Metric Geometry

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Contents

The material is collected mainly from books [AT], [BBI], and [BH] and from Lecture notes [La].

1 Metric spaces

We start by recalling the basic definitions related to metric spaces and introducing some examples and useful results.

1.1 Definitions and examples

Definition 1.2. Let X be a set. A function $d\colon X \times X \to [0, +\infty)$ is called a *pseudo metric* (in X) if

(1) $d(x, x) = 0$,

(2) $d(x, y) = d(y, x)$ and

(3) $d(x, z) \le d(x, y) + d(y, z)$ (triangle inequality)

for all $x, y, z \in X$. A pseudo metric d is called a *metric* if, in addition, $d(x, y) > 0$ for all $x, y \in X, x \neq y$. In that case the pair $(X, d))$ is called a *metric space*. Usually we say, for short, that X is a metric space, in particular, if the metric d is clear from the context.

Example 1.3. 1. The function $d\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$,

$$d\big((x, y), (x', y')\big) = |(x - x') + (y - y')|,$$

is a pseudo metric.

2. For any set X, the function

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y \end{cases}$$

is a metric.

3. For example, $d(x, y) = |x - y|$ and $d'(x, y) = \log(1 + |x - y|)$ are metrics in $\mathbb{R}$.

4. If (X, d) is a metric space, then $d_0\colon X \times X \to [0, 1)$,

$$d_0(x, y) = \frac{d(x, y)}{1 + d(x, y)},$$

is a metric in X. (Exercise: Verify the triangle inequality.)

5. If (X, d) is a metric space and $0 < \alpha < 1$, then (X, d^α), $d^\alpha(x, y) = (d(x, y))^\alpha$, is a metric space, too. (So called *snowflaked version* of (X, d).)

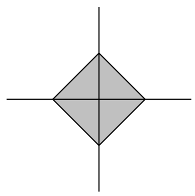
6. If (X_1, d_1) and (X_2, d_2) are metric spaces, then

$$d\big((x_1, x_2), (y_1, y_2)\big) = \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2}$$

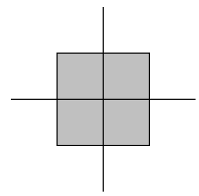
defines a metric in $X_1 \times X_2$.

7. If $(V, \|\cdot\|)$ is a normed space, then
$$d(x, y) = \|x - y\|$$
is a metric in V.

8. For example, norms $\|\cdot\|_1$,
$$\|(x_1, \dots, x_n)\|_1 = |x_1| + \cdots + |x_n|,$$
and $\|\cdot\|_\infty$,
$$\|(x_1, \dots, x_n)\|_\infty = \max\{|x_1|, \dots, |x_n|\},$$
defines metrics (denoted by d_1 and d_∞) in $\mathbb{R}^n$.



$\|x\|_1 \leq 1$



$\|x\|_\infty \leq 1$

9. If $\langle \cdot, \cdot \rangle$ is an inner product in V, then $\|x\| = \sqrt{\langle x, x \rangle}$ is a norm. In that case we say that $\|\cdot\|$ is an *inner product norm* (or Euclidean norm).

Example 1.4. For any set X we write
$$\ell^\infty(X) = \{f \colon X \to \mathbb{R} \mid \sup_{x \in X} |f(x)| < \infty\}$$
and
$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$
Then $(\ell^\infty(X), \|\cdot\|_\infty)$ is a normed space.

Problem 1.5. Prove that $(\mathbb{R}^n, \|\cdot\|_\infty) = (\ell^\infty(X), \|\cdot\|_\infty)$ for a suitable choice of X.

Lemma 1.6 (Parallelogram law)**.** *A norm $\|\cdot\|$ is an inner product norm in V if and only if*
$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \tag{1.7}$$
for all $x, y \in V$. If this is the case, then the inner product is given by the formula
$$\langle x, y \rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2).$$

Proof. If the norm is an inner product norm, a straightforward computation shows that (1.7) holds.

Suppose then that the norm $\|\cdot\|$ satisfies (1.7). We show that the formula
$$\langle x, y \rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2) \tag{1.8}$$
defines an inner product. Clearly $\langle x, y \rangle = \langle y, x \rangle$ and $\langle x, x \rangle = \|x\|^2 \geq 0$. Therefore, it suffices to show that, for each fixed y, the function
$$x \mapsto \langle x, y \rangle$$

is linear. Applying (1.7) to pairs $x' + y, x''$ and $x' - y, x''$ we obtain

$$\|x' + x'' + y\|^2 + \|x' - x'' + y\|^2 = 2\|x' + y\|^2 + 2\|x''\|^2,$$
$$\|x' + x'' - y\|^2 + \|x' - x'' - y\|^2 = 2\|x' - y\|^2 + 2\|x''\|^2.$$

Subtracting the second equation from the first one and using the definition (1.8) we get

$$\langle x' + x'', y\rangle + \langle x' - x'', y\rangle = 2\langle x', y\rangle.$$

Since $\langle 0, y\rangle = 0$, it follows (by choosing $x' = x''$) that

$$\langle 2x', y\rangle = 2\langle x', y\rangle.$$

Hence

$$\langle x' + x'', y\rangle + \langle x' - x'', y\rangle = \langle 2x', y\rangle.$$

Replacing here x' by $\frac{1}{2}(x' + x'')$ and x'' by $\frac{1}{2}(x' - x'')$ we obtain

$$\langle x', y\rangle + \langle x'', y\rangle = \langle x' + x'', y\rangle \tag{1.9}$$

for all $x', x'', y \in V$.

We have to show that $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$ for all $\lambda \in \mathbb{R}$. Repeating (1.9) we get

$$\langle nx, y\rangle = n\langle x, y\rangle \tag{1.10}$$

for all $n \in \mathbb{N}$. On the other hand,

$$\langle -x, y\rangle = \frac{1}{4}(\|-x + y\|^2 - \|-x - y\|^2) = -\frac{1}{4}(\|x + y\|^2 - \|x - y\|^2) = -\langle x, y\rangle,$$

and therefore (1.10) holds for all $n \in \mathbb{Z}$. It follows that

$$\langle qx, y\rangle = q\langle x, y\rangle$$

holds for all rational numbers $q = m/n$. Since $x \mapsto \langle x, y\rangle$ and multiplication by scalars are continuous functions (in the norm topology),

$$\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$$

holds for all $\lambda \in \mathbb{R}$. □

Remark 1.11. 1. Using Lemma 1.6 it is easy to see that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are not inner product norms in $\mathbb{R}^n$ for $n > 1$.

2. We will use the (Polish distance) notation

$$|x - y| := d(x, y) \tag{1.12}$$

in every metric space (even if X were not a vector space).

Example 1.13. If $\langle \cdot, \cdot\rangle$ is the (standard) inner product in $\mathbb{R}^{n+1}$ and

$$\mathbb{S}^n = \{x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \colon |x| = 1\}$$

is the unit sphere, the function $d\colon \mathbb{S}^n \times \mathbb{S}^n \to [0, \pi]$,

$$\cos d(x, y) = \langle x, y\rangle, \ x, y \in \mathbb{S}^n,$$

defines so called *angular metric* in $\mathbb{S}^n$. Then $d(x, y)$ is the angle between vectors x and y (and equals to the "length of the shortest arc on $\mathbb{S}^n$ joining x and y").

Definition 1.14. We say that a mapping $f\colon X \to Y$ between metric spaces X and Y is an *isometric embedding* if

$$|f(x) - f(y)| = |x - y|$$

for all $x, y \in X$. If, in addition, f is onto (surjective), we say that f is an *isometry*.

Problem 1.15. 1. Prove that every metric space (X, d) can be isometrically embedded into $\ell^\infty(X)$. (Hence the notation (1.12) makes sense.)

2. Study for which values of n the spaces $(\mathbb{R}^n, d_1)$ and $(\mathbb{R}^n, d_\infty)$ are isometric.

To study the second problem above one may use, for instance, the following theorem of Mazur and Ulam (1932)[1]. Recall that a mapping $f\colon V \to W$ is *affine* if the mapping $L\colon V \to W,\ L(x) = f(x) - f(0)$, is linear. Equivalently, f is affine if

$$f\big((1-t)x + ty\big) = (1-t)f(x) + tf(y) \tag{1.16}$$

for all $x, y \in V$ and $0 \le t \le 1$. Since every isometry is continuous, a sufficient condition for an isometry $f\colon V \to W$ to be affine is

$$f\big((x+y)/2\big) = \big(f(x) + f(y)\big)/2 \tag{1.17}$$

for all $x, y \in V$ (iteration of (1.17) gives (1.16) first for all dyadic rationals $t \in [0,1]$ and then (1.16) follows for all $t \in [0,1]$ by continuity).

Theorem 1.18 (Mazur-Ulam theorem)**.** *Suppose that V and W are normed spaces and that $f\colon V \to W$ is an isometry. Then f is affine.*

Proof. For $z \in V$, the *reflection of E in z* is the mapping $\psi\colon V \to V,\ \psi(x) = 2z - x$. Then $\psi \circ \psi = \mathrm{id}$, and hence ψ is bijective with $\psi^{-1} = \psi$. Moreover, ψ is an isometry, z is the only fixed point of ψ, and

$$|\psi(x) - z| = |x - z|, \quad |\psi(x) - x| = 2|x - z| \tag{1.19}$$

hold for all $x \in V$.

Let $x, y \in V$ and write $z = (x+y)/2$. In order to prove that f is affine, it suffices to show that $f(z) = (f(x) + f(y))/2 =: z'$. Let $\mathcal{F}$ be the family of all isometries $g\colon V \to V$ keeping the points x and y fixed. We will show first that also z is a fixed point for all $g \in \mathcal{F}$. Let $\lambda = \sup\{|g(z) - z| \colon g \in \mathcal{F}\}$. For $g \in \mathcal{F}$ we have $|g(z) - x| = |g(z) - g(x)| = |z - x|$. Hence $|g(z) - z| \le |g(z) - x| + |x - z| = 2|x - z|$, and so $\lambda < \infty$. Let ψ be the reflection of E in z. Then $\psi(x) = y$ and $\psi(y) = x$. If $g \in \mathcal{F}$, then also $g^* = \psi \circ g^{-1} \circ \psi \circ g \in \mathcal{F}$. Hence $|g^*(z) - z| \le \lambda$. Since g^{-1} is an isometry, this and (1.19) imply that

$$2|g(z) - z| = |\psi(g(z)) - g(z)| = |g^{-1} \circ \psi \circ g(z) - z| = |g^*(z) - z| \le \lambda$$

for all $g \in \mathcal{F}$. Hence $2\lambda \le \lambda$, and so $\lambda = 0$. This implies that $g(z) = z$ for all $g \in \mathcal{F}$. Let ψ' be the reflection of W in z'. Then $h = \psi \circ f^{-1} \circ \psi' \circ f \in \mathcal{F}$, and hence $h(z) = z$. This implies that $\psi'(f(z)) = f(z)$. Since z is the only fixed point of ψ, we have $f(z) = z'$ as desired $\square$

Remark 1.20. The surjectivity of f is essential in the Mazur-Ulam theorem: For example, $g\colon \mathbb{R} \to \mathbb{R}^2,\ g(t) = (t, |t|)$ is not affine, though it is an isometric embedding $(\mathbb{R}, |\cdot|) \to (\mathbb{R}^2, \|\cdot\|_\infty)$.

[1] The proof is taken from Väisälä: *A proof of the Mazur-Ulam theorem.* Amer. Math. Monthly 110 (2003) no. 7, 633-635.

Let us fix some notation. For a metric space X,

$B(x,r) = \{y \in X \colon |x-y| < r\}$ is the open ball of radius $r > 0$ centered at $x \in X$,

$\bar{B}(x,r) = \{y \in X \colon |x-y| \le r\}$ is the closed ball of radius r centered at x,

$S(x,r) = \{y \in X \colon |x-y| = r\}$ is the sphere of radius r centered at x,

$\operatorname{dist}(x,A) = \inf\{|x-y| \colon y \in A\}$ is the distance of $x \in X$ to $A \subset X$,

$\operatorname{dist}(A,B) = \inf\{|x-y| \colon x \in A, y \in B\}$ is the distance between sets $A, B \subset X$,

$\operatorname{diam}(A) = \sup\{|x-y| \colon x, y \in A\}$ is the diameter of $A \subset X$.

Note that $S(x,r)$ may be an empty set.

The metric determines the topology, denoted by $\mathcal{T}_d$, of X: A set $A \subset X$ is *open* (i.e. $A \in \mathcal{T}_d$) if, for each $x \in A$, there exists an open ball $B(x,r) \subset A$. Recall that a set $C \subset X$ is *closed* if its complement $C^c = X \setminus C$ is open. We denote the *closure* of a set $A \subset X$ by $\overline{A}$. Thus

$$\overline{A} = \{x \in X \colon B(x,r) \cap A \neq \emptyset \ \forall r > 0\}.$$

Note that the closure $\overline{B(x,r)}$ need not be the whole closed ball $\bar{B}(x,r)$.

A topological space $(X, \mathcal{T})$ is *Hausdorff* if disjoint points have disjoint neighborhoods. That is, for every $x, y \in X, x \neq y$, there exist open sets $x \in U$ and $y \in V$ such that $U \cap V = \emptyset$. In particular, every metric space is Hausdorff. Consequently, a sequence (x_i) in a metric space can have at most one limit.

A sequence (x_i) in a metric space X is called a *Cauchy sequence* if, for every $\varepsilon > 0$, there exists $i_0 \in \mathbb{N}$ such that

$$|x_i - x_j| < \varepsilon$$

for all $i, j \ge i_0$. A metric space X is *complete* if every Cauchy sequence in X converges. That is, if (x_i) is a Cauchy sequence in X, there exists $x \in X$ such that $|x_i - x| \to 0$ as $i \to \infty$.

For example, $\mathbb{R}^n$ is complete for all $n \in \mathbb{N}$, but $\mathbb{R}^n \setminus \{0\}$ is not (any sequence x_i converging to 0 (in $\mathbb{R}^n$) is a Cauchy sequence, but the limit 0 does not belong to the metric space $\mathbb{R}^n \setminus \{0\}$).

Problem 1.21. Prove that a metric space X is complete if and only if it has the following property: If (X_n) is a sequence of non-empty, closed subsets of X such that $X_{n+1} \subset X_n$ for every n and $\operatorname{diam}(X_n) \to 0$, then the sets X_n have a common point (i.e. $\cap_n X_n \neq \emptyset$). Note that the condition $\operatorname{diam}(X_n) \to 0$ is essential as an example $X_n = [n, \infty) \subset \mathbb{R}$ shows.

A mapping $f \colon X \to Y$ between metric spaces X and Y is Lipschitz if there exists a constant L such that

$$|f(x) - f(y)| \le L|x-y| \tag{1.22}$$

for all $x, y \in X$. In that case f is called L-Lipschitz. The smallest L for which (1.22) holds is denoted by $\operatorname{LIP}(f)$, i.e.

$$\operatorname{LIP}(f) = \inf\{L \colon f \ L\text{-Lipschitz}\}.$$

It is easy to see that f is then $\operatorname{LIP}(f)$-Lipschitz (i.e. "inf = min"). Every Lipschitz mapping is clearly continuous. A mapping $f \colon X \to Y$ is called *bi-Lipschitz* if there exists a constant $L \ge 1$ such that

$$\frac{1}{L}|x-y| \le |f(x) - f(y)| \le L|x-y|$$

for all $x, y \in X$. In this case we say that f is L-bi-Lipschitz. Every bi-Lipschitz mapping is a homeomorphism onto its image.

If $f \colon X \to Y$ is a bi-Lipschitz homeomorphism, then X and Y are complete simultaneously. Note that completeness is not a topological property: there are homeomorphic metric spaces X and Y such that X is complete while Y is not. (Exercise: construct an example.)

The following two theorems on complete metric spaces are very important in many contexts. We omit their proofs.

Theorem 1.23 (Banach's fixed point theorem). *Let X be a complete metric space and $f \colon X \to X$ an L-Lipschitz mapping, with $L < 1$. Then there exists a unique $x_0 \in X$ such that $f(x_0) = x_0$.*

Theorem 1.24 (Baire's theorem). *If X is a complete metric space, the intersection of every countable collection of dense open subsets of X is dense in X.*

Next we present useful extension and approximation results involving Lipschitz functions.

Theorem 1.25 (McShane-Whitney extension theorem). *Let X be a metric space, $A \subset X$, and $f \colon A \to \mathbb{R}$ L-Lipschitz. Then there exists an L-Lipschitz function $F \colon X \to \mathbb{R}$ such that $F|A = f$.*

Proof. For every $a \in A$ we define an L-Lipschitz function $f^a \colon X \to \mathbb{R}$

$$f^a(x) = f(a) + L|a - x|, \quad x \in X.$$

The function F is then defined by setting

$$F(x) = \inf_{a \in A} f^a(x), \quad x \in X.$$

Clearly $F(x) < \infty$ $\forall x \in X$. By fixing $a_0 \in A$ we see that

$$\begin{aligned} f(a) + L|a - x| &\geq f(a) + L|a - a_0| - L|a_0 - x| \\ &\geq f(a_0) - L|a_0 - x|. \end{aligned}$$

Hence $F(x) > -\infty$ for all $x \in X$. Since every f^a is L-Lipschitz and $F(x) > -\infty$ for all $x \in X$, F is L-Lipschitz. Moreover, for every $x \in A$

$$F(x) \leq f^x(x) = f(x) \leq f(y) + L|x - y| = f^y(x) \quad \forall y \in A,$$

and hence $F|A = f$. □

Corollary 1.26. *Let X be a metric space, $A \subset X$, and $f \colon A \to \mathbb{R}^n$ L-Lipschitz. Then there exists a $\sqrt{n}L$-Lipschitz mapping $F \colon X \to \mathbb{R}^n$ such that $F|A = f$.*

Proof. Apply Theorem 1.25 to the coordinate functions of f. □

Remark 1.27.

1. Theorem 1.25 holds (as such) in the case $X \subset \mathbb{R}^m$, $f \colon X \to \mathbb{R}^n$, but the proof is much harder. This is so called Kirszbraun's theorem.

2. It is a topic of quite active current research to study which pairs of metric spaces X, Y have a Lipschitz extension property (i.e. for every $A \subset X$ every Lipschitz mapping $f \colon A$ has a Lipschitz extension $F \colon X \to Y$).

Theorem 1.28. *Let X be a metric space and let $X' \subset X$ be dense. Suppose that Y is complete and that $f \colon X' \to Y$ is Lipschitz. Then there exists a unique continuous mapping $F \colon X \to Y$ such that $F|X' = f$. Moreover, F is Lipschitz and* $\mathrm{LIP}(F) = \mathrm{LIP}(f)$.

