $M_C > m_C$. Let $\varepsilon > 0$ be so small that $M_C - \varepsilon > m_C + \varepsilon$. Then the sets $\{x\colon u(x) \ge M_C - \varepsilon\}$ and $\{x\colon u(x) \le m_C + \varepsilon\}$ contain continua E and F, respectively, that join C and $\Omega \setminus D$. Write

$$w = \frac{\log u - \log(m_C + \varepsilon)}{\log(M_C - \varepsilon) - \log(m_C + \varepsilon)}$$

and observe that $w \ge 1$ in E and $w \le 0$ in F. Therefore $|\nabla w|$ is admissible for $\Gamma(E, F; D)$ and hence

$$\int_D |\nabla w|^p \ge M_p(\Gamma(E, F; D)) \ge \lambda_p(C, D).$$

On the other hand,

$$\int_D |\nabla \log u|^p dm \le c(p, \beta/\alpha) \operatorname{cap}_p(\Omega, \bar{D})$$

by the logarithmic Caccioppoli inequality (2.4), and

$$\nabla \log u = \left(\log \frac{M_C - \varepsilon}{m_C + \varepsilon} \right) \nabla w.$$

Hence

$$\log \frac{M_C - \varepsilon}{m_C + \varepsilon} \le c_0 \left(\frac{\operatorname{cap}_p(\Omega, \bar{D})}{\lambda_p(C, D)} \right)^{1/p}$$

and (4.3) follows by letting $\varepsilon \to 0$. $\square$

We can define $\lambda_p(C,D)$ analogously for $p \le n-1$, too. However, $\lambda_p(C,D)$ vanishes for $p \le n-1$. Consequently, Theorem 4.3 is useful only for $p > n-1$. The idea of the proof is basicly due to Granlund [**11**]. In the above form, (4.3) appeared first time in [**15**]. In general, it is difficult to obtain an effective lower bound for $\lambda_p(C,D)$ together with an upper bound for $\operatorname{cap}_p(\Omega,\bar{D})$. However, if $M = \mathbb{R}^n$ and $p = n$, one obtains a global Harnack inequality by choosing C, D, and Ω as concentric balls.

5. Liouville-type results for $\mathcal{A}$-harmonic functions

We have already mentioned that a global Harnack inequality

$$\max_{B(x,r)} u \le c \min_{B(x,r)} u$$

holds for nonnegative $\mathcal{A}$-harmonic functions on $B(x,2r)$ with a constant c independent of x, r, and u if M is complete and admits a global volume doubling condition and $(1,p)$-Poincaré's inequality. It follows from the global Harnack inequality that such manifold M cannot support nonconstant positive $\mathcal{A}$-harmonic functions for any $\mathcal{A}$ of type p. We say that M is *strong p-Liouville.*

Example 5.1. (1) Let M be complete with nonnegative Ricci curvature. Then it is well-known that M admits a global volume doubling property by the Bishop-Gromov comparison theorem (see [**1**], [**6**]). Furthermore, Buser's isoperimetric inequality [**4**] implies that M also admits a $(1,p)$-Poincaré inequality for every $p \ge 1$. Hence M is strong p-Liouville.

(2) Let $\mathbb{H}_n$ be the Heisenberg group. We write elements of $\mathbb{H}_n$ as (z,t), where $z = (z_1,\dots,z_n) \in \mathbb{C}^n$ and $t \in \mathbb{R}$. Furthermore, we assume that $\mathbb{H}_n$ is equipped with a left-invariant *Riemannian* metric in which the vector fields

$$\begin{aligned} X_j &= \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t},\\ Y_j &= \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t},\\ T &= \frac{\partial}{\partial t}, \end{aligned}$$