Proof. For every $x \in X$ choose a sequence (x_i) such that $x_i \in X'$ and $x_i \to x$. Then $(f(x_i))$ is a Cauchy sequence in Y since $|f(x_i) - f(x_j)| \le L|x_i - x_j| \to 0$ as $i, j \to \infty$. Here $L = \mathrm{LIP}(f)$. Since Y is complete, there exists $y \in Y$ such that $f(x_i) \to y$. We define

$$F(x) = y.$$

Then F is well-defined ($y = F(x)$ does not depend on the choice of the sequence (x_i)). To show that F is L-Lipschitz, let $x, y \in X$ and choose sequences $x_i \to x$ and $y_i \to y$. Then

$$|F(x) - F(y)| = \lim_{i\to\infty} |f(x_i) - f(y_i)| \le L \lim_{i\to\infty} |x_i - y_i| = L|x - y|.$$

The uniqueness of F is clear: if two continuous mappings coincide in a dense set, they must coincide everywhere. □

A function $f\colon X \to (-\infty, \infty]$ of a metric space (or, more generally, of a topological space) X is called *lower semicontinuous* if the set $\{x \in X : f(x) > a\}$ is open for each $a \in \mathbb{R}$. For example, the characteristic function of an open set is lower semicontinuous. A function f is called *upper semicontinuous* if $-f$ is lower semicontinuous.

Remark 1.29. A function $f\colon X \to (-\infty, \infty]$ is lower semicontinuous if and only if

$$\liminf_{y\to x} f(y) \ge f(x) \quad \forall x \in X$$

Theorem 1.30. *Let X be a metric space, $c \in \mathbb{R}$, and let $f\colon X \to [c, \infty]$ be lower semicontinuous. Then there exists an increasing sequence (f_i) of Lipschitz functions $f_i\colon X \to \mathbb{R}$ such that*

$$c \le f_i(x) \le f_{i+1}(x) \le f(x)$$

and

$$\lim_{i\to\infty} f_i(x) = f(x)$$

for every $x \in X$.

Proof. If $f(x) \equiv \infty$, we may choose $f_i(x) \equiv i$. Thus we may assume that $f(x) < \infty$ for some $x \in X$. For each $i \in \mathbb{N}$ we define an i-Lipschitz function f_i by

$$f_i(x) = \inf\{f(y) + i\,|x - y| : y \in X\}.$$

Then $c \le f_i(x) \le f_{i+1}(x) \le f(x)$ for all $x \in X$. Fix $x \in X$ and let $M \in [c, f(x))$. Choose $r > 0$ such that $f > M$ in $B(x, r)$. Then $f_i(x) \ge \min\{M, c + ir\}$. If $i \in \mathbb{N}$ is so large that $c + ir > M$, we have $f_i(x) \ge M$. Hence $\lim_{i\to\infty} f_i(x) = f(x)$. □

Every metric space can be isometrically embedded into a complete metric space. More precisely, we have the following theorem.

Theorem 1.31. *Let X be a metric space. Then there exists a complete metric space $\tilde{X}$ and an isometric embedding $f\colon X \to \tilde{X}$ such that $fX \subset \tilde{X}$ is dense. The space $\tilde{X}$ is unique up to an isometry and it is called the* completion *of X*

Proof. Let $\mathcal{X}$ be the set of all Cauchy sequences in X. If $\bar{x} = (x_i)$ and $\bar{y} = (y_i)$ are Cauchy sequences in X, we have

$$\big||x_i - y_i| - |x_j - y_j|\big| \leq |x_i - x_j| + |y_i - y_j|,$$

and hence $(|x_i - y_i|)$ is a Cauchy-sequence in $\mathbb{R}$. This sequence has a limit since $\mathbb{R}$ is complete, hence we may define $d\colon \mathcal{X} \times \mathcal{X} \to [0, \infty)$,

$$d(\bar{x}, \bar{y}) = \lim_{i\to\infty} |x_i - y_i|.$$

Clearly d is a pseudo metric in $\mathcal{X}$. Let $\sim$ be the equivalence relation

$$\bar{x} \sim \bar{y} \iff d(\bar{x}, \bar{y}) = 0$$

and let $\tilde{X} = \mathcal{X}/\!\sim$. Then d defines a metric in $\tilde{X}$ since

$$d([\bar{x}], [\bar{y}]) = d(\bar{x}, \bar{y})$$

is well-defined (i.e. is independent of the choice of representatives $\bar{x}, \bar{y}$). To show that $(\tilde{X}, d)$ is complete, let $(\bar{x}_i)$ be a Cauchy sequence in $(\tilde{X}, d)$. We have to show that there exists $\bar{x} \in \tilde{X}$ such that $d(\bar{x}_i, \bar{x}) \to 0$. By passing to a subsequence, if necessary, we may assume that, for all $i \in \mathbb{N}$,

$$d(\bar{x}_j, \bar{x}_k) < 1/i \quad \forall j, k \geq i.$$

For each i, $\bar{x}_i = \big[(x_{i,j})_{j=1}^{\infty}\big]$, where $(x_{i,j})_{j=1}^{\infty}$ is a Cauchy sequence in X. We may assume (again by passing to a subsequence) that the representative $(x_{i,j})_{j=1}^{\infty}$ satisfies

$$|x_{i,j} - x_{i,k}| < 1/n \quad \forall j, k \geq n.$$

Let $\bar{x}$ be the sequence of diagonal points $x_{j,j},\ j \in \mathbb{N}$. We claim that $\bar{x}$ is a Cauchy sequence in X (i.e. $[\bar{x}] \in \tilde{X}$) and that $d(\bar{x}_i, [\bar{x}]) \to 0$ as $i \to \infty$ which then shows that $(\tilde{X}, d)$ is complete. Suppose that $j \geq i$. Then for sufficiently large k we have

$$|x_{i,i} - x_{j,j}| \leq \underbrace{|x_{i,i} - x_{i,k}|}_{<1/i} + \underbrace{|x_{i,k} - x_{j,k}|}_{<2/i} + \underbrace{|x_{j,k} - x_{j,j}|}_{<1/j<1/i} < 4/i$$

which implies that $(x_{j,j})$ is Cauchy. Furthermore, for all sufficiently large k, we have

$$|x_{i,k} - x_{k,k}| \leq \underbrace{|x_{i,k} - x_{i,i}|}_{<1/i} + \underbrace{|x_{i,i} - x_{k,k}|}_{<4/i} < 5/i.$$

Hence

$$\lim_{i\to\infty} d(\bar{x}_i, [\bar{x}]) = \lim_{i\to\infty} \lim_{k\to\infty} |x_{i,k} - x_{k,k}| \leq \lim_{i\to\infty} 5/i = 0.$$

We define a mapping $f\colon X \to \tilde{X}$ by setting $f(x) = [(x_i)]$, where (x_i) is the Cauchy sequence $x_i \equiv x$. Then

$$d(f(x), f(y)) = \lim_{i\to\infty} |x_i - y_i| = |x - y|,$$

and so f is an isometric embedding.

To show that $f(X)$ is dense in $\tilde{X}$ we assume, on the contrary, that there exist $\bar{y} \in \tilde{X}$ and $\varepsilon > 0$ such that $d(\bar{y}, f(x)) \geq \varepsilon\ \forall x \in X$. Now $\bar{y} = [(y_i)]$, where (y_i) is a Cauchy sequence in X. Thus there exists $j \in \mathbb{N}$ such that $|y_i - y_j| < \varepsilon/2\ \forall i \geq j$. This leads to a contradiction since

$$0 < \varepsilon \leq d(\bar{y}, f(y_j)) = \lim_{i\to\infty} |y_i - y_j| \leq \varepsilon/2.$$

The uniqueness claim follows from Theorem 1.28. $\square$

1.32 Length spaces

Let X be a metric space and $I \subset \mathbb{R}$ an interval.

Definition 1.33. The *(total) variation* of a mapping $\gamma\colon I \to X$ on an interval $[a,b] \subset I$ is defined by

$$V_\gamma(a,b) = \sup\Big\{\sum_{i=1}^{k} |\gamma(t_i) - \gamma(t_{i-1})| \colon a \le t_0 < t_1 < \cdots < t_k \le b\Big\}.$$

We say that γ is *locally rectifiable* if $V_\gamma(a,b) < \infty$ for each (compact) $[a,b] \subset I$. The *length* of γ is

$$\ell(\gamma) := \sup_{[a,b]\subset I} V_\gamma(a,b).$$

and γ is called *rectifiable* if $\ell(\gamma) < \infty$. If $\gamma\colon [a,b] \to X$ is rectifiable, the function $s_\gamma\colon [a,b] \to [0,\ell(\gamma)]$,

$$s_\gamma(t) = V_\gamma(a,t) = \ell(\gamma|[0,t]),$$

is called the *length function* of γ.

Remark 1.34. Note that γ need not be continuous. If $V_\gamma(a,b) < \infty$, then

$$|\gamma(t+h) - \gamma(t)| \le s_\gamma(t+h) - s_\gamma(t)$$

for all $a \le t \le t+h \le b$. Hence γ is continuous if s_γ is continuous.

Definition 1.35. The *metric derivative* $|\dot\gamma|(t)$ of a mapping $\gamma\colon [a,b] \to X$ at $t \in (a,b)$ is defined as the limit

$$|\dot\gamma|(t) = \lim_{h\to 0} \frac{|\gamma(t+h) - \gamma(t)|}{|h|}$$

whenever the limit exists.

Example 1.36. Let X be $\mathbb{R}^n$ with the standard metric and write $\gamma = (\gamma_1, \ldots, \gamma_n)$. If the derivative $\gamma'(t) = (\gamma_1'(t), \ldots, \gamma_n'(t)) \in \mathbb{R}^n$ exists, then $|\dot\gamma|(t) = |\gamma'(t)|$.

Definition 1.37. We say that a mapping $\gamma\colon [a,b] \to X$ is *absolutely continuous* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{i=1}^{k} |\gamma(b_i) - \gamma(a_i)| < \varepsilon$$

whenever $]a_i, b_i[,\ i = 1, \ldots, k$, are disjoint subintervals of $[a,b]$ with

$$\sum_{i=1}^{k} |b_i - a_i| \le \delta.$$

Remark 1.38.

1. An absolutely continuous mapping $\gamma\colon [a,b] \to X$ is clearly continuous.
2. Every Lipschitz mapping $\gamma\colon [a,b] \to X$ is absolutely continuous.

Definition 1.39. A continuous mapping $\gamma\colon I \to X$ of an interval $I \subset \mathbb{R}$ is called a *path*.

Theorem 1.40. *If $\gamma\colon [a,b] \to X$ is L-Lipschitz, the metric derivative $|\dot\gamma|(t)$ exists for a.e. $t \in [a,b]$ and*

$$\ell(\gamma) = \int_a^b |\dot\gamma|(t)\, dt. \tag{1.41}$$

Proof. Let $\{t_n : n \in \mathbb{N}\} \subset [a,b]$ be dense. Then $\{x_n = \gamma(t_n)\}$ is dense in $\gamma[a,b]$ since γ is Lipschitz. For each $n \in \mathbb{N}$, define $\varphi_n\colon [a,b] \to \mathbb{R}$,

$$\varphi_n(t) = |\gamma(t) - x_n|.$$

Given $n \in \mathbb{N}$ and $t, s \in [a,b]$, we have

$$|\varphi_n(t) - \varphi_n(s)| = \big||\gamma(t) - x_n| - |\gamma(s) - x_n|\big| \le |\gamma(t) - \gamma(s)| \le L|t-s|.$$

Hence each φ_n is L-Lipschitz and the derivative $\varphi_n'(t)$ exists for a.e. $t \in [a,b]$. It follows that, for a.e. $t \in [a,b]$, $\varphi_n'(t)$ exists for all $n \in \mathbb{N}$. For these t we define

$$m(t) = \sup_n |\varphi_n'(t)|.$$

Note that $t \mapsto m(t)$ is measurable and $m(t) \le L$ a.e., hence m is integrable on $[a,b]$.

We will show that

$$|\dot\gamma|(t) = m(t) \quad \text{for a.e. } t \in [a,b]. \tag{1.42}$$

Since $|\gamma(t+h) - \gamma(t)| \ge |\varphi_n(t+h) - \varphi_n(t)|$ for all $n \in \mathbb{N}$, we have

$$\liminf_{h\to 0} \frac{|\gamma(t+h) - \gamma(t)|}{|h|} \ge \liminf_{h\to 0} \frac{|\varphi_n(t+h) - \varphi_n(t)|}{|h|} = |\varphi_n'(t)|.$$

Taking the supremum over all $n \in \mathbb{N}$ yields

$$\liminf_{h\to 0} \frac{|\gamma(t+h) - \gamma(t)|}{|h|} \ge m(t) \quad \text{for a.e. } t \in [a,b]. \tag{1.43}$$

By the Lebesgue differentiation theorem

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} m(s)\, ds = m(t) \quad \text{for a.e. } t \in [a,b].$$

Since $\{x_n\} \subset \gamma[a,b]$ is dense, we have $|\gamma(t) - \gamma(s)| = \sup_n \big||\gamma(t) - x_n| - |\gamma(s) - x_n|\big|$, and so

$$|\gamma(t) - \gamma(s)| \le \sup_n \Big| \int_s^t |\varphi_n'(\tau)|\, d\tau \Big| \le \Big| \int_s^t m(\tau)\, d\tau \Big|. \tag{1.44}$$

Hence for a.e. $t \in [a,b]$

$$\limsup_{h\to 0} \frac{|\gamma(t+h) - \gamma(t)|}{|h|} \le \limsup_{h\to 0} \Big| \frac{1}{h} \int_t^{t+h} m(\tau)\, d\tau \Big| = m(t)$$

which together with (1.43) proves (1.42).

It remains to prove (1.41). By (1.42) and (1.44) we have

$$\sum_{i=1}^{k}|\gamma(t_i)-\gamma(t_{i-1})| \le \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}|\dot\gamma|(t)\,dt \le \int_a^b|\dot\gamma|(t)\,dt$$

for all $a \le t_0 \le t_1 \le \cdots \le t_k \le b$. Taking the supremum over all such partitions yields

$$V_\gamma(a,b) \le \int_a^b |\dot\gamma|(t)\,dt.$$

To prove the converse inequality, fix $\varepsilon > 0$ and then an integer $n \ge 2$ such that $h_n = (b-a)/n \le \varepsilon$. Writing $t_i = a + ih_n$ and performing, for each i, the change of variables $s = t - t_i$ we get

$$\begin{aligned}
\frac{1}{h_n}\int_a^{b-\varepsilon}|\gamma(t+h_n)-\gamma(t)|\,dt &\le \frac{1}{h_n}\int_{t_0}^{t_{n-1}}|\gamma(t+h_n)-\gamma(t)|\,dt\\
&= \frac{1}{h_n}\sum_{i=0}^{n-2}\int_{t_i}^{t_{i+1}}|\gamma(t+h_n)-\gamma(t)|\,dt\\
&= \frac{1}{h_n}\int_0^{h_n}\sum_{i=0}^{n-2}|\gamma(s+t_{i+1})-\gamma(s+t_i)|\,ds\\
&\le \frac{1}{h_n}\int_0^{h_n} V_\gamma(a,b)\\
&= V_\gamma(a,b).
\end{aligned}$$

Hence

$$\begin{aligned}
\int_a^{b-\varepsilon}|\dot\gamma|(t)\,dt &= \int_a^{b-\varepsilon}\liminf_{n\to\infty}\frac{|\gamma(t+h_n)-\gamma(t)|}{h_n}\,dt\\
&\le \liminf_{n\to\infty}\frac{1}{h_n}\int_a^{b-\varepsilon}|\gamma(t+h_n)-\gamma(t)|\,dt\\
&\le V_\gamma(a,b)
\end{aligned}$$

by Fatou's lemma. The inequality $\int_a^b|\dot\gamma|(t)\,dt \le V_\gamma(a,b)$ follows by letting $\varepsilon \to 0$. □

Remark 1.45. It can be shown that Theorem 1.40 holds for absolutely continuous paths $\gamma\colon [a,b] \to X$. Indeed, given an absolutely continuous path $\gamma\colon [a,b] \to X$ there exists a unique Radon-measure μ_γ on $[a,b]$ such that $\mu_\gamma(]c,d[) = V_\gamma(c,d)$ for all open intervals $]c,d[\subset [a,b]$. Furthermore, μ_γ is absolutely continuous with respect to the Lebesgue measure m (since γ is absolutely continuous) and

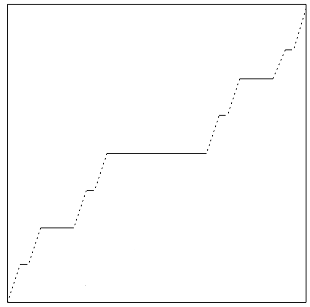
$$D_m\mu(t) = |\dot\gamma|(t) \quad \text{for a.e. } t \in [a,b],$$

where $D_m\mu$ is the Radon-Nikodym derivative of μ with respect to m. The equation (1.41) follows then from the Radon-Nikodym theorem. However, we will not prove these statements here.

Problem 1.46. Let $f\colon [0,1] \to [0,1]$ be the Cantor 1/3-function ("devil's staircase") [see e.g. Holopainen: Reaalianalyysi I, Esim. 1.21] and let $\gamma\colon [0,1] \to \mathbb{R}^2$ be the *path*

$$\gamma(t) = \big(t, f(t)\big).$$

Compute $V_\gamma(0,t)$, $t \in [0,1]$, and study the existence and values of $|\dot\gamma|(t)$. Conclusions?



Lemma 1.47. *The length function s_γ of a rectifiable mapping $\gamma\colon [a,b] \to X$ is increasing. Furthermore,*

(a) γ is continuous if and only if s_γ is continuous, and

(b) γ is absolutely continuous if and only if s_γ is absolutely continuous.

Proof. It is clear that s_γ is increasing. As noticed in Remark 1.34

$$|\gamma(t+h) - \gamma(t)| \leq s_\gamma(t+h) - s_\gamma(t)$$

for all $a \leq t \leq t+h \leq b$. Hence γ is (absolutely) continuous if s_γ is (absolutely) continuous.

Suppose then that $\gamma\colon [a,b] \to X$ is continuous, $c \in [a,b)$, and that $\varepsilon > 0$. Since γ is uniformly continuous, there exists $\delta > 0$ such that $|\gamma(x) - \gamma(y)| < \varepsilon/3$ for all $x, y \in [a,b]$, $|x-y| < \delta$. Furthermore, there exists a partition $a \leq x_0 < x_1 < \cdots < x_k \leq b$ such that $x_{j-1} = c$ for some j, $x_j - x_{j-1} < \delta$, and that

$$\sum_{i=1}^{k} |\gamma(x_i) - \gamma(x_{i-1})| > s_\gamma(b) - \varepsilon/3.$$

Since $|\gamma(x_j) - \gamma(x_{j-1})| < \varepsilon/3$, we have

$$\sum_{i \neq j} |\gamma(x_i) - \gamma(x_{i-1})| > s_\gamma(b) - 2\varepsilon/3.$$

Hence

$$V_\gamma(x_{j-1}, x_j) + \underbrace{V_\gamma(a, x_{j-1}) + V_\gamma(x_j, b)}_{> s_\gamma(b) - 2\varepsilon/3} = s_\gamma(b),$$

and so

$$V_\gamma(x_{j-1}, x_j) < 2\varepsilon/3.$$

It follows that s_γ is right-continuous at c. Similarly, s_γ is left-continuous at every point $c \in (a,b]$. Hence s_γ is continuous.

Suppose then that γ is absolutely continuous. Fix $\varepsilon > 0$ and let $\delta > 0$ be as in the definition of the absolute continuity of γ. Let $]c_1, d_1[, \ldots,]c_k, d_k[\subset [a,b]$ be disjoint intervals such that

$$\sum_{i=1}^{k} (d_i - c_i) < \delta.$$

For each $i = 1, \ldots, k$ there exists a partition $c_i = x_0^i < x_1^i < \cdots < x_l^i = d_i$ of $[c_i, d_i]$ such that

$$s_\gamma(d_i) - s_\gamma(c_i) = V_\gamma(c_i, d_i) < \sum_{j=1}^{l} |\gamma(x_j^i) - \gamma(x_{j-1}^i)| + \varepsilon/k.$$

Since

$$\sum_i \sum_j (x_j^i - x_{j-1}^i) < \delta,$$

we have, by absolute continuity of γ,

$$\sum_i \sum_j |\gamma(x_j^i) - \gamma(x_{j-1}^i)| < \varepsilon,$$

and so

$$\sum_{i=1}^k |s_\gamma(d_i) - s_\gamma(c_i)| < \varepsilon + k\varepsilon/k = 2\varepsilon.$$

Hence s_γ is absolutely continuous. $\square$

Definition 1.48. The *arc length parameterization* of a rectifiable path $\gamma\colon [a,b] \to X$ is the path $\gamma_s\colon [0,\ell(\gamma)] \to X$ defined by

$$\gamma_s(t) = \gamma\big(s_\gamma^{-1}(t)\big),$$

where

$$s_\gamma^{-1}(t) = \sup\{s\colon s_\gamma(s) = t\}.$$

Thus $\gamma(t) = \gamma_s\big(s_\gamma(t)\big)$ for all $t \in [a,b]$. It follows from the definitions that

$$\ell(\gamma_s|[t,t']) = t' - t \tag{1.49}$$

for all $0 \le t \le t' \le \ell(\gamma)$.

Theorem 1.50. *The arc length parameterization γ_s of a rectifiable path γ is 1-Lipschitz and*

$$|\dot\gamma_s|(t) = 1 \quad \textit{for a.e. } t \in [0,\ell(\gamma)].$$

Proof. The 1-Lipschitz property follows from (1.49). By Theorem 1.40, $|\dot\gamma_s|(t)$ exists and $|\dot\gamma_s|(t) \le 1$ for a.e. $t \in [0,\ell(\gamma)]$. Suppose, on the contrary, that $|\dot\gamma| < 1$ on a set of positive measure. Then there exist $\varepsilon > 0$ and a set $E \subset [a,b]$, with $m(E) > 0$, such that $|\dot\gamma_s|(t) < 1 - \varepsilon$ for all $t \in E$. Then

$$\begin{aligned}
\ell(\gamma) = \ell(\gamma_s) &= \int_0^{\ell(\gamma_s)} |\dot\gamma_s|(t)\,dt \\
&= \int_{[0,\ell(\gamma_s)]\setminus E} |\dot\gamma_s|(t)\,dt + \int_E |\dot\gamma_s|(t)\,dt \\
&\le \ell(\gamma) - m(E) + (1-\varepsilon)m(E) \\
&< \ell(\gamma),
\end{aligned}$$

which is a contradiction. $\square$

Definition 1.51. Let $\gamma\colon [a,b] \to X$ be a rectifiable path and let $\rho\colon X \to [0,\infty]$ be a Borel-function. The *line integral of ρ over γ* is

$$\int_\gamma \rho\,ds := \int_0^{\ell(\gamma)} \rho\big(\gamma_s(t)\big)\,dt.$$

The integral exists ($\in [0,\infty]$) since $\rho\circ\gamma_s$ is Borel. If $\gamma\colon I\to X$ is locally rectifiable, the integral of ρ over γ is defined as

$$\int_\gamma \rho\, ds = \sup_{[a,b]\subset I} \int_{\gamma|[a,b]} \rho\, ds.$$

Definition 1.52. Let $G\subset\mathbb{R}^n$, $G\neq\mathbb{R}^n$, be a domain (i.e. open and connected). For each $z\in G$ we write

$$\delta(z) = \operatorname{dist}(z, \mathbb{R}^n\setminus G)$$

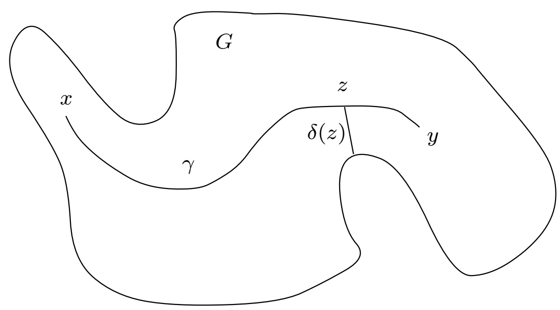
for the distance of z to the complement of G. Let $\gamma\colon [a,b]\to G$ be a rectifiable path. The *quasihyperbolic length* of γ is defined as

$$\ell_k(\gamma) = \int_\gamma \frac{1}{\delta(z)}\, ds(z) = \int_0^{\ell(\gamma)} \frac{1}{\delta\big(\gamma_s(t)\big)}\, dt.$$

The *quasihyperbolic distance* between points $x,y\in G$ is

$$k_G(x,y) := \inf_\gamma \ell_k(\gamma),$$

where the infimum is taken over all rectifiable paths $\gamma\colon [a,b]\to G$, with $\gamma(a)=x$ and $\gamma(b)=y$.



Problem 1.53. Prove that for a domain $G\subset\mathbb{R}^n$, $G\neq\mathbb{R}^n$, the quasihyperbolic distance k_G is a metric.

Definition 1.54. Let (X,d) be a metric space. Define $d_s\colon X\times X\to[0,\infty]$ by setting

$$d_s(x,y) = \inf_\gamma \ell(\gamma), \tag{1.55}$$

where the infimum is taken over all paths $\gamma\colon I\to X$ joining x and y, i.e. $x,y\in\gamma(I)$. If no such path exists, we set $d_s(x,y)=\infty$.

We assume from now on that each pair of points $x,y\in X$ can be joined by a rectifiable path and we call (X,d) *rectifiably connected.*

Theorem 1.56. *Let (X,d) be a rectifiably connected metric space and let d_s be defined by* (1.55). *Then*

(a) d_s defines a metric in X,

(b) $d_s(x,y)\ge d(x,y)$ for all $x,y\in X$,

(c) $\mathcal{T}_d\subset\mathcal{T}_{d_s}$,

(d) *if* $\gamma\colon [a,b] \to X$ *is a rectifiable path in* (X,d)*, it is also a rectifiable path in* (X,d_s)*,*

(e) *the length of a path in* (X,d_s) *is the same as its length in* (X,d)*,*

(f) $(d_s)_s = d_s$*.*

Proof. Claims (a) and (b) are clear and (c) follows from (b). If $\gamma\colon [a,b] \to X$ is a rectifiable path in (X,d),

$$d_s\big(\gamma(t+h),\gamma(t)\big) \le \ell\big(\gamma|[t,t+h]\big) = s_\gamma(t+h) - s_\gamma(t) \to 0$$

as $h \to 0+$ by Lemma 1.47 (a). Hence γ is a path (i.e. continuous) in (X,d_s). Furthermore,

$$\sum_{i=1}^{k} d_s\big(\gamma(t_i),\gamma(t_{i+1})\big) \le \sum_{i=1}^{k}\big(s_\gamma(t_i) - s_\gamma(t_{i-1})\big) = s_\gamma(t_k) - s_\gamma(t_0) \le \ell(\gamma)$$

for all $a \le t_0 < t_1 \cdots < t_k \le b$. Hence the length of γ in (X,d_s) satisfies $\ell_s(\gamma) \le \ell(\gamma) < \infty$ and (d) follows. If γ is a path in (X,d_s), it is continuous also in (X,d) by (c). We proved above that $\ell_s(\gamma) \le \ell(\gamma)$. On the other hand, $\ell_s(\gamma) \ge \ell(\gamma)$ by (b), and hence (e) holds. The claim (f) follows from (d) and (e) since γ is a rectifiable path in (X,d_s) if and only if it is a rectifiable path in (X,d). Moreover, $\ell_s(\gamma) = \ell(\gamma)$. □

Problem 1.57. Construct a rectifiably connected metric space (X,d) such that $\mathcal{T}_{d_s} \not\subset \mathcal{T}_d$.

The metric d_s is called the *length metric* (or the *inner metric*) associated to d.

Definition 1.58. A metric space (X,d) is called a *length space* (or an *inner metric space*) if $d = d_s$, that is

$$d(x,y) = \inf_\gamma \ell(\gamma),$$

where the infimum is taken over all paths $\gamma\colon I \to X$ joining x and y.

We say that (X,d) is a *local length space* if each point of X has a neighborhood U such that $d(x,y) = d_s(x,y)$ for all $x,y \in U$.

Example 1.59. 1. If $G \subset \mathbb{R}^n$, $G \ne \mathbb{R}^n$, is a domain , then (G,k_G) is a length space.

2. Let $\mathbb{S}^n(r) = \{x \in \mathbb{R}^{n+1}\colon |x| = r\}$ and let d be the standard metric of $\mathbb{R}^{n+1}$. Then

$$d(x,y) = |x-y| \quad \text{and}$$
$$d_s(x,y) = r \arccos\left(\frac{\langle x,y\rangle}{r^2}\right)$$

for $x,y \in \mathbb{S}^n(r)$. We notice that $d_s(x,y) > d(x,y)$ if $x \ne y$. The angular metric (in Example 1.13) of $\mathbb{S}^n$ is an inner metric.

Definition 1.60. Let X be a metric space. A path $\gamma\colon I \to X$ is called a *geodesic* if it is an isometric embedding, i.e.

$$|\gamma(t) - \gamma(s)| = |t-s|$$

for all $t,s \in I$. A path $\gamma\colon I \to X$ is a *local geodesic* if for all $t \in I$ there exists $\varepsilon > 0$ such that $\gamma|(I \cap [t-\varepsilon, t+\varepsilon])$ is a geodesic.

Every geodesic is, of course, a local geodesic but the converse need not be true.

Problem 1.61. Construct an example to verify the previous statement.

Definition 1.62. A metric space X is a *(uniquely) geodesic space* if each pair of points $x, y \in X$ can be joined by a (unique) geodesic $\gamma\colon [0, |x-y|] \to X$, with $\gamma(0) = x$ and $\gamma(|x-y|) = y$.

Every geodesic space if a length space but not converse.

Example 1.63. Every normed space $(V, \|\cdot\|)$ equipped with the metric $d(x, y) = \|x - y\|$ is a geodesic space. It is uniquely geodesic if and only if it is *strictly convex*, the latter meaning that if $x, y \in V,\ x \neq y,\ \|x\| = \|y\| = 1$, then

$$\|(1-t)x + ty\| < 1 \quad \forall t \in (0, 1).$$

Given $x, y \in V,\ x \neq y$, consider the path $\sigma\colon [0, 1] \to V,\ \sigma(t) = (1-t)x + ty$. Then σ has a constant speed

$$\begin{aligned} |\dot\sigma|(t) &= \lim_{h\to 0} \frac{|\sigma(t+h) - \sigma(t)|}{|h|} \\ &= \lim_{h\to 0} \frac{\|(1-t-h)x + (t+h)y - (1-t)x - ty\|}{|h|} \\ &= \lim_{h\to 0} \frac{|h|\|y-x\|}{|h|} \\ &= \|y - x\|, \end{aligned}$$

and so $\ell(\sigma) = \|y - x\| = d(x, y)$. Hence the path $\gamma\colon [0, \|x-y\|] \to V$,

$$\gamma(t) = \sigma\big(t/\|y - x\|\big),$$

is a geodesic from x to y. The other part of the claim is left as an exercise. For instance, metric spaces $(\mathbb{R}^n, d_1)$ and $(\mathbb{R}^n, d_\infty)$ are not uniquely geodesic if $n > 1$. Here d_1 and d_∞ are metrics defined by norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$, respectively.

Theorem 1.64. *Let X be a complete metric space. Then:*

(a) X is a geodesic space if and only if, for all $x, y \in X$, there exists $z \in X$ ("midpoint") such that

$$|x - z| = |y - z| = \frac{1}{2}|x - y|; \tag{1.65}$$

(b) X is a length space if and only if, for all $x, y \in X$ and all $\varepsilon > 0$, there exists $z \in X$ ("ε-midpoint") such that

$$\max\{|x - z|, |y - z|\} \le \frac{1}{2}|x - y| + \varepsilon. \tag{1.66}$$

Proof. We will prove only (a). The claim (b) can be proved by modifying the argument below. This is left as an exercise.

If X is a geodesic space and $x, y \in X$, there exists a geodesic $\gamma\colon [0, |x-y|] \to X$ with $x = \gamma(0)$ and $y = \gamma(|x-y|)$. Then the point $z = \gamma(|x-y|/2)$ satisfies (1.65). Suppose then that X satisfies the "midpoint" property (1.65). Fix $x, y \in X$. To construct a geodesic $\gamma\colon [0, |x-y|] \to X$ from $x = \gamma(0)$ to $y = \gamma(|x-y|)$, we first define a path $\sigma\colon [0, 1] \to X$ as follows. We set $\sigma(0) = x$ and $\sigma(1) = y$. For $\sigma(1/2)$ we choose a midpoint of x and y given by (1.65). For $\sigma(1/4)$ we choose a midpoint of x and $\sigma(1/2)$, for $\sigma(3/4)$ a midpoint of $\sigma(1/2)$ and y, and so forth. By this way we define $\sigma(t)$ for all dyadic rational numbers $t \in [0,1]$ (of the form $k/2^m$, for $m \in \mathbb{N},\ k = 0, 1, \dots, 2^m$). Thus σ is defined in a dense subset of $[0,1]$ and it is 1-Lipschitz. Since X is assumed to be complete, we can extend σ to a 1-Lipschitz path $\sigma\colon [0,1] \to X$ by Theorem 1.28. It follows (from Theorem 1.40) that $\ell(\sigma) \le |x-y|$. On the other hand, $\ell(\sigma) \ge |x-y|$, and hence $\ell(\sigma) = |x-y|$. Now $\gamma\colon [0, |x-y|] \to X,\ \gamma(t) = \sigma(t/|x-y|)$, is a geodesic joining x and y. $\square$

Problem 1.67. We may ask whether every *complete* length space is a geodesic space. Construct an example to show that this is not the case.

Definition 1.68. Let (f_n) be a sequence of mappings of a metric space X into another metric space Y. We say that (f_n) is *equicontinuous at* $x_0 \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f_n(x) - f_n(y)| < \varepsilon$$

for all $n \in \mathbb{N}$ and for all $x, y \in B(x_0, \delta)$. The sequence (f_n) is called *equicontinuous* if it is equicontinuous at each point $x \in X$ with $\delta > 0$ independent of x. More precisely, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f_n(x) - f_n(y)| < \varepsilon$$

for all $n \in \mathbb{N}$ and all $x, y \in X$, with $|x-y| \le \delta$.

Lemma 1.69 (Arzelà-Ascoli)**.** *Suppose that X is a separable metric space and that Y is a compact metric space. If (f_n) is equicontinuous at every point $x \in X$, then it has a subsequence that converges uniformly on compacts subsets of X to a continuous mapping $f\colon X \to Y$.*

Proof. Let $Q = \{q_1, q_2, \dots\} \subset X$ be a countable dense set. Since Y is compact, we can choose a subsequence $(f_{1,n})$ of (f_n) such that $\big(f_{1,n}(q_1)\big)$ converges. Denote the limit by $f(q_1)$. Next we choose a subsequence $(f_{2,n})$ of $(f_{1,n})$ such that $\big(f_{2,n}(q_2)\big)$ converges. Denote the limit by $f(q_2)$. Continuing this way we choose, for each $k \in \mathbb{N}$, a subsequence (f_{k+1}, n) of $(f_{k,n})$ such that $(f_{k+1,n})$ converges at all points $q_i,\ i \le k+1$. The diagonal sequence $(f_{n,n})$ converges pointwise in Q to a mapping $f\colon Q \to Y$. Let $x \in X$ and $\varepsilon > 0$. Since (f_n) is equicontinuous at x, there exists $\delta > 0$ such that $|f_{n,n}(q) - f_{n,n}(q')| < \varepsilon$ for all $n \in \mathbb{N}$ and all $q, q' \in B(x,\delta) \cap Q$. Hence

$$|f(q) - f(q')| \le \varepsilon \tag{1.70}$$

for all $q, q' \in B(x,\delta) \cap Q$. Since Q is dense in X and Y is compact (hence complete), f has a unique continuous extension $f\colon X \to Y$ defined as follows (cf. Theorem 1.28). Let $x \in X \setminus Q$ and choose a sequence $x_i \to x$ of points $x_i \in Q$. We get from (1.70) that $\big(f(x_i)\big)$ is a Cauchy-sequence. Hence it has a limit which we denote by $f(x) = \lim_{i\to\infty} f(x_i)$. Moreover, $f(x)$ is well-defined.