$j = 1,\dots,n$, form an orthonormal frame. Harnack's inequality for nonnegative $\mathcal{A}$-harmonic functions on $\mathbb{H}_n$ was proved in [**16**] by using Jerison's version of Poincaré's inequality. Jerison proved in [**24**] that (1,1)-Poincaré's inequality holds for the horizontal gradient

$$\nabla_0 u = \sum_{j=1}^{n}\left((X_j u)X_j + (Y_j u)Y_j\right)$$

and for balls in so-called Carnot-Carathéodory metric. Since the L^p-norm of the Riemannian gradient is larger than that of the horizonal gradient, we have (1,1)-Poincaré's inequality for the Riemannian gradient as well if geodesic balls are replaced by Carnot-Carathéodory balls or Heisenberg balls $B_H(r) = \{(z,t) \in \mathbb{H}_n \colon (|z|^4 + t^2)^{1/4} < r\}$ and their left-translations.

5.1. p-parabolicity, p-hyperbolicity. Classically, a Riemannian manifold M is called parabolic if it does not support a positive Green's function for the Laplace equation.

DEFINITION 5.2. We say that a Riemannian manifold M is p-parabolic, with $1 < p < \infty$, if

$$\operatorname{cap}_p(M, C) = 0$$

for all compact sets $C \subset M$. Otherwise, we say that M is p-hyperbolic.

EXAMPLE 5.3. (1) A compact Riemannian manifold is p-parabolic for all $p \geq 1$.

(2) In the Euclidean space $\mathbb{R}^n$ we have precise formulas for p-capacities of balls:

$$\operatorname{cap}_p(\mathbb{R}^n, \bar{B}(r)) = \begin{cases} c\, r^{n-p}, & \text{if } 1 \leq p < n, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $\mathbb{R}^n$ is p-parabolic if and only if $p \geq n$.

(3) If the Heisenberg group $\mathbb{H}_n$ is equipped with the left-invariant Riemannian metric, we do not have precise formulas for capacities of balls. However, for $r \geq 1$,

$$\operatorname{cap}_p(\mathbb{H}_n, \bar{B}_H(r)) \approx r^{2n+2-p}$$

if $1 \leq p < 2n+2$, and $\operatorname{cap}_p(\mathbb{H}_n, \bar{B}_H(r)) = 0$ if $p \geq 2n+2$. Hence $\mathbb{H}_n$ is p-parabolic if and only if $p \geq 2n+2$.

(4) Any complete Riemannian manifold M with finite volume $\operatorname{Vol}(M) < \infty$ is p-parabolic for all $p \geq 1$. This is easily seen by fixing a point $o \in M$ and taking a function $\varphi \in C_0^\infty\big(B(o,R)\big)$, with $\varphi \mid \bar{B}(o,r) = 1$ and $|\nabla\varphi| \leq c/(R-r)$. We obtain an estimate

$$\operatorname{cap}_p\big(B(o,R), \bar{B}(o,r)\big) \leq c \operatorname{Vol}(M)/(R-r)^p \to 0$$

as $R \to \infty$.

(5) Let M^n be a Cartan-Hadamard n-manifold, i.e. a complete, simply connected Riemannian manifold of nonpositive sectional curvature and dimension n. If the sectional curvature has a negative upper bound $K_M \leq -a^2 < 0$, then M is p-hyperbolic for all $p \geq 1$. This follows since M^n satisfies an isoperimetric inequality

$$\operatorname{Vol}(D) \leq \frac{a}{n-1} \operatorname{Area}(\partial D)$$

for all domains $D \Subset M$, with smooth boundary; see [**35**], [**5**]. If $p > 1$, then $v(x) = \exp(-\delta d(x,o))$ is a positive supersolution of the p-Laplace equation for some $\delta = \delta(n,p) > 0$ (see [**18**]). Hence the p-hyperbolicity of M also follows from the theorem below for $p > 1$.