To show that the convergence is uniform on compact subsets, we fix a compact set $C \subset Y$ and $\varepsilon > 0$. For each $x \in C$ there exists $\delta_x > 0$ such that $|f_n(z) - f_n(y)| < \varepsilon$ for all $n \in N$ and all $z, y \in B(x, \delta_x)$. By compactness, C may be covered by finitely many balls $B(x, \delta_x)$. Let $\delta > 0$ be the minimum of these finitely many δ_x's. Then

$$|f_n(x) - f_n(y)| < \varepsilon \quad \text{and} \quad |f(x) - f(y)| \le \varepsilon$$

for all $x, y \in C$, with $|x-y| < \delta$. For each $y \in C$ there exists $j(y) \in \mathbb{N}$ such that $|y - q_{j(y)}| < \delta/2$. Again, we may cover C by finitely many balls $B(y, \delta/2)$. Let N be the maximum of the corresponding finitely many $j(y)$'s. Then for each $y \in C$ there exists $j(y) \le N$ such that $|y - q_{j(y)}| < \delta$. Finally, we choose $M \in \mathbb{N}$ so large that

$$|f_{n,n}(q_j) - f(q_j)| < \varepsilon$$

for all $n \geq M$ and all $j = 1, \dots, N$. Now for all $y \in C$ and $n \geq M$ we have

$$|f_{n,n}(y) - f(y)| \leq |f_{n,n}(y) - f_{n,n}(q_{j(y)})| + |f_{n,n}(q_{j(y)}) - f(q_{j(y)})| + |f(q_{j(y)}) - f(y)| \leq 3\varepsilon.$$

Hence $f_{n,n} \to f$ uniformly in C. $\square$

Lemma 1.71. *Let (γ_j) be a sequence of mappings $\gamma_j\colon [a,b] \to X$ converging uniformly to a mapping $\gamma\colon [a,b] \to X$. If γ is rectifiable, then for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that*

$$\ell(\gamma) \leq V_{\gamma_n}(a,b) + \varepsilon$$

for all $n \geq n_\varepsilon$.

Proof. Choose a partition $a \leq t_0 \leq t_1 \leq \cdots \leq t_k \leq b$ such that

$$\ell(\gamma) \leq \sum_{i=1}^{k} |\gamma(t_i) - \gamma(t_{i-1})| + \varepsilon/2.$$

Then we choose n_ε so large that $|\gamma(t) - \gamma_n(t)| < \varepsilon/4k$ for all $n \geq n_\varepsilon$ and all $t \in [a,b]$. Now for all $n \geq n_\varepsilon$ we have

$$\begin{aligned} |\gamma(t_i) - \gamma(t_{i-1})| &\leq |\gamma(t_i) - \gamma_n(t_i)| + |\gamma_n(t_i) - \gamma_n(t_{i-1})| + |\gamma_n(t_{i-1}) - \gamma(t_{i-1})| \\ &\leq \varepsilon/2k + |\gamma_n(t_i) - \gamma_n(t_{i-1})|. \end{aligned}$$

Hence

$$\ell(\gamma) \leq \varepsilon/2 + \sum_{i=1}^{k} |\gamma_n(t_i) - \gamma_n(t_{i-1})| + \varepsilon/2 \leq \varepsilon + V_{\gamma_n}(a,b)$$

for all $n \geq n_\varepsilon$. $\square$

Definition 1.72. We say that a metric space X is *proper* (or *boundedly compact*) if every bounded closed set is compact. Equivalently, X is proper if all closed (bounded) balls are compact. Recall also that a topological space Y is *locally compact* if each point of Y has a neighborhood U such that the closure $\overline{U}$ (i.e. the intersection of all closed sets containing U) is compact.

Theorem 1.73 (Hopf-Rinow)**.** *Every complete locally compact length space X is a proper geodesic space.*

Proof. To prove that X is proper, it suffices to show that, for a fixed $z \in X$, balls $\bar{B}(z,r)$ are compact for every $r \geq 0$. Let

$$I = \{r \geq 0\colon \bar{B}(z,r) \text{ is compact}\}.$$

Then $0 \in I$ and I is an interval. Indeed, if $\bar{B}(z,r)$ is compact for some $r > 0$, then $\bar{B}(z,s) \subset \bar{B}(z,r)$ is a closed subset of a compact set $\bar{B}(z,r)$ for all $0 \leq s \leq r$. Hence $\bar{B}(z,s)$ is compact and I is an interval. We will show that $I = [0,\infty)$. Fix $r \in I$. Since X is locally compact, we may cover the compact set $\bar{B}(z,r)$ by finitely many open balls $B(x_i,\varepsilon_i)$ such that $\bar{B}(x_i,\varepsilon_i)$ is compact. Then the finite union $\cup_i \bar{B}(x_i,\varepsilon_i)$ is compact and contains $\bar{B}(z,r+\delta)$ for some $\delta > 0$. This shows that I is open in $[0,\infty)$.

Next we will show that I is also closed in $[0,\infty)$. Suppose that $[0,\rho) \subset I$, $\rho > 0$. To prove that $\rho \in I$, it suffices to show that any sequence $(y_j)_{j\in\mathbb{N}}$ in $\bar{B}(z,\rho)$ has a subsequence converging to a point of $\bar{B}(z,\rho)$. Let $(\varepsilon_j)_{j\in\mathbb{N}}$, $0 < \varepsilon_j < \rho$, be a decreasing sequence tending to 0. Since X is a length space, there exists, for each $i, j \in \mathbb{N}$, a point $x_j^i \in \bar{B}(z, \rho - \varepsilon_i/2)$ such that $|x_j^i - y_j| \leq \varepsilon_i$. Such a point x_j^i exists since otherwise $\bar{B}(y_j, \varepsilon_i) \cap \bar{B}(z, \rho - \varepsilon_i/2) = \emptyset$ and consequently all paths from z to y_j would be of length at least $\rho + \varepsilon_i/2$ which is a contradiction since X is a length space and $|z - y_j| \leq \rho$. (To find x_j^i choose a path from z to y_j of length $< |z - y_j| + \varepsilon_i/2$ and then choose an appropriate point x_j^i on this path.) Since $\bar{B}(z, \rho - \varepsilon_1/2)$ is compact, the sequence (x_j^1) has a convergent subsequence $(x_{j_k^1}^1)$. Similarly, the sequence $(x_{j_k^1}^2)$ has a convergent subsequence $(x_{j_k^2}^2)$ and the sequence $(x_{j_k^2}^3)$ has a convergent subsequence $(x_{j_k^3}^3)$, and so forth. Continuing this way, we obtain by a diagonal process an increasing sequence of $n_k \nearrow \infty$ such that $(x_{n_k}^i)$ converges for every $i \in \mathbb{N}$. We claim that the associated sequence $y_{n_k} \in X$ is a Cauchy-sequence. Let $\varepsilon > 0$ and choose $i \in \mathbb{N}$ such that $\varepsilon_i < \varepsilon/3$. Since $(x_{n_k}^i)$ converges, it is a Cauchy-sequence and hence there exists $m \in \mathbb{N}$ such that

$$|x_{n_k}^i - x_{n_l}^i| < \varepsilon/3 \quad \text{for all } n_k, n_l \geq m.$$

Then

$$|y_{n_k} - y_{n_l}| \leq |y_{n_k} - x_{n_k}^i| + |x_{n_k}^i - x_{n_l}^i| + |x_{n_l}^i - y_{n_l}| < \varepsilon$$

for $n_k, n_l \geq m$. Hence (y_{n_k}) is a Cauchy-sequence. It converges (to a point in $\bar{B}(z, \rho)$) since X is complete. We have proved that every sequence in $\bar{B}(z, \rho)$ has a convergent subsequence. Hence $\bar{B}(z, \rho)$ is compact, and so $\rho \in I$. Thus I is both open and closed in $[0, \infty)$, so $I = [0, \infty)$.

It remains to prove that X is geodesic. Let $x, y \in X$. Since X is a length space, there exists for each $j \in \mathbb{N}$ a point $z_j \in X$ such that

$$\frac{1}{2}|x - y| \leq \max\{|x - z_j|, |y - z_j|\} \leq \frac{1}{2}|x - y| + 1/j.$$

The points z_j belong to a compact set $\bar{B}(x, \frac{1}{2}|x - y| + 1)$ and hence there exists a subsequence converging to a point z which satisfies

$$|x - z| = |y - z| = \frac{1}{2}|x - y|.$$

It follows now from Theorem 1.64 that X is geodesic. □

Theorem 1.74. *A length space is proper if and only if it is complete and locally compact.*

Proof. Suppose that X is a proper length space. Since each closed ball $\bar{B}(x, r)$ is compact, X is locally compact. Let $(x_j)_{j \in \mathbb{N}}$ be a Cauchy-sequence in X. Then $x_j \in \bar{B}(x, r)$ for some $x \in X$ and $r > 0$. Since $\bar{B}(x, r)$ is compact, there exists a convergent subsequence of (x_j). But (x_j) is Cauchy, so the whole sequence converges. Thus X is complete. The other direction of the claim follows from Theorem 1.73. □

Let $I \subset \mathbb{R}$ be an interval. We call a path $\gamma \colon I \to X$ a *linearly reparameterized geodesic* or a *constant speed geodesic* if there exists a constant $\lambda \geq 0$ such that

$$|\gamma(t) - \gamma(s)| = \lambda |t - s|$$

for all $t, s \in I$. Similarly, $\gamma \colon I \to X$ is called a *local linearly reparameterized geodesic* (or a *local constant speed geodesic*) if for each $t \in I$ there exists $\delta > 0$ such that $\gamma | I \cap [t - \delta, t + \delta]$ is a linearly reparameterized geodesic.

Theorem 1.75. *Let x and y be points in a proper metric spaces X. Suppose that there exists a unique geodesic $\sigma\colon [0,\ell] \to X$ joining x and y in X. Let $\gamma\colon [0,1] \to X,\ \gamma(t) = \sigma(t\ell)$, be a linearly reparameterized geodesic. Let $\gamma_k\colon [0,1] \to X,\ k \in \mathbb{N}$, be linearly reparameterized geodesics such that $\gamma_k(0) \to x$ and $\gamma_k(1) \to y$. Then $\gamma_k \to \gamma$ uniformly.*

Proof. Let $R > 0$ be so large that the images $\gamma_k([0,1])$ are contained in the compact ball $\bar{B}(x,R)$. For all $t,s \in [0,1]$

$$|\gamma_k(t) - \gamma_k(s)| = \lambda_k|t-s|, \tag{1.76}$$

where

$$\lambda_k = |\gamma_k(0) - \gamma_k(1)| \le 2R. \tag{1.77}$$

Hence (γ_k) is equicontinuous. Assume first that the sequence γ_k does not converge pointwise to γ. Then there exist $t_0 \in (0,1),\ \varepsilon > 0$, and a subsequence (γ_{k_j}) such that

$$|\gamma_{k_j}(t_0) - \gamma(t_0)| \ge \varepsilon \quad \text{for all } k_j.$$

By the Arzelà-Ascoli theorem (Lemma 1.69) there exists a subsequence of (γ_{k_j}) converging uniformly to a path $\bar\gamma\colon [0,1] \to X$ joining x to y with

$$|\bar\gamma(t_0) - \gamma(t_0)| \ge \varepsilon.$$

From (1.76) and (1.77) we get that

$$|\bar\gamma(t) - \bar\gamma(s)| = |x-y||t-s|$$

for all $t,s \in [0,1]$. Hence $\bar\gamma$ is a linearly reparameterized geodesic from x to y and $\bar\gamma \neq \gamma$ which contradicts with the uniqueness of γ. (Note that the uniqueness of σ implies the uniqueness of $\gamma\colon [0,1] \to X$.) Hence γ_k converges pointwise to γ. The convergence is uniform by equicontinuity (see the end of the proof of the Arzelà-Ascoli theorem). □

A path $\gamma\colon [0,\ell] \to X$ is called a *loop* (or *closed*) if $\gamma(0) = \gamma(\ell)$. It can be extended to a periodic path $\tilde\gamma\colon \mathbb{R} \to X$ by setting $\tilde\gamma(t + k\ell) = \gamma(t)$ for $t \in [0,\ell]$ and $k \in \mathbb{Z}$. A loop $\gamma\colon [0,1] \to X$ is called a *closed (linearly reparameterized) geodesic* if $\tilde\gamma$ is a local (linearly reparameterized) geodesic.

We say that a path-connected topological space Y is *semi-locally simply connected* if each point $y \in Y$ has a neighborhood U such that each closed path in U is homotopic to a constant path in Y. Recall that paths $\gamma\colon [0,1] \to Y$ and $\sigma\colon [0,1] \to Y$, with $\gamma(0) = \sigma(0),\ \gamma(1) = \sigma(1)$, are *homotopic in* Y, denoted by $\gamma \simeq \sigma$, if there exists a continuous map $h\colon [0,1] \times [0,1] \to Y$ such that $h(\cdot,0) = \gamma,\ h(\cdot,1) = \sigma,\ h(0,\cdot) = \gamma(0) = \sigma(0)$, and $h(1,\cdot) = \gamma(1) = \sigma(1)$.

Theorem 1.78. *If X is a compact length space that is semi-locally simply connected, then every closed path $\sigma\colon [0,1] \to X$ is homotopic to a closed linearly reparameterized geodesic or homotopic to a constant path.*

Proof. The assumption that X is compact and semi-locally simply connected implies the existence of $r > 0$ such that every loop of length less that r is homotopic to a constant path. Indeed, for each $x \in X$ there is $r_x > 0$ such that every loop in $B(x, 2r_x)$ is homotopic to a constant path. By compactness, $X = \cup_{i=1}^k B(x_i, r_{x_i})$. If $r = \min\{r_{x_i}\colon i = 1,\dots,k\}$, then every loop of length $< r$ belongs to some $B(x_i, 2r_{x_i})$ and thus is homotopic to a constant path.

Suppose that σ is a loop which is not homotopic to a constant path. Then

$$\ell = \inf\{\ell(\gamma)\colon \gamma\colon [0,1] \to X, \gamma \simeq \sigma\} \geq r > 0.$$

Furthermore, $\ell < \infty$ since there exists a rectifiable loop which is homotopic to σ. (This holds since X is a semi-locally simply connected length space [Exercise: Prove this statement.].) We want to show that σ is homotopic to a closed linearly reparameterized geodesic. Choose a sequence of loops $\sigma_i\colon [0,1] \to X$ such that $\sigma_i \simeq \sigma$, $\ell(\sigma_i) \to \ell$, and that each σ_i has a constant speed (i.e. $|\dot{\sigma}_i|(t) \equiv \ell(\sigma_i)$). Then the sequence (σ_i) is equicontinuous since for all $t, s \in [0,1]$

$$|\sigma_i(t) - \sigma_i(s)| \leq \ell(\sigma_i)|t - s|$$

and $\ell(\sigma_i) \to \ell$. By the Arzelà-Ascoli theorem (σ_i) has a subsequence, still denoted by (σ_i), converging uniformly to an ℓ-Lipschitz path $\bar{\sigma}\colon [0,1] \to X$. It remains to show that $\bar{\sigma} \simeq \sigma$ and that $\bar{\sigma}$ is a closed linearly reparameterized geodesic. Choose $n \in \mathbb{N}$ such that $|\sigma_n(t) - \bar{\sigma}(t)| < r/4$ for all $t \in [0,1]$. Then we choose $0 = t_0 < t_1 < \cdots < t_m = 1$ such that $\ell(\sigma_n|[t_{k-1}, t_k]) < r/4$ and $\ell(\bar{\sigma}|[t_{k-1}, t_k]) < r/4$ for $k = 1, \ldots, m$. Since X is a length space, we may then choose paths γ_k from $\sigma_n(t_k)$ to $\bar{\sigma}(t_k)$ of length $< r/4$. We obtain m loops of length $< r$. Hence they are all homotopic to constant paths, and consequently $\bar{\sigma} \simeq \sigma_n \simeq \sigma$. Furthermore, $\ell(\bar{\sigma}) \geq \ell$ by definition. On the other hand, $\ell(\bar{\sigma}) \leq \ell$ since $\bar{\sigma}$ is ℓ-Lipschitz. Hence $\bar{\sigma}$ has the minimum length among paths homotopic to σ. It follows that $\bar{\sigma}$ is a closed linearly reparameterized geodesic. □

1.79 Constructions

Next we describe some basic constructions of new metric (length) spaces from given ones.

We say that $d\colon X \times X \to [0, \infty]$ is a *generalized metric* if it satisfies all the axioms of the metric except that the value $d(x, y) = \infty$ is allowed. The pair (X, d) is then called a *generalized metric space*. In particular, if (X, d) is a metric space, then (X, d_s) is (always) a generalized metric space. Here d_s is given by Definition 1.54 and called the *generalized length (inner) metric* associated to d.

The *product* of metric spaces (X_1, d_1) and (X_2, d_2) is the set $X = X_1 \times X_2$ with the metric

$$d\big((x_1, x_2), (y_1, y_2)\big) = \big(d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2\big)^{1/2}.$$

Theorem 1.80. *Let X_1 and X_2 be metric spaces and let X be their product (metric space). Then we have:*

(1) X is complete if and only if X_1 and X_2 are complete,

(2) X is a length space if and only if X_1 and X_2 are length spaces,

(3) X is a geodesic spaces if and only if X_1 and X_2 are geodesic spaces,

(4) a path $\gamma\colon I \to X$, $\gamma = (\gamma_1, \gamma_2)$, is a constant speed geodesic if and only if γ_1 and γ_2 are constant speed geodesics.

Proof. The claim (1) is obvious.

To prove (2) and (3), suppose first that X is a length space. Fix $x, y \in X_1$, $\varepsilon > 0$, and $z_2 \in X_2$. Since X is a length space, there is a path $\gamma\colon [a,b] \to X$ from (x, z_2) to (y, z_2) of length $\leq d((x, y_0), (y, y_0)) + \varepsilon = d_1(x, y) + \varepsilon$. The projection $\pi\colon X \to X_1$, $\pi(x_1, x_2) = x_1$, is 1-Lipschitz and therefore $\pi \circ \gamma$ is a path in X_1 from x to y of length $\leq d_1(x, y) + \varepsilon$. This shows that X_1 is a length space. Similarly, X_2 is a length space. If X is a geodesic space, the argument above with $\varepsilon = 0$ shows that X_1 and X_2 are geodesic spaces.

Suppose then that X_1 and X_2 are length spaces. Let $(x_1, x_2) \in X$ and $(y_1, y_2) \in X$. There are sequences $(\gamma_{i,1})$ and $(\gamma_{i,2})$ of unit speed paths $\gamma_{i,j}\colon [0, \ell_{i,j}] \to X_j,\ j = 1, 2$, such that $\gamma_{i,j}(0) = x_j,\ \gamma_{i,j}(\ell_{i,j}) = y_j$ and that $\ell_{i,j} \to d_j(x_j, y_j),\ j = 1, 2$. Now the mappings

$$\gamma_i\colon [0, \ell_{i,1}] \times [0, \ell_{i,2}] \to X, \quad \gamma_i(t_1, t_2) = \big(\gamma_{i,1}(t_1), \gamma_{i,2}(t_2)\big),$$

are 1-Lipschitz. For each $i \in \mathbb{N}$ define

$$\sigma_i\colon [0, 1] \to [0, \ell_{i,1}] \times [0, \ell_{i,2}], \quad \sigma_i(t) = (t\ell_{i,1}, t\ell_{i,2}).$$

Then $\gamma_i \circ \sigma_i\colon [0, 1] \to X$ is a path from (x_1, x_2) to (y_1, y_2) of length $\le \big(\ell_{i,1}^2 + \ell_{i,2}^2\big)^{1/2}$, where

$$\big(\ell_{i,1}^2 + \ell_{i,2}^2\big)^{1/2} \to d\big((x_1, x_2), (y_1, y_2)\big)$$

as $i \to \infty$. Thus X is a length space.

If X_1 and X_2 are geodesic spaces, we choose geodesics $\gamma_j\colon [0, \ell_j] \to X_j,\ \ell_j = d_j(x_j, y_j)$, from x_j to $y_j,\ j = 1, 2$, and apply the method above to find a geodesic in X from (x_1, x_2) to (y_1, y_2) of length $\sqrt{\ell_1^2 + \ell_2^2} = d\big((x_1, x_2), (y_1, y_2)\big)$. This shows that X is a geodesic space.

To prove (4) we first observe that an easy calculation shows that $\gamma = (\gamma_1, \gamma_2)$ is a constant speed geodesic if γ_1 and γ_2 are constant speed geodesics. Suppose then that $\gamma = (\gamma_1, \gamma_2)$ is a constant speed geodesic. We use the following characterization (whose proof is left as an exercise): A path $\sigma\colon I \to X$ is a constant speed geodesic if and only if

$$|\sigma(t) - \sigma(s)| = 2|\sigma(t) - \sigma((t + s)/2)|$$

for all $t, s \in I$. Given $t, s \in I$ we denote

$$\begin{aligned}
x &= (x_1, x_2) = \big(\gamma_1(t), \gamma_2(t)\big),\\
y &= (y_1, y_2) = \big(\gamma_1(s), \gamma_2(s)\big),\\
m &= (m_1, m_2) = \big(\gamma_1((t + s)/2), \gamma_2((t + s)/2)\big),\\
a_i &= d_i(x_i, m_i),\\
b_i &= d_i(m_i, y_i),\\
c_i &= d_i(x_i, y_i).
\end{aligned}$$

We have

$$\frac{1}{2} d(x, y) = d(x, m) = d(m, y),$$

and hence

$$\frac{1}{2}(c_1^2 + c_2^2) = a_1^2 + b_1^2 + a_2^2 + b_2^2.$$

On the other hand,

$$a_i^2 + b_i^2 \ge \frac{1}{2}(a_i + b_i)^2 \ge \frac{1}{2} c_i^2, \tag{1.81}$$

(where the last inequality follows from the triangle inequality) and therefore

$$\frac{1}{2}(c_1^2 + c_2^2) \le a_1^2 + b_1^2 + a_2^2 + b_2^2 = \frac{1}{2}(c_1^2 + c_2^2).$$

Hence there must be an equality in (1.81) for $i = 1, 2$ which is possible if and only if $a_i = b_i = \frac{1}{2} c_i$. Thus γ_1 and γ_2 are constant speed geodesics and (4) follows. □

Let $(X_\alpha, d_\alpha), \alpha \in A$, be a family of (generalized) metric spaces. Their *disjoint union* is the generalized metric space (X, d), where

$$X = \bigsqcup_{\alpha \in A} X_\alpha = \bigcup_{\alpha \in A} X_\alpha \times \{\alpha\}$$

equipped with the generalized metric

$$d\big((x, \alpha), (x', \alpha')\big) = \begin{cases} d_\alpha(x, x'), & \text{if } \alpha = \alpha', \\ \infty, & \text{otherwise.} \end{cases}$$

Let X be a generalized metric space, $\sim$ an equivalence relation in X, and let

$$\bar{X} = X/\!\sim$$

be the set of equivalence classes. We define $\bar{d} \colon \bar{X} \times \bar{X} \to [0, \infty]$ by setting

$$\bar{d}(\bar{x}, \bar{y}) = \inf \sum_{i=1}^{k} |x_i - y_i|,$$

where the infimum is taken over all sequences $x_1, y_1, \ldots, x_k, y_k,\ k \in \mathbb{N}$, with $x_1 \in \bar{x},\ y_k \in \bar{y}$, and $y_j \sim x_{j+1}$ for $j = 1, \ldots, k-1$. (Think of equivalence classes as islands, pairs x_j, y_j as bridges, and $\bar{d}(\bar{x}, \bar{y})$ as the infimum of total lengths of bridges needed to connect the island $\bar{x}$ to the island $\bar{y}$.)

It is obvious that $\bar{d}$ is symmetric and satisfies the triangle inequality, but in general $\bar{d}$ is only a (generalized) pseudometric rather than a metric. For instance, if there exists an equivalence class which is dense in X, then $\bar{d}$ is identically zero. We call $\bar{d}$ the *(generalized) quotient pseudometric* on $\bar{X}$ associated to $\sim$. Note that $\bar{d}(\bar{x}, \bar{y}) \le |x - y|$ for all $x, y \in X$ and $\bar{d}(\bar{x}, \bar{y}) \le \operatorname{dist}(\bar{x}, \bar{y})$.

Theorem 1.82. *Suppose that (X, d) is a generalized metric space, $\sim$ is an equivalence relation in X, and let $\bar{d}$ be the generalized quotient pseudometric on $\bar{X} = X/\!\sim$.*

(1) Suppose that for every equivalence class $\bar{x} \subset X$ there exists $\varepsilon(\bar{x}) > 0$ such that

$$B(\bar{x}, \delta) := \{y \in X \colon\ \operatorname{dist}(y, \bar{x}) < \delta\}$$

is a union of equivalence classes for all $0 < \delta \le \varepsilon(\bar{x})$. Then

$$\bar{d}(\bar{x}, \bar{y}) = \operatorname{dist}(\bar{x}, \bar{y}) \quad \textit{whenever } \bar{x}, \bar{y} \in \bar{X} \textit{ and } \bar{d}(\bar{x}, \bar{y}) < \varepsilon(\bar{x}). \tag{1.83}$$

If, in addition, every equivalence class $\bar{x} \subset X$ is closed, then $\bar{d}$ is a generalized metric on $\bar{X}$.

(2) If (X, d) is a length space and $\bar{d}$ is a metric, then $(\bar{X}, \bar{d})$ is a length space.

Proof. (1) Let $\bar{x}, \bar{y}, \in \bar{X}, \bar{x} \ne \bar{y}, \bar{d}(\bar{x}, \bar{y}) < \varepsilon(\bar{x})$. First we show that

$$\operatorname{dist}(z, \bar{x}) = \operatorname{dist}(z', \bar{x}) \tag{1.84}$$

whenever $z, z' \in \bar{z}$ and $\operatorname{dist}(\bar{x}, \bar{z}) < \varepsilon(\bar{x})$. Choose $z \in \bar{z}$ and $\delta > 0$ such that

$$\operatorname{dist}(\bar{x}, z) < \delta < \varepsilon(\bar{x}).$$

Since $B(\bar{x}, \delta)$ is a union of equivalence classes and $z \in B(\bar{x}, \delta)$, it follows that $z' \in B(\bar{x}, \delta)$, hence $\operatorname{dist}(\bar{x}, z') < \delta$, for all $z' \sim z$. This holds for all $\delta > \operatorname{dist}(\bar{x}, z)$, and so

$$\operatorname{dist}(\bar{x}, z') \le \operatorname{dist}(\bar{x}, z) \quad \text{for all } z' \sim z.$$

Thus (1.84) holds.

Choose $\varepsilon > 0$ such that $\bar{d}(\bar{x}, \bar{y}) + \varepsilon < \varepsilon(\bar{x})$ and then points $x_1, y_1, \ldots, x_k, y_k$ in X such that $x_1 \in \bar{x}, y_k \in \bar{y}, y_i \sim x_{i+1}$ for $i = 1, \ldots, k-1$, and

$$\sum_{i=1}^{k} |x_i - y_i| \leq \bar{d}(\bar{x}, \bar{y}) + \varepsilon < \varepsilon(\bar{x}).$$

Next we show by induction that

$$\operatorname{dist}(y_j, \bar{x}) \leq \sum_{i=1}^{j} |x_i - y_i| \tag{1.85}$$

for all $j = 1, \ldots, k$. If $j = 1$, then $x_j = x_1 \in \bar{x}$ and hence $\operatorname{dist}(y_1, \bar{x}) \leq |x_1 - y_1|$. Suppose that

$$\operatorname{dist}(y_{j-1}, \bar{x}) \leq \sum_{i=1}^{j-1} |x_i - y_i|.$$

Since $x_j \sim y_{j-1}$ and $\operatorname{dist}(y_{j-1}, \bar{x}) < \varepsilon(\bar{x})$, we get from (1.84) that

$$\begin{aligned} \operatorname{dist}(y_j, \bar{x}) &\leq |x_j - y_j| + \operatorname{dist}(x_j, \bar{x}) \\ &= |x_j - y_j| + \operatorname{dist}(y_{j-1}, \bar{x}) \\ &\leq \sum_{i=1}^{j} |x_i - y_i|. \end{aligned}$$

By (1.85), we now have

$$\bar{d}(\bar{x}, \bar{y}) \leq \operatorname{dist}(\bar{x}, \bar{y}) \leq \operatorname{dist}(\bar{x}, y_k) \leq \sum_{i=1}^{k} |x_i - y_i| \leq \bar{d}(\bar{x}, \bar{y}) + \varepsilon.$$

This holds for every $\varepsilon > 0$, and hence

$$\bar{d}(\bar{x}, \bar{y}) = \operatorname{dist}(\bar{x}, \bar{y}).$$

If $\bar{x} \subset X$ is closed and $y \notin \bar{x}$, then $\operatorname{dist}(y, \bar{x}) > 0$. If $\operatorname{dist}(\bar{x}, \bar{y}) = \bar{d}(\bar{x}, \bar{y}) < \varepsilon(\bar{x})$, then

$$\operatorname{dist}(\bar{x}, \bar{y}) = \inf_{y' \in \bar{y}} \operatorname{dist}(y', \bar{x}) = \operatorname{dist}(y, \bar{x}) > 0.$$

Thus $\bar{d}$ is a generalized metric.

(2) Suppose that $0 < \bar{d}(\bar{x}, \bar{y}) < \infty$. Let $\varepsilon > 0$ and choose $x_1, y_1, \ldots, x_k, y_k \in X$ such that $x_1 \in \bar{x}, y_k \in \bar{y}, y_j \sim x_{j+1}$ for $j = 1, \ldots, k-1$, and

$$\sum_{j=1}^{k} |x_j - y_j| < \bar{d}(\bar{x}, \bar{y}) + \varepsilon.$$

Since X is a length space, there are paths $\sigma_j \colon [j-1, j] \to X$ from x_j to y_j, with

$$\ell(\sigma_j) \leq |x_j - y_j| + \varepsilon/k, \quad j = 1, \ldots, k.$$

Let $\pi\colon X \to \bar{X}$ be the canonical projection and let $\bar{\sigma}\colon [0,k]$ be defined by $\bar{\sigma}|[j-1,j] = \pi\circ\sigma_j$. Then $\bar{\sigma}$ is a path in $\bar{X}$ from $\bar{x}$ to $\bar{y}$. Furthermore, π is 1-Lipschitz and hence

$$\begin{aligned}\ell(\bar{\sigma}) &= \sum_{j=1}^{k}\ell(\pi\circ\sigma_j) \le \sum_{j=1}^{k}\ell(\sigma_j)\\ &\le \sum_{j=1}^{k}|x_j - y_j| + \varepsilon\\ &< \bar{d}(\bar{x},\bar{y}) + 2\varepsilon.\end{aligned}$$

Hence $(\bar{X},\bar{d})$ is a length space. ◻

Example 1.86. 1. Let $\bar{X} = \mathbb{R}/\sim$, where $x \sim y \iff y - x \in \mathbb{Z}$ (i.e. $\bar{X} = \mathbb{R}/\mathbb{Z}$). Then $\bar{d}(\bar{x},\bar{y}) = \operatorname{dist}(\bar{x},\bar{y})$ and $(\bar{X},\bar{d})$ is a geodesic space isometric to a circle of length 1 (equipped with the inner metric).

2. *Metric graphs.* A *combinatorial graph* consists of two set V (vertices) and E (edges), where each edge $e \in E$ connects a pair of vertices. More precisely, consider two set E and V and (endpoint) maps $\partial_j\colon E \to V,\ j = 0,1$, such that $V = \partial_0 E \cup \partial_1 E$. Let $\sim$ be the equivalence relation in

$$\bigsqcup_{e\in E}[0,1] = \bigcup_{e\in E}[0,1]\times\{e\} = [0,1]\times E$$

such that

$$(i,e) \sim (j,e') \quad \text{if } i,j \in \{0,1\},\ e,e' \in E, \text{ and } \partial_i e = \partial_j e',$$

and that $(t,e) \sim (t,e)$ for all $(t,e) \in [0,1]\times E$. Let $X = [0,1]\times E/\sim$ and let $\pi\colon [0,1]\times E \to X$ be the canonical projection. We identify V with $\pi(\{0,1\}\times E)$.

To define a metric in X, fix a mapping $\ell\colon E \to (0,\infty)$. It assigns to each edge $e \in E$ a length $\ell(e)$. A *piecewise linear path* is a map $\gamma\colon [0,1] \to X$ such that for some partition $0 = t_0 \le t_1 \le \cdots \le t_n = 1$,

$$\gamma|[t_i,t_{i+1}](t) = \pi(c_i(t),e_i),$$

where $e_i \in E$ and $c_i\colon [t_i,t_{i+1}] \to [0,1]$ is affine such that $c_i(t_i) = 0$ and $c_{i-1}(t_i) = 1$ for $i = 1,\dots,n-1$. Note that

$$\pi(c_i(t_{i+1}),e_i) = \gamma(t_{i+1}) = \pi(c_{i+1}(t_{i+1}),e_{i+1})$$

for $i = 0,\dots,n-2$. Hence e_i and e_{i+1} have a common endpoint. We say that γ *joins x to y* if $\gamma(0) = x$ and $\gamma(1) = y$. We assume that X is *connected*, that is any two points in X can be joined by such γ. The *length* of γ is defined by

$$\ell(\gamma) = \sum_{i=0}^{n-1}\ell(e_i)|c_i(t_i) - c_{i+1}(t_{i+1})|.$$

We define a pseudometric $d\colon X\times X \to [0,\infty)$ by setting

$$d(x,y) = \inf_{\gamma}\ell(\gamma),$$

where the infimum is taken over all piecewise linear paths γ joining x to y. The pseudometric space (X, d) is called a *metric graph*. If, for all $v \in V$,

$$\inf\{\ell(e)\colon e \in E,\ v \in \{\partial_0 e, \partial_1 e\}\} > 0,$$

then (X, d) is a metric space, in fact, a length space. If the set $\{\ell(e)\colon e \in E\}$ is finite, then (X, d) is a complete geodesic space. A simply connected metric graph, with $\ell(e) \equiv 1$, is called a *tree*.

Let $(X_\alpha, d_\alpha)_{\alpha\in\mathcal{A}}$ be a family of metric spaces. Suppose that there exist a metric space Z and isometries $i_\alpha\colon Z \to Z_\alpha$ onto closed subsets $Z_\alpha \subset X_\alpha$ for each $\alpha \in \mathcal{A}$. Let $\sim$ be the equivalence relation in

$$\bigsqcup_{\alpha\in\mathcal{A}} X_\alpha$$

such that $i_\alpha(z) \sim i_\beta(z)$ for all $z \in Z$ and $\alpha, \beta \in \mathcal{A}$. The quotient space

$$\bar{X} = \bigsqcup_{\alpha\in\mathcal{A}} X_\alpha/{\sim}$$

equipped with the quotient pseudometric $\bar{d}$ is called the *isometric gluing of X_α's along Z.*

Theorem 1.87. *Let $(X_\alpha, d_\alpha)_{\alpha\in\mathcal{A}}$, Z, $i_\alpha\colon Z \to Z_\alpha$, and $(\bar{X}, \bar{d})$ be as above. Then we have:*

(1) $\bar{d}$ is a metric in $\bar{X}$.

(2) For all $x \in X_\alpha$, $y \in X_\beta$

$$\bar{d}(\bar{x}, \bar{y}) = \begin{cases} d_\alpha(x, y) & \text{if } \alpha = \beta, \\ \inf\{d_\alpha(x, i_\alpha(z)) + d_\beta(i_\beta(z), y)\colon z \in Z\} & \text{if } \alpha \neq \beta. \end{cases} \tag{1.88}$$

(3) If each X_α is a length space, then $\bar{X}$ is a length space.

(4) If each X_α is a geodesic space and Z is proper, then $\bar{X}$ is a geodesic space.

Proof. Suppose that $\bar{x} \subset \sqcup_\alpha X_\alpha$. If $\bar{x} = [x_\alpha]$ for some $x_\alpha \in X_\alpha \setminus Z_\alpha$, then $\bar{x}$ is the singleton $\{x_\alpha\}$. Furthermore, there exists an open ball $B(x_\alpha, r) \subset X_\alpha \setminus Z_\alpha$ since Z_α is closed. Now the δ-neighborhood $B(\bar{x}, \delta)$ is the ball $B(x_\alpha, \delta)$ which is a union of equivalence classes (= singletons) for all $0 < \delta \leq r$. On the other hand, if $\bar{x} = [i_\alpha(z)]$ for some $z \in Z$, then

$$\bar{x} = \bigcup_{\alpha\in\mathcal{A}} \{i_\alpha(z)\}$$

and the δ-neighborhood of $\bar{x}$ can be expressed as

$$B(\bar{x}, \delta) = \bigcup_{\alpha\in\mathcal{A}} B(i_\alpha(z), \delta) = \bigcup_{\alpha\in\mathcal{A}} \big(B(i_\alpha(z), \delta) \setminus Z_\alpha\big) \cup \bigcup_{\alpha\in\mathcal{A}} \big(B(i_\alpha(z), \delta) \cap Z_\alpha\big).$$

Here the first union is a union of equivalence classes (= singletons). The second union can be expressed as

$$\bigcup_{\alpha\in\mathcal{A}} B(i_\alpha(z), \delta) \cap Z_\alpha = \bigcup_{x\in B(z,\delta)} [i_\alpha(x)],$$

where $B(z, \delta)$ is a ball in Z. Thus $B(\bar{x}, \delta)$ is a union of equivalence classes for all $\delta > 0$. Next we show that all equivalence classes are closed in $(\sqcup_\alpha X_\alpha, d)$. If $\bar{x} = [x]$ for some $x \in X_\alpha \setminus Z_\alpha$, then $\bar{x} = \{x\}$, which is closed. If $\bar{x} = [i_\alpha(z)]$ for some $z \in Z$, then $\bar{x} = \cup_\alpha \{i_\alpha(z)\}$ and

$$\bigsqcup_\alpha X_\alpha \setminus \bar{x} = \bigsqcup_\alpha \big(X_\alpha \setminus i_\alpha(z)\big),$$

which is open. Hence $\bar{x}$ is closed. It now follows from Theorem 1.82 (1) that $\bar{d}$ is a metric.