THEOREM 5.4. *Let M be a Riemannian manifold and $1 < p < \infty$. Then the following conditions are equivalent:*

(1) *M is p-parabolic.*

(2) *$M_p(\Gamma_\infty) = 0$, where Γ_∞ is the family of all paths $\gamma\colon [0,\infty) \to M$ such that $\gamma(t) \to \infty$ as $t \to \infty$.*

(3) *Every nonnegative supersolution of*

$$-\operatorname{div} \mathcal{A}_x(\nabla u) = 0 \tag{5.1}$$

on M is constant for all $\mathcal{A}$ of type p.

(4) *M does not support a positive Green's function $g(\cdot, y)$ for (5.1) for any $\mathcal{A}$ of type p and $y \in M$.*

Here $\gamma(t) \to \infty$ means that $\gamma(t)$ eventually leaves any compact set. For the proof of Theorem 5.4 as well as for the discussion below we refer to [**15**]. Let us explain what is Green's function for (5.1). We define it first in a "regular" domain $\Omega \Subset M$, where regular means that the Dirichlet problem for $\mathcal{A}$-harmonic equation is solvable with continuous boundary data. For this notion, see [**14**].

We need a concept of $\mathcal{A}$-capacity. Let $C \subset \Omega$ be compact, and assume for simplicity that $\Omega \setminus C$ is regular. Thus there exists a unique $\mathcal{A}$-harmonic function in $\Omega \setminus C$ with continuous boundary values $u = 0$ on $\partial\Omega$ and $u = 1$ in C. Call u the $\mathcal{A}$-potential of (Ω, C). We define

$$\operatorname{cap}_{\mathcal{A}}(\Omega, C) = \int_\Omega \langle \mathcal{A}_x(\nabla u), \nabla u \rangle.$$

Then

$$\operatorname{cap}_{\mathcal{A}}(\Omega, C) \approx \operatorname{cap}_p(\Omega, C)$$

and furthermore,

$$\operatorname{cap}_{\mathcal{A}}(\Omega_1, C_1) \geq \operatorname{cap}_{\mathcal{A}}(\Omega_2, C_2) \tag{5.2}$$

if $C_2 \subset C_1$ and/or $\Omega_1 \subset \Omega_2$. Note that this property is obvious for variational capacities but $\operatorname{cap}_{\mathcal{A}}$ is not necessary a variational capacity.

The definition of Green's function, and in particular its uniqueness when $p = n$, relies on the following observation.

Lemma 5.5. *Let $\Omega \Subset M$ be a domain and let $C \subset \Omega$ be compact such that $\Omega \setminus C$ is regular. Let u be the $\mathcal{A}$-potential of (Ω, C). Then, for every $0 \leq a < b \leq 1$,*

$$\operatorname{cap}_{\mathcal{A}}(\{u > a\}, \{u \geq b\}) = \frac{\operatorname{cap}_{\mathcal{A}}(\Omega, C)}{(b-a)^{p-1}}.$$

Definition 5.6. Suppose that $\Omega \Subset M$ is a regular domain and let $y \in \Omega$. A function $g = g(\cdot, y)$ is called a *Green's function for (5.1) in* Ω if

(1) g is positive and $\mathcal{A}$-harmonic in $\Omega \setminus \{y\}$,
(2) $\lim_{x \to z} g(x) = 0$ for all $z \in \partial\Omega$,
(3)
$$\lim_{x \to y} g(x) = \operatorname{cap}_{\mathcal{A}}(\Omega, \{y\})^{1/(1-p)},$$
which we interprete to mean $\lim_{x \to y} g(x) = \infty$ if $p \leq n$,
(4) for all $0 \leq a < b < \operatorname{cap}_{\mathcal{A}}(\Omega, \{y\})^{1/(1-p)}$,
$$\operatorname{cap}_{\mathcal{A}}(\{g > a\}, \{g \geq b\}) = (b-a)^{1/(1-p)}.$$

Theorem 5.7. *Let $\Omega \Subset M$ be a regular domain and $y \in \Omega$. Then there exists a Green's function for (5.1) in Ω. Furthermore, it is unique at least if $p \geq n$.*

Monotonicity properties (5.2) of $\mathcal{A}$-capacity and the so-called Loewner property, i.e. $\operatorname{cap}_n C > 0$ if C is a continuum, are crucial in proving the uniqueness when $p = n$. Indeed, we can show that on sufficiently small spheres $S(y, r)$

$$|g(x, y) - \operatorname{cap}_{\mathcal{A}}(\Omega, \bar{B}(y, r))^{1/(1-n)}| \leq c, \quad x \in S(y, r),$$

which then easily implies the uniqueness.