To verify the equation (1.88) for $\bar{d}$, it suffices to notice that any sequence $x'_1, y'_1, \ldots, x'_k, y'_k$ in the definition of $\bar{d}(\bar{x}, \bar{y})$, with

$$\sum_{i=1}^{k} |x'_i - y'_i| < \infty,$$

can be replaced by a sequence x_1, y_1, x_2, y_2, with $x_1 \in \bar{x}, y_1 \sim x_2, y_2 \in \bar{y}$, and

$$|x_1 - y_1| + |x_2 - y_2| \le \sum_{i=1}^{k} |x'_i - y'_i|.$$

Thus (1.88) holds.

(3) To show that $(\bar{X}, \bar{d})$ is a length space, let $\bar{x}, \bar{y}$ and $\varepsilon > 0$. Then $\bar{x} = [x]$ and $\bar{y} = [y]$ for some $x \in X_\alpha$ and $y \in X_\beta$. If $\alpha = \beta$, we may join x and y in X_α by a path γ of length $\le d_\alpha(x, y) + \varepsilon$. Then $\pi \circ \gamma$ is a path of length $\le \bar{d}(\bar{x}, \bar{y}) + \varepsilon$ in $\bar{X}$ joining $\bar{x}$ and $\bar{y}$. If $\alpha \neq \beta$, choose $z \in Z$ such that $d_\alpha(x, i_\alpha(z)) + d_\beta(i_\beta(z), y) \le \bar{d}(\bar{x}, \bar{y}) + \varepsilon/2$ and then paths γ_α and γ_β joining x and $i_\alpha(z)$ in X_α and, respectively, $i_\beta(z)$ and y in X_β such that $\ell(\gamma_\alpha) \le d_\alpha(x, i_\alpha(z)) + \varepsilon/4$ and $\ell(\gamma_\beta) \le d_\beta(i_\beta(z), y) + \varepsilon/4$. Composing these paths with π gives a path in $\bar{X}$ of length $\le \bar{d}(\bar{x}, \bar{y}) + \varepsilon$ joining $\bar{x}$ and $\bar{y}$. Hence $\bar{X}$ is a length space.

(4) Let $\bar{x} = [x]$ and $\bar{y} = [y] \in \bar{X}$, with $x \in X_\alpha,\ y \in X_\beta$. If $\alpha = \beta$, there exists a geodesic γ in X_α joining x and y. Then $\pi \circ \gamma$ is a geodesic in $\bar{X}$ joining $\bar{x}$ and $\bar{y}$. Suppose then that $\alpha \neq \beta$. For each $j \in \mathbb{N}$, choose $z_j \in Z$ such that

$$\bar{d}(\bar{x}, \bar{y}) \le d_\alpha(x, i_\alpha(z_j)) + d_\beta(i_\beta(z_j), y) \le \bar{d}(\bar{x}, \bar{y}) + 1/j.$$

The points z_j belong to a closed bounded set

$$i_\alpha^{-1}\big(\bar{B}(x, 2\bar{d}(\bar{x}, \bar{y})) \cap Z_\alpha\big)$$

which is compact since Z is proper. Hence there exists a subsequence of (z_j) converging to a point $z \in Z$ which satisfies

$$\bar{d}(\bar{x}, \bar{y}) = d_\alpha(x, i_\alpha(z)) + d_\beta(i_\beta(x), y).$$

Since X_α and X_β are geodesic spaces, there are geodesics γ_α and γ_β joining x_α and $i_\alpha(z)$ in X_α and $i_\beta(z)$ and y in X_β, respectively. Composing these geodesics with π gives a geodesic in $\bar{X}$ joining $\bar{x}$ and $\bar{y}$. $\square$

1.89 Group actions and coverings

[Lectures (Feb. 20, 22) were given by Pekka Pankka. Notes are written by him.]

In this section we study quotient spaces, which arise from group actions, and then an inverse question which leads us to covering mappings and covering spaces.

Let us begin with some examples on groups defined on a metric space. Let X be a metric space, and let

$$\begin{aligned} \mathrm{Isom}(X) &= \{f\colon X \to X \colon f \text{ is a surjective isometry}\} \\ \mathrm{BL}(X) &= \{f\colon X \to X \colon f \text{ is surjective and bilipschitz}\} \\ \mathrm{Homeo}(X) &= \{f\colon X \to X \colon f \text{ is homeomorphism}\}. \end{aligned}$$

These sets have a natural group structure given by the composition of mappings. Furthermore, $\mathrm{Isom}(X) \subset \mathrm{BL}(X) \subset \mathrm{Homeo}(X)$ also as groups. In this section we concentrate on subgroups of $\mathrm{Isom}(X)$ and their quotient spaces. Let us begin with some terminology.

Definition 1.90. Let G be a subgroup of $\mathrm{Homeo}(X)$. We call the map

$$G \times X \to X, \quad (g, x) \mapsto g(x),$$

the action of G *on* X. In general, if G is a group and $\Psi\colon G \to \mathrm{Homeo}(X)$ is a homomorphism, we call the map

$$G \times X \to X, \quad (g, x) \mapsto (\Psi(g))(x),$$

the action of G *(via* Ψ*) on* X. In this case we usually identify group elements with their images and denote $g(x) := (\Psi(g))(x)$.

Convention: We denote the neutral element of the group always by e.

Definition 1.91. An action of G on X is

(1) *free*, if $g(x) \neq x$ for every $x \in X$ and $g \in G \setminus \{e\}$.

(2) *proper*, if for every $x \in X$ there exists a neighborhood U of x in X such that $gU \cap U \neq \emptyset$ for only finitely many elements in G.

Example 1.92. (1) Let $X = \mathbb{R}^2$ and let G be the group spanned by mappings $(x, y) \mapsto (x+1, y)$ and $(x, y) \mapsto (x, y+1)$. Then G is a subgroup of $\mathrm{Isom}(X)$ isomorphic to $\mathbb{Z}^2$ and it acts on X freely and properly. *Exercise:* Check this statement.

(2) Let $X = \mathbb{R}^2 = \mathbb{C}$ and let G be the subgroup of $\mathrm{Isom}(X)$ spanned by the mapping (in complex notation) $z \mapsto e^{i2\pi/3}z$. Since G has three elements, the action of G on X is necessarily proper, but it is not free. *Exercise:* Check.

(3) Let $X = \mathbb{R}^2 = \mathbb{C}$ and let G be the group of mappings $z \mapsto e^{it}z$, where $t \in \mathbb{R}$. Then G is isomorphic to $\mathbb{R}$ and the action of G on X is neither free nor proper. *Exercise:* Check.

Definition 1.93. Let X be a (generalized) metric space and G a group acting on X. For $x \in X$ we say that

$$Gx = \{g(x) \colon g \in G\}$$

is the G*-orbit of* x. Furthermore, we say that x and y are equivalent under G, written as $x \sim_G y$, if $Gx = Gy$.

Lemma 1.94. *(1)* $\sim_G$ *is an equivalence relation in* X.

(2) $Gx = Gy$ *if and only if* $Gx \cap Gy \neq \emptyset$.

Proof. Clearly (1) holds. In (2) the "only if" part is trivial. Let us now assume that there exists $z \in Gx \cap Gy$. Thus $z = g(x)$ and $z = h(y)$ for some $g, h \in G$. Hence $y = (h^{-1} \circ g)(x)$ and $x = (g^{-1} \circ h)(y)$. Therefore, by the definition of G-orbit, $Gy \subset Gx$ and $Gx \subset Gy$. □

Let X be a (generalized) metric space. We denote by X/G the quotient space $X/{\sim_G}$. Let $\bar{d}$ be the quotient (generalized) pseudometric in X/G as in Section 1.79. We denote elements (equivalence classes) in X/G either by $\bar{x}$ or by Gx depending on which notation suits better to the context.

Lemma 1.95. *Let X be a generalized metric space and $G \subset \mathrm{Isom}(X)$ a subgroup. Then*

(1) $B(\bar{x}, \delta) = \bigcup_{y \in B(x,\delta)} \bar{y}$ for every $\bar{x} \in X/G$ and $\delta > 0$.

(2) If G acts properly on X then every G-orbit is closed in X.

Proof. (1) Let $x \in X$ and $\delta > 0$. Let $y \in B(x, \delta)$. Then $d(g(y), g(x)) = d(y, x) < \delta$ for every $g \in G$, since $G \subset \mathrm{Isom}(X)$. Thus $\bar{y} \subset B(\bar{x}, \delta)$. Therefore $\bigcup_{y \in B(x,\delta)} \bar{y} \subset B(\bar{x}, \delta)$.

Let us now show the other direction. Let $w \in B(\bar{x}, \delta)$. Then there exists $z \in \bar{x}$ such that $d(z, w) < \delta$. Fix $g \in G$ such that $g(x) = z$ and let $y = g^{-1}(w)$. Then $d(y, x) = d(g(y), g(x)) = d(w, z) < \delta$, since g is an isometry. Thus $w \in \bigcup_{y \in B(x,\delta)} \bar{y}$.

(2) Let $x \in X$ and let y be a point in the closure of $\bar{x}$. We show that $y \in \bar{x}$. Since G acts on X properly, we may fix $r > 0$ such that

$$\Gamma = \{g \in G \colon g(B(x, r)) \cap B(x, r) \neq \emptyset\}$$

is finite. Let $x_i \in \bar{x}$ be such that $d(x_i, y) \to 0$ as $i \to \infty$. Then there exists $g_i \in G$ such that $x_i = g_i(x)$ for every i. Fix i_0 such that $d(g_i(x), y) < r/2$ for $i \geq i_0$, and let $h_i = g_i^{-1} \circ g_{i_0}$ for every $i \geq i_0$. Then

$$\begin{aligned} d(h_i(x), x) &= d(g_i(h_i(x)), g_i(x)) = d(g_{i_0}(x), g_i(x)) \\ &\leq d(g_{i_0}(x), y) + d(y, g_i(x)) < r \end{aligned}$$

for $i \geq i_0$, since g_i is an isometry. Thus $h_i \in \Gamma$ for $i \geq i_0$. Since Γ is finite, also sets $\{h_i\}_{i \geq i_0}$ and $\{g_i\}_{i \geq i_0}$ are finite. Thus $y = g_i(x)$ for some i, and $y \in \bar{x}$.

□

Theorem 1.96. *Let $G \subset \mathrm{Isom}(X)$ be a subgroup and X a (generalized) metric space. Then*

(1) $\bar{d}(\bar{x}, \bar{y}) = \mathrm{dist}(\bar{x}, \bar{y})$ for every $\bar{x}, \bar{y} \in X/G$.

(2) If the action of G is proper, then $(X/G, \bar{d})$ is a (generalized) metric space.

(3) If $(X/G, \bar{d})$ is a metric space and X is a length space, then X/G is a length space.

Proof. Since (2) and (3) follow directly from (1) and (2) in Theorem 1.82, it is sufficient to prove (1). Let $\bar{x}, \bar{y} \in X/G$. If $\bar{d}(\bar{x}, \bar{y}) < \infty$, let $\delta = 1 + \bar{d}(\bar{x}, \bar{y})$. Then, by Lemma 1.95, $B(\bar{x}, \delta) = \bigcup_{z \in B(x,\delta)} \bar{z}$. Thus, by (1) in Theorem 1.82, $\bar{d}(\bar{x}, \bar{y}) = \mathrm{dist}(\bar{x}, \bar{y})$. If $\bar{d}(\bar{x}, \bar{y}) = \infty$, then $\mathrm{dist}(\bar{x}, \bar{y}) = \infty$ by the definition of $\bar{d}$. □

By Theorem 1.96, we know that under some assumptions on G, X/G is a length space whenever X is. Therefore it is natural to ask whether the same holds for being geodesic, that is, if X is a geodesic space, is X/G also a geodesic space when G satisfies some (additional) assumptions? We do not answer this question directly, but we show that X/G is complete and locally compact whenever X is. Then, by the Hopf-Rinow theorem, we have that if X is a complete and locally compact length space then also X/G is complete and locally compact length space and that they both are geodesic. The main tool is to show that under additional assumptions on G the quotient map $\pi\colon X \to X/G$, $x \mapsto \bar{x}$, is a local isometry and a covering map.

Definition 1.97. We say that a continuous map $f\colon X \to Y$ is a *covering map* if f is surjective and each $y \in Y$ has a neighborhood U such that

$$f^{-1}U = \bigcup_{x \in f^{-1}(y)} V_x$$

where V_x is a neighborhood of x and $f|V_x\colon V_x \to U$ is a homeomorphism for every $x \in f^{-1}(y)$. We also assume that sets V_x are pairwise disjoint, that is, $V_x \cap V_{x'} = \emptyset$ for $x \neq x'$.

We say that X is a *covering space* of (the base space) Y.

Example 1.98. (1) Let $f\colon \mathbb{R} \to S^1$, $t \mapsto (\cos t, \sin t)$, where $S^1 = \{x \in \mathbb{R}^n \colon |x| = 1\}$. Then f is a covering map. *Exercise:* Check this.

(2) Let $f\colon \mathbb{R} \to S^1$ be as in (1). Then $f|[0, 2\pi)$ is not a covering map, since for all $r < 2$ $f^{-1}(B((1,0), r) \cap S^1)$ consists of two components, but $f^{-1}((1,0)) = \{0\}$. (Alternatively show that f is not a local homeomorphism at the origin.)

(3) Let $f\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto z^2$. Then f is not a covering map, since f is not a local homeomorphism at the origin, but $f|\mathbb{C} \setminus \{0\}\colon \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ is a covering map. *Exercise:* Check.

Lemma 1.99. *Let $G \subset \mathrm{Isom}(X)$ act on X freely and properly. Then $\pi\colon X \to X/G$ is a local isometry, i.e. for every $x \in X$ there exists a neighborhood U such that $\pi|U$ is an isometry. Moreover, π is a covering map.*

Proof. π is a local isometry: Let $x \in X$. Since G acts properly, there exists $r > 0$ such that

$$\Gamma_x = \{g \colon gB(x,r) \cap B(x,r) \neq \emptyset\}$$

is finite. Since G acts freely, $g(x) \neq x$ for every $g \in \Gamma \setminus \{e\}$. We set

$$r_x = \begin{cases} \min\left\{\min\{d(g(x), x) \colon g \in \Gamma \setminus \{e\}\}, r\right\}/4, & \Gamma_x \neq \{e\}, \\ r/4, & \Gamma_x = \{e\}. \end{cases}$$

We show that π is an isometry on $B(x, r_x)$. Let $y, z \in B(x, r_x)$, $y \neq x$. Since $\mathrm{dist}(\bar{y}, \bar{z}) \leq d(y,z)$, we have, by (1) in Theorem 1.96, that

$$\bar{d}\big(\pi(y), \pi(z)\big) = \bar{d}(\bar{y}, \bar{z}) = \mathrm{dist}(\bar{y}, \bar{z}) \leq d(y,z).$$

Suppose $\mathrm{dist}(\bar{y}, \bar{z}) < d(y,z)$. Then there exists $g, h \in G$ such that $d(g(y), h(z)) < d(y,z)$. Thus $d(y, g^{-1}(h(z))) < d(y,z)$ and

$$\begin{aligned} d\big(x, g^{-1}(h(x))\big) &\leq d(x,y) + d\big(y, g^{-1}(h(z))\big) + d\big(g^{-1}(h(z)), g^{-1}(h(x))\big) \\ &< r_x + 2r_x + d(z,x) < 4r_x \leq r. \end{aligned}$$

Thus $g^{-1}(h(x)) \in g^{-1}(h(B(x,r))) \cap B(x,r)$ and $g^{-1} \circ h \in \Gamma_x$. Since

$$d(x, g^{-1}(h(x))) < 2r_x \le d(x, g'(x))$$

for every $g' \in \Gamma_x \setminus \{e\}$, $g^{-1} \circ h = e$. Thus $g = h$ and $d(g(x), h(y)) = d(x,y)$. This is a contradiction with $d(g(x), h(y)) < d(x,y)$. Therefore $\operatorname{dist}(\bar{x}, \bar{y}) = d(x,y)$. Thus $d(\pi(x), \pi(y)) = d(x,y)$ in $B(x, r_x)$.

π is a local homeomorphism: Let $x \in X$ and $r_x > 0$ be as above. We show that $\pi | B(x, r_x)\colon B(x, r_x) \to B(\bar{x}, r_x)$ is a homeomorphism. Since π is a local isometry in $B(x, r_x)$, $\pi|B(x, r_x)$ is an injection. Let $\bar{y} \in B(\bar{x}, r_x)$. Then $\bar{y} \in B(\bar{x}, r_x)$ and there exists, by Lemma 1.95, $z \in B(x, r_x)$ such that $\bar{z} = \bar{y}$. Since $\pi(z) = \pi(y)$, by the definition of π, $\pi(B(x, r_x)) = B(\bar{x}, r_x)$. (We use here also the first part of the proof.) Thus $\pi|B(x, r_x)\colon B(x, r_x) \to B(\bar{x}, r_x)$ is a bijection. Since $\pi|B(x, r_x)$ is an isometry, $(\pi|B(x, r_x))^{-1}$ is also an isometry and hence continuous.

π is a covering map: Let $x \in X$. We show first that for every $h \in G$ we may take $r_{h(x)} \ge r_x$. Then $\pi|B(h(x), r_x)$ is a local isometry and a local homeomorphism for every $h \in G$.

Let $h \in G$. We show first that

$$\Gamma = \{g \colon gB(h(x), 2r_x) \cap B(h(x), 2r_x) \ne \emptyset\}$$

is finite. Let $g \in \Gamma$. Since h is an isometry, we have that

$$gh^{-1}(B(h(x), 2r_x)) \cap B(x, 2r_x) = gh^{-1}h(B(x, 2r_x)) \cap B(x, 2r_x) = gB(x, 2r_x) \cap B(x, 2r_x).$$

Since $gB(x, 2r_x) \cap B(x, 2r_x) \ne \emptyset$ and $2r_x < r$ in the definition of Γ_x, $g \in \Gamma_x$. Therefore Γ is finite. Thus we may take $r = 2r_x$ in the definition of $\Gamma_{h(x)}$. Since

$$d(g(h(x)), h(x)) = d(h^{-1}gh(x), x)$$

for every $g \in \Gamma_{h(x)}$, we have that $r_{h(x)} \ge r_x$.