Next take an exhaustion of M by regular domains $\Omega_i \subset \Omega_{i+1} \Subset M,\ M = \cup_i \Omega_i$. We can construct an increasing sequence of Green's functions $g_i(\cdot, y)$ on Ω_i. Then the limit is either identically $+\infty$ or

$$g(\cdot, y) := \lim_{i\to\infty} g_i(\cdot, y)$$

is a positive $\mathcal{A}$-harmonic function on $M \setminus \{y\}$. In the latter case we call the limit function $g(\cdot, y)$ a Green's function for (5.1) on M.

5.2. Liouville-type properties. We have the following list of Liouville-type properties of M (which may or may not hold for M):

(1) M is p-parabolic.
(2) Every nonnegative $\mathcal{A}$-harmonic function on M is constant for every $\mathcal{A}$ of type p. (*Strong p-Liouville.*)
(3) Every bounded $\mathcal{A}$-harmonic function on M is constant for every $\mathcal{A}$ of type p. (*p-Liouville.*)
(4) Every $\mathcal{A}$-harmonic function u on M with $\nabla u \in L^p(M)$ is constant for every $\mathcal{A}$ of type p. (*D_p-Liouville.*)

We refer to [**15**] for the proof of the following general result, and to [**16**] and [**21**] for studies concerning the converse directions.

THEOREM 5.8.

$$(1) \ \Rightarrow \ (2) \ \Rightarrow \ (3) \ \Rightarrow \ (4).$$

5.3. Applications to quasiregular maps. Here we give applications of the above results on n-parabolicity and various Liouville properties to the existence of non-constant quasiregular mappings between given Riemannian manifolds.

Let us start with the Gromov-Zorich "global homeomorphism theorem" that is a generalization of Zorich's theorem we mentioned in the introduction; see [**12**], [**37**].

THEOREM 5.9. *Suppose that M is n-parabolic, $n = \dim M \geq 3$, and that N is simply connected. Let $f\colon M \to N$ be a locally homeomorphic quasiregular map. Then f is injective and fM is n-parabolic.*

PROOF. We give here a very rough idea of the proof. First one observes that fM is n-parabolic (see Theorem 5.10 below), and so $N \setminus fM$ is of n-capacity zero. Then one shows, again by using the n-parabolicity of M, that the set E of all asymptotic limits of f is of zero capacity. Consequently, E is of Hausdorff dimension zero. Recall that an asymptotic limit of f is a point $y \in N$ such that $f\big(\gamma(t)\big) \to y$ as $t \to \infty$ for some path $\gamma \in \Gamma_\infty$ in M. Removing $E \cup (N \setminus FM)$ from N has no effect on the simply connectivity for dimensions $n \geq 3$. That is, $fM \setminus E$ remains simply connected. Thus one can extend uniquely any branch of local inverses of f and obtain a homeomorphism $g\colon fM \setminus E \to g(fM \setminus E)$ such that $f \circ g = \mathrm{id} \mid (fM \setminus E)$. Finally, g can be extended to E to obtain the inverse of f. □

We want to mention that in [**20**] the author and Pankka generalized the global homeomorphism theorem for mappings of finite distortion under mild conditions on the distortion. See also [**38**] for a related result for locally quasiconformal mappings.