To show that h is a covering map, it now suffices to note that

$$\begin{aligned} \pi^{-1} B_{\bar{d}}(\bar{x}, r_x) &= B(\bar{x}, r_x) = \bigcup_{y \in B(x, r_x)} \bar{y} = \bigcup_{y \in B(x, r_x)} \bigcup_{g \in G} \{g(y)\} \\ &= \bigcup_{g \in G} g(B(x, r_x)) = \bigcup_{g \in G} B(g(x), r_x). \end{aligned}$$

Since the sets $B(g(x), r_x)$ are disjoint (by the definition of r_x) and $\pi|B(g(x), r_x)$ is a local homeomorphism onto $B_{\bar{d}}(\bar{x}, r_x)$ for every $g \in G$, π is a covering map. □

Theorem 1.100. *Let X be a complete and locally compact metric space, let $G \subset \operatorname{Isom}(X)$ act freely and properly on X. Then X/G is complete and locally compact.*

Proof. X/G is locally compact: Let $\bar{x} \in X/G$. Since X is locally compact, there exists a neighborhood U of x such that $\overline{U}$ is compact. Since π is a local homeomorphism, πU is a neighborhood of $\bar{x}$. Since $\overline{\pi U} = \pi \overline{U}$, $\overline{\pi U}$ is compact. Thus X/G is locally compact.

X/G is complete: Let $(\bar{x}_i)$ be a Cauchy-sequence in X/G. Fix an increasing sequence of indices $n_j \in \mathbb{N}$ such that

$$\bar{d}(\bar{x}_i, \bar{x}_{n_j}) < \frac{1}{2^j}$$

for every $i \ge n_j$. We fix $y_j \in \bar{x}_{n_j}$ as follows. For $j = 1$, let $y_1 \in \bar{x}_{n_j}$. Then, by Lemma 1.95(1), we may inductively fix $y_j \in \bar{x}_{n_j} \cap B(y_{j-1}, 2^{-(j-1)})$ for $j \ge 2$. Clearly (y_j) is a Cauchy-sequence. Since X is complete, there exists $y \in X$ such that $y_j \to y$ as $j \to \infty$. Thus $\bar{y}_j \to \bar{y}$. Since $(\bar{x}_i)$ is a Cauchy-sequence, $\bar{x}_i \to \bar{y}$ as $i \to \infty$. Therefore X/G is complete. □

When we combine Theorems 1.96 and 1.100 with the Hopf-Rinow theorem, we have the following corollary.

Corollary 1.101. *If X is complete and locally compact length space, and G acts properly and freely on X, then X/G is complete and locally compact length space. Moreover, both spaces are geodesic.*

Problem 1.102. Can X geodesic $\Rightarrow X/G$ geodesic be proved directly without assuming that X is complete and locally compact?

We devote the end of this section to metric properties of covering spaces of length spaces. First we give a candidate for a metric in the covering space, and then show that with respect to this metric the covering map is a local isometry.

Definition 1.103. Let Y be a local length space, X a topological space, and $f\colon X \to Y$ a local homeomorphism. We define $\tilde{d}\colon X \times X \to \mathbb{R}$ by

$$\tilde{d}(x, y) = \inf_{\gamma} \ell(f \circ \alpha),$$

where the infimum is taken over all paths $\gamma\colon I \to X$ joining x to y. If there are no paths connecting points x and y in X, we set $\tilde{d}(x, y) = \infty$.

Lemma 1.104. *Let X be a connected topological space, Y a local length space, and $f\colon X \to Y$ a local homeomorphism. Then X is path-connected.*

Proof. It is sufficient to show that X is locally path-connected. Let $x \in X$. Since f is a local homeomorphism and Y is a local length space, there exists $r > 0$ and a neighborhood V of x such that $f|V$ is a homeomorphism from V onto $B(f(x), 2r)$ and that $d(z, z') = d_s(z, z')$ for all $z, z' \in B(f(x), r)$. Let $y \in V \cap f^{-1}B(f(x), r)$. Since Y is a local length space and $f(y) \in B(f(x), r)$, there exists a path $\alpha\colon [0, 1] \to B(f(x), 2r)$ such that $\alpha(0) = f(x)$ and $\alpha(1) = f(y)$. Then $\tilde{\alpha} = (f|V)^{-1} \circ \alpha$ is a path connecting x to y in V. $\square$

Theorem 1.105. *Let Y be a local length space and $f\colon X \to Y$ a covering map. Then $\tilde{d}$ is a generalized metric. Furthermore, if X is connected, then $\tilde{d}$ is a metric.*

Proof. $\tilde{d}$ is a generalized metric: We show that $\tilde{d}(x, y) > 0$ for $x \neq y$. The proof of Theorem 1.56 can be adapted to obtain the other properties of $\tilde{d}$.

Let $x \in X$. Since f is a local homeomorphism, we may fix $r > 0$ and a neighborhood U of x such that $f|U\colon U \to B(f(x), r)$ is a homeomorphism. Let $y \in X$, $y \neq x$, and suppose that there exists a path $\alpha\colon [0, 1] \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$. If such a path does not exists, then $\tilde{d}(x, y) = \infty$ and we are done.

If $f \circ \alpha$ is not contained in $B(f(x), r)$, i.e. there exists $t \in [0, 1]$ such that $f(\alpha(t)) \notin B(f(x), r)$, then there exists $s \in [0, 1]$ such that $f(\alpha(s)) \in \partial B(f(x), r)$. Then $\ell(f \circ \alpha) \geq d(f(\alpha(s)), f(\alpha(0)) = r$. On the other hand, if $f \circ \alpha$ is contained in $B(f(x), r)$, then α is contained in U since f is a covering map. Since $x \neq y$ and f is a homeomorphism in U, $f(x) \neq f(y)$. Thus $\ell(f \circ \alpha) \geq d(f(x), f(y)) > 0$.

if X is connected then $\tilde{d}$ is a metric: We only need to show that every pair of points in X can be joined by a path of finite length. Let $x, y \in X$. Since X is path-connected by Lemma 1.104, there exists a path $\alpha\colon [0, 1] \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$. By compactness, we can cover $\alpha[0, 1]$ by open sets $V_1, \ldots, V_k$ such that $f|V_i\colon V_i \to B(y_i, r_i)$ is a homeomorphism for some $y_i \in Y$ and $r_i > 0$ and that $d(z, z') = d_s(z, z')$ for all $z, z' \in B(y_i, r_i)$, $i = 1, \ldots, k$. Fix $0 = t_0 < \ldots < t_m = 1$ such that for every $1 \leq i \leq m$ there exists $1 \leq k_i \leq k$ such that $\alpha[t_{i-1}, t_i] \subset V_{k_i}$. Since $f(\alpha(t_{i-1})$ and $f(\alpha(t_i))$ belong to $B(y_i, r_i)$ for every i, there exists a path

$\beta_i'\colon [t_{t-1}, t_i] \to B(y_i, r_i)$ such that $\beta_i'(t_{i-1}) = f(\alpha(t_{i-1}))$, $\beta_i'(t_i) = f(\alpha(t_i))$, and $\ell(\beta_i') \le 2r_i$. We define $\beta\colon [0,1] \to X$ by $\beta|[t_{i-1}, t_i] = (f|V_{k_i})^{-1} \circ \beta_i'$. Then β is a path from x to y and

$$\ell(f \circ \beta) = \sum_{i=1}^{m} \ell(\beta_i') \le \sum_{i=1}^{m} 2r_i < \infty.$$

□

Theorem 1.106. *Let X be a connected topological space, Y a local length space, and $f\colon X \to Y$ a covering map. Then $f\colon (X, \tilde{d}) \to Y$ is a local isometry.*

Proof. Let $x \in X$. Fix a neighborhood V of x and $r > 0$ such that $f|V\colon V \to B(f(x), r)$ is a homeomorphism and that $d(z, z') = d_s(z, z')$ for all $z, z' \in B(f(x), r)$. Let $W = V \cap f^{-1}(B(f(x), r/4)$. We show that $f|W$ is an isometry.

Let $y, z \in W$. Since $\ell(f \circ \tilde{\alpha}) \ge d(f(y), f(w))$ for all paths $\tilde{\alpha}\colon [0,1] \to X$ connecting y and z, we have, by the definition of $\tilde{d}$, that $\tilde{d}(y, z) \ge d(f(y), f(z))$.

Let $\varepsilon \in (0, r/4)$. Since Y is a local length space, there exists a path $\alpha\colon [0,1] \to Y$ such that $\alpha(0) = f(y)$, $\alpha(1) = f(z)$, and $\ell(\alpha) \le d(f(y), f(z)) + \varepsilon$. Since $y, z \in W$, $d(f(y), f(z)) + \varepsilon < 2r/4 + r/4 = 3r/4$. Since $\ell(\alpha) < 3r/4$ and $\alpha(0) = f(y) \in B(f(x), r/4)$, α is contained in $B(f(x), r)$. Hence we may define $\tilde{\alpha} = (f|V)^{-1} \circ \alpha$. Since $\tilde{\alpha}$ is a path connecting y to z, we have

$$\tilde{d}(y, z) \le \ell(f \circ \tilde{\alpha}) = \ell(\alpha) \le d(f(y), f(z)) + \varepsilon.$$

Thus $\tilde{d}(y, z) \le d(f(y), f(z))$. □

If we assume that X is Hausdorff and f is a local homeomorphism, we get the following version of Theorems 1.105 and 1.106.

Theorem 1.107. *Let Y be a local length space, X a connected Hausdorff space, $f\colon X \to Y$ a local homeomorphism, and let $\tilde{d}$ be as above. Then $\tilde{d}$ is a metric and*

(a) f is a local isometry,

(b) $(X, \tilde{d})$ is a length space, and

(c) $\tilde{d}$ is the only metric on X with properties (a) and (b).

Proof. We leave the proof as an exercise. □

Theorem 1.108. *Let X be a connected metric space, $\tilde{X}$ a complete metric space, and $\pi\colon \tilde{X} \to X$ a local homeomorphism. Suppose that*

(1) $\ell(\tilde{\alpha}) \le \ell(\pi \circ \tilde{\alpha})$ for every path $\tilde{\alpha}\colon [0,1] \to \tilde{X}$ and

(2) for every $x \in X$ there exists $r > 0$ such that every $y \in B(x, r)$ can be connected to x by a unique constant speed geodesic $\gamma_y\colon [0,1] \to B(x, r)$ and that γ_y varies continuously with y.

Then π is a covering map.

In particular, if π is a local isometry, then it is a local homeomorphism and satisfies (1).

Proof. First we show that, for every rectifiable path $\alpha\colon [0,1] \to X$ and for every $\tilde{x} \in \pi^{-1}(\alpha(0))$ there exists a unique maximal lift of α starting at $\tilde{x}$, i.e. a path $\tilde{\alpha}\colon [0,1] \to \tilde{X}$ such that $\tilde{\alpha}(0) = \tilde{x}$ and $\pi \circ \tilde{\alpha} = \alpha$. Fix such a rectifiable path $\alpha\colon [0,1] \to X$ and $\tilde{x} \in \pi^{-1}(\alpha(0))$. Since π is a local homeomorphism, there exists a unique lift of $\alpha|[0,\varepsilon]$ starting at $\tilde{x}$ for some $\varepsilon > 0$. Suppose that $\tilde{\alpha}\colon [0,a) \to \tilde{X}$ is the unique lift of $\alpha|[0,a)$ starting at $\tilde{x}$, with $0 < a \le 1$. Choose a sequence $0 < t_1 < t_2 < \cdots$ converging to a. By the assumption (1)

$$|\tilde{\alpha}(t_i) - \tilde{\alpha}(t_j)| \le \ell\big(\tilde{\alpha}|[t_i,t_j]\big) \le \ell\big(\pi \circ \tilde{\alpha}|[t_i,t_j]\big) = \ell\big(\alpha|[t_i,t_j]\big)$$

for $i < j$. Since α is rectifiable, $\big(\tilde{\alpha}(t_i)\big)$ is a Cauchy-sequence in $\tilde{X}$, and hence has a limit. We define $\tilde{\alpha}(a)$ to be the limit. Hence $\alpha|[0,a]$ has the unique lift starting at $\tilde{x}$. This shows that the maximal interval $I \subset [0,1]$ such that $0 \in I$ and that $\alpha|I$ has the unique lift starting at $\tilde{x}$ is closed. Since π is a local homeomorphism, I is also open. Hence $I = [0,1]$.

The assumption (2) and the connectedness of X imply that every pair of points in X can be joined by a rectifiable path. Combining this with the existence of lifts yields that $\pi|V\colon V \to X$ is surjective for all components V of $\tilde{X}$. It remains to prove that every point $x \in X$ has a neighborhood U such that the restriction of π to each component of $\pi^{-1}U$ is a homeomorphism onto U.

Let $x \in X$ and choose $r > 0$ as in (2). Fix $\tilde{x} \in \pi^{-1}(x)$. For $y \in B(x,r)$, let $\tilde{\gamma}_y\colon [0,1] \to \tilde{X}$ be the unique maximal lift of $\gamma_y\colon [0,1] \to B(x,r)$ starting at $\tilde{x}$. We define a mapping $g_{\tilde{x}}\colon B(x,r) \to \tilde{X}$ by $g_{\tilde{x}}(y) = \tilde{\gamma}_y(1)$. Denote $B(\tilde{x}) = g_{\tilde{x}}B(x,r)$. We claim that $g_{\tilde{x}}\colon B(x,r) \to B(\tilde{x})$ is a homeomorphism. Since $(\pi|B(\tilde{x})) \circ g_{\tilde{x}} = \mathrm{id}_{B(x,r)}$, $g_{\tilde{x}} \circ (\pi|B(\tilde{x})) = \mathrm{id}_{B(\tilde{x})}$, and π is a local homeomorphism, it suffices to show that $g_{\tilde{x}}$ is continuous.

Since π is a local homeomorphism, we may cover $\gamma_y[0,1]$ by open balls $B_1, \ldots, B_k \subset B(x,r)$ such that

$$\gamma_y\big[\tfrac{j-1}{k}, \tfrac{j}{k}\big] \subset B_j \quad \text{for } j = 1, \ldots, k$$

and that there are continuous mappings $g^j\colon B_j \to \tilde{X}$, with $\pi \circ g^j = \mathrm{id}_{B_j}$ and $g^j\big(\gamma_y(t)\big) = g_{\tilde{x}}\big(\gamma_y(t)\big)$ for all $t \in [(j-1)/k, j/k]$. If $\delta > 0$ is small enough and $z \in B(y,\delta) \subset B(x,r)$, we have

$$\gamma_z\big[\tfrac{j-1}{k}, \tfrac{j}{k}\big] \subset B_j \quad \text{for } j = 1, \ldots, k$$

since γ_z varies continuously with z. Thus we may define a mapping $g\colon B(y,\delta) \times [0,1]$ by setting

$$g(z,t) = g^j\big(\gamma_z(t)\big) \quad \text{whenever } (z,t) \in B(y,\delta) \times \big[\tfrac{j-1}{k}, \tfrac{j}{k}\big].$$

Since the definitions of g using g^j and g^{j+1} agree at $(y, j/k)$ they agree in the connected set $B(y,\delta) \times \{j/k\}$. Hence g is well-defined and continuous. Now $t \mapsto g(z,t)$ is a lift of γ_z starting at $\tilde{x}$, and so $g(z,t) = \tilde{\gamma}_z(t)$. In particular, $g(z,1) = \tilde{\gamma}_z(1) = g_{\tilde{x}}(z)$ for all $z \in B(y,\delta)$, and hence $g_{\tilde{x}}$ is continuous.

We have shown that $\pi^{-1}\big(B(x,r)\big)$ is the union of open sets $B(\tilde{x}) = g_{\tilde{x}}B(x,r)$, where $\tilde{x} \in \pi^{-1}(x)$, and that $\pi|B(\tilde{x})$ is a homeomorphism onto $B(x,r)$. Finally we observe that the sets $B(\tilde{x})$ are disjoint. Indeed, if $\tilde{y} \in B(\tilde{x}) \cap B(\tilde{x}')$, then the lifts of $\gamma_{\pi(\tilde{y})}$ starting at $\tilde{x}$ and $\tilde{x}'$ both end at $\tilde{y}$, thus they must coincide and $\tilde{x} = \tilde{x}'$. Hence π is a covering map. $\square$

2 Alexandrov spaces

In this section we will define and study Alexandrov spaces which are metric spaces with *curvature bounded from below* (or *from above*). The definition is based on comparisons with *model spaces*. It is worth noting that we will not define a curvature on a metric space.

2.1 Model spaces

We start with the definition of model spaces and then study the sphere and the hyperbolic space in detail.

Definition 2.2. Model spaces M^n_κ, where $n \in \mathbb{N}$ and $\kappa \in \mathbb{R}$, are the following metric spaces:

(1) If $\kappa = 0$, then M^n_0 is the Euclidean space $\mathbb{R}^n$ equipped with the standard metric.

(2) If $\kappa > 0$, then M^n_κ is obtained from the sphere $\mathbb{S}^n$ by multiplying the angular metric by the constant $\frac{1}{\sqrt{\kappa}}$. See Example 1.13 and (2.3) below.

(3) If $\kappa < 0$, then M^n_κ is obtained from the hyperbolic space $\mathbb{H}^n$ by multiplying the hyperbolic metric by the constant $\frac{1}{\sqrt{-\kappa}}$. See 2.7 below for the definition of the hyperbolic space.

The sphere $\mathbb{S}^n$

The *n-dimensional sphere* is the set

$$\mathbb{S}^n = \{x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \colon \langle x, x\rangle = 1\},$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^{n+1}$. We equip $\mathbb{S}^n$ with the (angular) metric

$$d \colon \mathbb{S}^n \times \mathbb{S}^n \to [0, \pi]$$

defined by the formula

$$\cos d(x, y) = \langle x, y\rangle \tag{2.3}$$

for $x, y \in \mathbb{S}^n$. Clearly $d(x, y) = d(y, x) \geq 0$ with equality if and only if $x = y$. The triangle inequality will be proved later (see Theorem 2.6). Thus $(\mathbb{S}^n, d)$ is a metric space.