THEOREM 5.10. *If N is n-hyperbolic and M is n-parabolic, then every quasiregular mapping $f\colon M \to N$ is constant.*

Proof. Suppose that $f\colon M \to N$ is a nonconstant quasiregular mapping. Then $fM \subset N$ is open. If $fM \neq N$, pick a point $y \in \partial(N \setminus fM)$ and let $g = g(\cdot, y)$ be the Green's function on N for the n-Laplacian. Then $g \circ f$ is a nonconstant positive $\mathcal{A}$-harmonic function on M which gives a contradiction with the n-parabolicity of M and Theorem 5.8. If $fM = N$, let u be a nonconstant positive supersolution on N for the n-Laplacian. Then $u \circ f$ is a nonconstant supersolution on M for some $\mathcal{A}$ of type n which is again a contradiction. □

Example 5.11.
(1) If N is a Cartan-Hadamard manifold, with $K_N \leq -a^2 < 0$, then every quasiregular mapping $f\colon \mathbb{R}^n \to N$ is constant.
(2) Let $\mathbb{H}_n$ be the Heisenberg group with a left-invariant Riemannian metric, then every quasiregular mapping $f\colon \mathbb{R}^{2n+1} \to \mathbb{H}_n$ is constant.

Theorem 5.12. *Suppose that M is strong n-Liouville while N is not. Then every quasiregular map $f\colon M \to N$ is constant.*

Proof. If N is not strong n-Liouville, then it is n-hyperbolic by Theorem 5.8. Suppose that $f\colon M \to N$ is a nonconstant quasiregular mapping. Then $fM \subset N$ is open. If $fM \neq N$, choose a point $y \in \partial(N \setminus fM)$ and let $g = g(\cdot, y)$ be the Green's function for n-Laplacian on N. Then $g \circ f$ is a nonconstant positive $\mathcal{A}$-harmonic function, with $\mathcal{A}$ of type n. This is a contradiction. If $fM = N$, we choose a nonconstant positive n-harmonic function u on N and get a contradiction as above. □

Theorem 5.13. *Let N be a Cartan-Hadamard n-manifold, with $-b^2 \leq K \leq -a^2 < 0$, and let M be a complete Riemannian n-manifold admitting a global doubling property and a global $(1, n)$-Poincaré inequality. Then every quasiregular mapping $f\colon M \to N$ is constant.*

Proof. By [**18**], N admits nonconstant positive n-harmonic functions. Hence N is not strong n-Liouville. On the other hand, the assumptions on M imply that a global Harnack's inequality for positive $\mathcal{A}$-harmonic functions of type n holds on M. Thus M is strong n-Liouville, and the claim follows from Theorem 5.12. □

Theorem 5.14 ("One-point Picard"). *Suppose that N is n-hyperbolic and M is strong n-Liouville. Then every quasiregular mapping $f\colon M \to N \setminus \{y\}$, with $y \in N$, is constant.*

Proof. Suppose that $f\colon M \to N \setminus \{y\}$ is a nonconstant quasiregular mapping. Then $\partial(N \setminus fM) \neq \emptyset$. Choose a point $z \in \partial(N \setminus fM)$, and let $g = g(\cdot, z)$ be the Green's function on N for the n-Laplacian. Then $g \circ f$ is a non-constant positive $\mathcal{A}$-harmonic function for some $\mathcal{A}$ of type n leading to a contradiction. □

In [**22**] the author and Rickman applied a method of Lewis ([**25**]) that relies on Harnack's inequality to prove the following general version of Picard's theorem on the number of omitted values of a quasiregular mapping. See [**31**, IV.2.1] and [**8**] for earlier versions of Picard's theorem.

Theorem 5.15. *Suppose that M admits a global Harnack's inequality for positive $\mathcal{A}$-harmonic functions of type n. Assume, furthermore, that M has the following covering property: for each $0 < k < 1$ there exists an integer $m = m(k)$ such that every ball $B(x, r) \subset M$ can contain at most m disjoint balls of radius kr. Suppose that N has at least two ends, i.e. there exists a compact set $C \subset N$ such that*

$N \setminus C$ has at least 2 unbounded components. Then, for every $K \geq 1$, there exists a constant q such that every K-quasiregular mapping $f\colon M \to N$ must be constant if N has at least q ends.

5.4. p-parabolicity and volume growth. Suppose that M is complete. Fix a point $o \in M$ and write $V(t) = \mathrm{Vol}\big(B(o,t)\big)$.

THEOREM 5.16. *Let $1 < p < \infty$ and suppose that*

$$\int^{\infty} \left(\frac{t}{V(t)}\right)^{1/(p-1)} dt = \infty,$$

or

$$\int^{\infty} \frac{dt}{V'(t)^{1/(p-1)}} = \infty.$$

Then M is p-parabolic.

PROOF. One can either construct a test function involving the integrals above, or use a p-modulus estimate for separating (spherical) rings. More precisely, write $B(t) = B(o,t)$ and $S(t) = S(o,t) = \partial B(o,t)$. For $R > r > 0$ and integers $k \geq 1$, we write $t_i = r + i(R-r)/k,\ i = 0, 1, \ldots, k$. Then, by a well-known property of modulus,

$$M_p\left(\Gamma\big(S(r), S(R); \bar{B}(R)\big)\right)^{1/(1-p)} \geq \sum_{i=0}^{k-1} M_p\left(\Gamma\left(S(t_i), S(t_{i+1}); \bar{B}(t_{i+1})\right)\right)^{1/(1-p)};$$

see e.g. [**31**, II.1.5]. Here $\Gamma\big(S(r), S(R); \bar{B}(R)\big)$ is the family of all paths joining $S(r)$ and $S(R)$ in $\bar{B}(R)$. For each $i = 0, \ldots, k-1$ we have an estimate

$$M_p\left(\Gamma\left(S(t_i), S(t_{i+1}); \bar{B}(t_{i+1})\right)\right) \leq \left(V(t_{i+1}) - V(t_i)\right)(t_{i+1} - t_i)^{-p}.$$

Hence

$$M_p\big(\Gamma\big(S(r), S(R); \bar{B}(R)\big)\big)^{1/(1-p)} \geq \sum_{i=0}^{k-1} \left(\frac{V(t_{i+1}) - V(t_i)}{t_{i+1} - t_i}\right)^{1/(1-p)} (t_{i+1} - t_i). \tag{5.3}$$

Thus the right-hand side of (5.3) tends to the integral

$$\int_r^R \frac{dt}{V'(t)^{1/(p-1)}}$$

as $k \to \infty$. We obtain an estimate

$$M_p\left(\Gamma\left(S(r), S(R); \bar{B}(R)\right)\right) \leq \left(\int_r^R \frac{dt}{V'(t)^{1/(p-1)}}\right)^{1-p}.$$

In particular, if

$$\int_r^{\infty} \frac{dt}{V'(t)^{1/(p-1)}} = \infty$$

for some $r > 0$, then M is p-parabolic. □

The converse is not true in general. That is, M can be p-parabolic even if

$$\int^{\infty} \left(\frac{t}{V(t)}\right)^{1/(p-1)} dt < \infty$$

or

$$\int^{\infty} \frac{dt}{V'(t)^{1/(p-1)}} < \infty;$$

see [**33**].

It is interesting to study when the converse is true.

Suppose that we have polar coordinates (ϑ, t) on M, with the the pole at o. Let $J(\vartheta, t)$ be the Jacobian determinant for the change of variables associated to these polar coordinates. We want to estimate $\operatorname{cap}_p(B(o,R), \bar{B}(o,r))$ from below. For that purpose, let us take a test function $u \in C_0^\infty(B(o,R))$, with $u \mid \bar{B}(o,r) = 1$. Then

$$\begin{aligned} 1 &\le \int_r^R |\nabla u(\gamma(t))| dt = \int_r^R |\nabla u(\gamma(t)| J(\vartheta,t)^{1/p} J(\vartheta,t)^{-1/p} \\ &\le \left(\int_r^R |\nabla u(\gamma(t)|^p J(\vartheta,t) dt \right)^{1/p} \left(\int_r^R J(\vartheta,t)^{1/(1-p)} dt \right)^{(p-1)/p}. \end{aligned}$$

Suppose that there is "homogeneity" on $S(t)$ so that $J(\vartheta,t) \approx \operatorname{Area} S(t) = V'(t)$ for all ϑ. Then

$$\int_r^R |\nabla u|^p J(\vartheta,t) dt \ge c \left(\int_r^R \frac{dt}{V'(t)^{1/(p-1)}} \right)^{1-p}.$$

Integrating with respect to ϑ and using Fubini's theorem we obtain

$$\int |\nabla u|^p \ge c \left(\int_r^R \frac{dt}{V'(t)^{1/(p-1)}} \right)^{1-p}$$

for all test functions u. Hence

$$\begin{aligned} \operatorname{cap}_p(B(o,R), \bar{B}(o,r)) &\ge c \left(\int_r^R \frac{dt}{V'(t)^{1/(p-1)}} \right)^{1-p} \\ &\ge c \left(\int_r^\infty \frac{dt}{V'(t)^{1/(p-1)}} \right)^{1-p}. \end{aligned}$$

Some kind of symmetry on M is provided if M admits a global doubling property and global $(1,p)$-Poincaré inequality. We refer to [**17**] for the proofs of the following two theorems.

Theorem 5.17. *Suppose that M is complete and admits a global doubling property and global $(1,p)$-Poincaré inequality for $1 < p < \infty$. Then*

$$M \text{ is } p\text{-hyperbolic if and only if } \int^\infty \left(\frac{t}{V(t)} \right)^{1/(p-1)} dt < \infty.$$

In some cases, we can estimate Green's functions:

Theorem 5.18. *Suppose that M is complete and has nonnegative Ricci curvature everywhere. Let $1 < p < \infty$. Then*

$$M \text{ is } p\text{-hyperbolic if and only if } \int^\infty \left(\frac{t}{V(t)} \right)^{1/(p-1)} dt < \infty.$$

Furthermore, we have estimates for Green's functions for (5.1)

$$c^{-1}\int_{2r}^{\infty}\left(\frac{t}{V(t)}\right)^{1/(p-1)}dt \le g(x,o) \le c\int_{2r}^{\infty}\left(\frac{t}{V(t)}\right)^{1/(p-1)}dt$$

for every $x \in \partial M(r)$, where $M(r)$ is the union of all unbounded components of $M \setminus \bar{B}(o,r)$. The constant c depends only on n, p, α, and β.

Theorem 5.17 follows also from the following sharper result; see [**19**].

Theorem 5.19. *Suppose that M is complete and that there exists a geodesic ray $\gamma\colon [0,\infty) \to M$ such that for all $t > 0$,*

$$|B(\gamma(t), 2s)| \le c|B(\gamma(t), s)|,$$

whenever $0 < s \le t/4$, and that

$$\fint_{B_\gamma(t)} |u - u_{B_\gamma(t)}|dm \le c\left(\fint_{2B_\gamma(t)} |\nabla u|^p dm\right)^{1/p}$$

for all $u \in C^\infty\big(2B_\gamma(t)\big)$, where $B_\gamma(t) = B(\gamma(t), t/8)$. Then M is p-hyperbolic if

$$\int^{\infty}\left(\frac{t}{|B(\gamma(t), t/4)|}\right)^{1/(p-1)} dt < \infty.$$

Theorem 5.19 can be applied to obtain the following.

Theorem 5.20. *Let M be a complete Riemannian n-manifold whose Ricci curvature is nonnegative outside a compact set. Suppose that M has maximal volume growth ($V(o,t) \approx r^n$). Then M is p-parabolic if and only if $p \ge n$.*

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Department of Mathematics, P.O. Box 4 (Yliopistonkatu 5), FIN-00014 University of Helsinki, Finland
E-mail address: ilkka.holopainen@helsinki.fi