An intersection of $\mathbb{S}^n$ with a 2-dimensional subspace (i.e. a plane passing through 0) is called a *great circle*. Given $x \in \mathbb{S}^n$ the *orthogonal complement* of x (with respect to $\langle \cdot, \cdot \rangle$) is the n-dimensional subspace

$$x^\perp = \{y \in \mathbb{R}^{n+1} \colon \langle x, y\rangle = 0\}.$$

Great circles can be parameterized as follows. Given $x \in \mathbb{S}^n$ and a unit vector $u \in x^\perp$, the image of the path $\gamma \colon \mathbb{R} \to \mathbb{S}^n$,

$$\gamma(t) = (\cos t)x + (\sin t)u,$$

is a great circle, more precisely, the intersection of $\mathbb{S}^n$ and the 2-dimensional subspace spanned by x and u. We note that

$$d\big(\gamma(t), \gamma(s)\big) = |t - s| \tag{2.4}$$

for all $t, s \in \mathbb{R}$, with $|t - s| \leq \pi$. This holds since

$$\begin{aligned}
\cos d\big(\gamma(t), \gamma(s)\big) &= \big\langle (\cos t)x + (\sin t)u, (\cos s)x + (\sin s)u \big\rangle \\
&= \cos t \cos s + \sin t \sin s \\
&= \cos(t - s).
\end{aligned}$$

Hence γ is a local geodesic and $\gamma|[a,b] \to \mathbb{S}^n$ is a geodesic for all $a,b \in \mathbb{R}$, with $0 < b-a \leq \pi$. The vector $u = \gamma'(0)$ is called the *initial vector* of γ. If $y \in \mathbb{S}^n \setminus \{x\}$ and $d(x,y) < \pi$, there is a unique[2] geodesic $\gamma|[0, d(x,y)]$ from x to y. It is determined by the initial vector

$$u = \lambda(y - \langle x,y\rangle x), \ \ \lambda = \frac{1}{\sqrt{1-\langle x,y\rangle^2}}.$$

If $d(x,y) = \pi$, then $y = -x$ and any choice of an initial vector yields a geodesic from x to y.

Suppose that $v \in x^\perp$ is another unit vector and let $\sigma\colon \mathbb{R} \to \mathbb{S}^n$ be the path

$$\sigma(t) = (\cos t)x + (\sin t)v.$$

Then the *spherical angle* between γ and σ at x is the angle between u and v, i.e. the unique $\alpha \in [0,\pi]$ such that $\cos\alpha = \langle u,v\rangle$. The *spherical triangle* Δ in $\mathbb{S}^n$ consists of three distinct points $x,y,z \in \mathbb{S}^n$ (*vertices* of Δ) and three geodesics (*sides* of Δ) joining each pair of vertices. We denote the sides of Δ by $[x,y], [x,z]$, and $[y,z]$. The *vertex angle* of Δ at x is the spherical angle between sides $[x,y]$ and $[x,z]$.

Theorem 2.5 (The spherical law of cosines)**.** *Let Δ be a spherical triangle in $\mathbb{S}^n$ with vertices A,B,C. Let $a = d(B,C), b = d(A,C), c = d(A,B)$, and let γ be the vertex angle of Δ at C. Then*

$$\cos c = \cos a \cos b + \sin a \sin b \cos\gamma.$$

Proof. Suppose that there are no antipodal pairs among A, B, and C. Let $u \in C^\perp$ and $v \in C^\perp$ be the initial vectors of $[C,A]$ and $[C,B]$, respectively. Then, by definition, $\cos\gamma = \langle u,v\rangle$. Hence

$$\begin{aligned}\cos c &= \cos d(A,B) = \langle A,B\rangle \\ &= \big\langle (\cos b)C + (\sin b)u, (\cos a)C + (\sin a)v\big\rangle \\ &= \cos a \cos b\,\langle C,C\rangle + \sin a \sin b\,\langle u,v\rangle \\ &= \cos a\cos b + \sin a \sin b \cos\gamma.\end{aligned}$$

The special case where A and B (or A and C, or B and C) are antipodal is easy and will be omitted. ☐

Theorem 2.6. *For all $A,B,C \in \mathbb{S}^n$*

$$d(A,B) \leq d(A,C) + d(C,B),$$

with equality if and only if C lies on a geodesic joining A and B. Hence $(\mathbb{S}^n, d)$ is a geodesic metric space.

Proof. First we observe that for fixed $a \in [0,\pi]$ and $b \in [0,\pi]$, the function

$$\gamma \mapsto \cos a \cos b + \underbrace{\sin a \sin b}_{\geq 0} \cos\gamma$$

decreases from $\cos(a-b)$ to $\cos(a+b)$ as γ increases from 0 to π.

To prove the triangle inequality we may assume that A,B, and C are distinct points. Let $a = d(B,C)$, $b = d(A,C)$, $c = d(A,B)$, and let Δ be a spherical triangle with vertices A,B,C. Let γ be the vertex angle of Δ at C. Then the spherical law of cosines and the observation above imply that

$$\cos c \geq \cos(a+b).$$

Hence $c \leq a+b$, with the equality if and only if $\gamma = \pi$ and $a+b \leq \pi$, i.e. C belongs to a geodesic joining A and B. ☐

[2] See Theorem 2.6 for the uniqueness.

The hyperbolic space $\mathbb{H}^n$

We approach the *hyperbolic geometry* from a metric point of view, and therefore we use the following *hyperboloid model* for $\mathbb{H}^n$.

Consider $\mathbb{R}^{n+1}$ equipped with a symmetric bilinear form

$$\langle x, y\rangle_{n,1} = -x_{n+1}y_{n+1} + \sum_{i=1}^{n} x_i y_i, \quad x = (x_1, \dots, x_{n+1}),\ y = (y_1, \dots, y_{n+1}).$$

Given $x \in R^{n+1}$ the *orthogonal complement* of x with respect to $\langle\cdot,\cdot\rangle_{n,1}$ is the n-dimensional subspace

$$x^{\perp} = \{y \in \mathbb{R}^{n+1} \colon \langle x, y\rangle_{n,1} = 0\}.$$

If $\langle x, x\rangle_{n,1} < 0$, then (by linear algebra) $\langle\cdot,\cdot\rangle_{n,1}|x^{\perp}$ is positive definite, i.e. an inner product. This can be seen also by a direct computation.

Definition 2.7. The (real) *hyperbolic n-space* $\mathbb{H}^n$ is the set

$$\mathbb{H}^n = \{x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \colon \langle x, x\rangle_{n,1} = -1,\ x_{n+1} > 0\}$$

equipped with the metric $d \colon \mathbb{H}^n \times \mathbb{H}^n \to [0, \infty)$ defined by the formula

$$\cosh d(x, y) = -\langle x, y\rangle_{n,1}, \quad x, y \in \mathbb{H}^n. \tag{2.8}$$

Remark 2.9. The hyperbolic space is the upper sheet of the hyperboloid

$$\{x \in \mathbb{R}^{n+1} \colon \langle x, x\rangle_{n,1} = -1\}.$$

For all $x, y \in \mathbb{H}^n$,

$$\begin{aligned} &\langle x, y\rangle_{n,1} \le -1 \quad \text{and} \\ &\langle x, y\rangle_{n,1} = -1 \iff x = y. \end{aligned}$$

Thus $d(x, y) = d(y, x) \ge 0$, with the equality if and only if $x = y$. The triangle inequality will be proved later (see Theorem 2.12).

Let $x \in \mathbb{H}^n$ and let $u \in x^{\perp}$ be a unit vector with respect to $\langle\cdot,\cdot\rangle_{n,1}$, that is

$$\langle u, u\rangle_{n,1} = 1 \quad \text{and} \quad \langle u, x\rangle_{n,1} = 0.$$

Consider the path $\gamma \colon \mathbb{R} \to \mathbb{R}^{n+1}$,

$$\gamma(t) = (\cosh t)x + (\sinh t)u. \tag{2.10}$$

Since γ is continuous, $\gamma(0) = x \in \mathbb{H}^n$, and

$$\begin{aligned} \big\langle \gamma(t), \gamma(t)\big\rangle_{n,1} &= \big\langle (\cosh t)x + (\sinh t)u, (\cosh t)x + (\sinh t)u\big\rangle_{n,1} \\ &= \cosh^2 t\,\langle x, x\rangle_{n,1} + 2\cosh t\,\sinh t\langle x, u\rangle_{n,1} + \sinh^2 t\,\langle u, u\rangle_{n,1} \\ &= \sinh^2 t - \cosh^2 t \\ &= -1 \end{aligned}$$

we have $\gamma(t) \in \mathbb{H}^n$ for all $t \in \mathbb{R}$ (i.e. $\gamma(t)$ belongs to the upper sheet of the hyperboloid). Note that $\gamma\mathbb{R}$ is the intersection of $\mathbb{H}^n$ and the 2-dimensional subspace of $\mathbb{R}^{n+1}$ spanned by x and u. Next we observe that for all $t, s \in \mathbb{R}$,

$$\begin{aligned}\cosh d\big(\gamma(t), \gamma(s)\big) &= -\big\langle \gamma(t), \gamma(s)\big\rangle_{n,1} \\ &= -\big\langle (\cosh t)x + (\sinh t)u, (\cosh s)x + (\sinh s)u\big\rangle_{n,1} \\ &= \cosh t\, \cosh s - \sinh t\, \sinh s \\ &= \cosh(t-s).\end{aligned}$$

Hence

$$d\big(\gamma(t), \gamma(s)\big) = |t-s|$$

for all $t, s \in \mathbb{R}$, and therefore γ is a geodesic.

Given $x, y \in \mathbb{H}^n,\ x \neq y$, let $u \in x^\perp$ be the unit vector

$$u = \lambda(y + \langle x, y\rangle_{n,1} x), \quad \lambda = \frac{1}{\sqrt{\langle x, y\rangle_{n,1}^2 - 1}}$$

and let γ be defined by (2.10). Then u is the unique unit vector in $x^\perp$ such that

$$\begin{aligned}&y = \gamma(t) = (\cosh t)x + (\sinh t)u, \quad \text{with} \\ &t = d(x, y).\end{aligned}$$

We call u the *initial vector* (at x) of the *hyperbolic segment* (or *geodesic segment*) $[x, y] = \gamma[0, d(x, y)]$. Thus any two points of $\mathbb{H}^n$ can be joined by a unique[3] geodesic segment. The *hyperbolic angle* between two hyperbolic segments with initial vectors u and v (at x) is the unique angle $\alpha \in [0, \pi]$ such that

$$\cos\alpha = \langle u, v\rangle_{n,1}.$$

A *hyperbolic triangle* Δ consists of three distinct points $x, y, z \in \mathbb{H}^n$ (*vertices* of Δ) and the geodesic segments (*sides* of Δ) joining each pair of vertices. The *vertex angle* at x is the hyperbolic angle between $[x, y]$ and $[x, z]$.

Theorem 2.11 (The hyperbolic law of cosines)**.** *Let Δ be a hyperbolic triangle in $\mathbb{H}^n$ with vertices A, B, C. Let $a = d(B, C), b = d(A, C), c = d(A, B)$, and let γ be the vertex angle of Δ at C. Then*

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\gamma.$$

Proof. Let $u \in C^\perp$ and $v \in C^\perp$ be the initial vectors of $[C, A]$ and $[C, B]$, respectively. Then, by definition, $\cos\gamma = \langle u, v\rangle_{n,1}$. Hence

$$\begin{aligned}\cosh c = \cosh d(A, B) &= -\langle A, B\rangle_{n,1} \\ &= -\big\langle (\cosh b)C + (\sinh b)u, (\cosh a)C + (\sinh a)v\big\rangle \\ &= -\cosh a \cosh b\, \langle C, C\rangle_{n,1} - \sinh a \sinh b\, \langle u, v\rangle_{n,1} \\ &= \cosh a \cosh b - \sinh a \sinh b \cos\gamma.\end{aligned}$$

□

[3]See Theorem 2.12 for the uniqueness.

Theorem 2.12. *For all $A, B, C \in \mathbb{H}^n$*

$$d(A,B) \leq d(A,C) + d(C,B),$$

with equality if and only if C lies on the geodesic segment joining A and B. Hence $(\mathbb{H}^n, d)$ is a uniquely geodesic metric space.

Proof. Again we first we observe that for fixed $a > 0$ and $b > 0$, the function

$$\gamma \mapsto \cosh a \cosh b - \underbrace{\sinh a \sinh b}_{\geq 0} \cos \gamma$$

increases from $\cosh(a-b)$ to $\cosh(a+b)$ as γ increases from 0 to π.

To prove the triangle inequality we may assume that A, B, and C are distinct points. Let $a = d(B,C),\ b = d(A,C),\ c = d(A,B)$, and let Δ be the hyperbolic triangle with vertices A, B, C. Let γ be the vertex angle of Δ at C. Then the hyperbolic law of cosines and the observation above imply that

$$\cosh c \leq \cosh(a+b).$$

Hence $c \leq a + b$, with the equality if and only if $\gamma = \pi$, i.e. C belongs to a geodesic joining A and B. □

Remark 2.13. It might be interesting for those who are familiar with differential geometry (and Riemannian geometry) to note that $\mathbb{H}^n$ is the level set $\{x \in \mathbb{R}^{n+1} \colon f(x) = 0\}$ of a smooth function $f \colon \mathbb{R}^{n+1} \to \mathbb{R}$,

$$f(x) = \langle x, x \rangle_{n,1} + 1,$$

with

$$\nabla f(x) = 2(x_1, \ldots, x_n, -x_{n+1}) \neq 0$$

for all $x \in \mathbb{H}^n$. Thus $\mathbb{H}^n$ is a differentiable n-manifold (see e.g. [Ho, Esim. 2.28]). Furthermore, we have the equality

$$\begin{aligned} \langle \nabla f(x), y \rangle &= 2\Big(\sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1} \Big) \\ &= 2\langle x, y \rangle_{n,1}, \end{aligned}$$

where $\langle \nabla f(x), y \rangle$ is the standard inner product. Hence $x^\perp$ is tangent to $\mathbb{H}^n$ at x for all $x \in \mathbb{H}^n$, i.e it is a tangent space of $\mathbb{H}^n$ at x. Finally,

$$x \mapsto \langle \cdot, \cdot \rangle_{n,1} | x^\perp$$

is smooth, and hence it is a Riemannian metric on $\mathbb{H}^n$. Thus $\mathbb{H}^n$ is a Riemannian n-manifold.

2.14 Angles in metric spaces

We want to define a notion of an angle in a metric space. Therefore, we first recall how to express an angle in the plane in purely metric terms. Suppose that p, x, y are three distinct points in $\mathbb{R}^2$. Denote by $\angle_p xy$ the angle ($\in [0, \pi]$) at p between the segments $[p, x]$ and $[p, y]$. Applying the usual law of cosines ("$c^2 = a^2 + b^2 - 2ab \cos \gamma$"), we see that

$$\angle_p xy = \arccos \frac{|p-x|^2 + |p-y|^2 - |x-y|^2}{2|p-x||p-y|}.$$

Let then (X,d) be a metric space and let p,x,y be three distinct points in X. A *comparison triangle* of the triple (p,x,y) is a triangle in the Euclidean plane $\mathbb{R}^2$ with vertices $\bar{p},\bar{x},\bar{y}$ such that $|\bar{p}-\bar{x}| = d(p,x)$, $|\bar{p}-\bar{y}| = d(p,y)$, and $|\bar{x}-\bar{y}| = d(x,y)$. It follows from the triangle inequality in X that a comparison triangle always exists. It is unique up to an isometry of $\mathbb{R}^2$ and we denote any of them by $\bar{\Delta}(p,x,y)$. The *comparison angle between x and y at p*, denoted by $\bar{\angle}_p xy$ (or $\bar{\angle}_p(x,y)$, is defined by

$$\bar{\angle}_p xy = \arccos \frac{d(p,x)^2 + d(p,y)^2 - d(x,y)^2}{2d(p,x)d(p,y)}.$$

Hence $\bar{\angle}_p xy = \angle_{\bar{p}}\bar{x}\bar{y}$.

Next we define an angle between two geodesic segments emanating from the same point.

Definition 2.15. Let X be a metric space and let $\alpha\colon [0,a] \to X$ and $\beta\colon [0,b] \to X$ be two geodesics with $\alpha(0) = \beta(0) = p$. Given $t \in (0,a]$ and $s \in (0,b]$ consider the comparison triangle $\bar{\Delta}(p,\alpha(t),\beta(s))$ and the comparison angle $\bar{\angle}_p(\alpha(t),\beta(s))$. The *(Alexandrov) angle* (or the *upper angle*) between α and β (at p) is the number $\angle_p(\alpha,\beta) \in [0,\pi]$ defined by

$$\begin{aligned}\angle_p(\alpha,\beta) &= \limsup_{t,s\to 0} \bar{\angle}_p(\alpha(t),\beta(s)) \\ &= \lim_{r\to 0} \sup_{0<t,s<r} \bar{\angle}_p(\alpha(t),\beta(s)).\end{aligned}$$

If the limit

$$\lim_{t,s\to 0} \bar{\angle}_p(\alpha(t),\beta(s))$$

exists, we say that the *angle exists in strong sense*.

Remark 2.16.

1. The Alexandrov angle between α and β at p depends only on germs[4] of α and β at 0. That is, if $\tilde{\alpha}\colon [0,\tilde{a}] \to X$ and $\tilde{\beta}\colon [0,\tilde{b}]$ are geodesics such that $\tilde{\alpha}|[0,\varepsilon] = \alpha|[0,\varepsilon]$ and $\tilde{\beta}|[0,\varepsilon] = \beta|[0,\varepsilon]$ for some $\varepsilon > 0$, then
$$\angle_p(\alpha,\beta) = \angle_p(\tilde{\alpha},\tilde{\beta}).$$
2. If $\gamma\colon [a,b] \to X$ is a geodesic, with $a < 0 < b$, and if $\alpha\colon [0,-a] \to X$, $\alpha(t) = \gamma(-t)$, and $\beta = \gamma|[0,b]$, then $\angle_{\gamma(0)}(\alpha,\beta) = \pi$.
3. Angles do not, in general, exist in strong sense. For example, let $(V,\|\cdot\|)$ be a normed space. Then angles exist at 0 in strong sense if and only if the norm is an inner product norm.
4. In $(\mathbb{R}^2, d_\infty)$ paths $\gamma_n\colon [0,1/n] \to (\mathbb{R}^2, d_\infty)$,
$$\gamma_n(t) = \big(t, t^n(1-t)^n\big),\ n \in \mathbb{N},\ n \geq 2,$$
are geodesics emanating from the origin and their germs are pairwise disjoint. However, the Alexandrov angle between any two of them at 0 is always zero.

Clearly, $\angle_p(\alpha,\beta) = \angle_p(\beta,\alpha) \geq 0$ and the next theorem shows that the mapping $(\alpha,\beta) \mapsto \angle_p(\alpha,\beta)$ satisfies the triangle inequality. However, as the last remark above shows, this mapping does not, in general, define a metric in the set of (germs of) geodesics emanating from p.

[4] Let Y be a set and let X be a topological space. Consider the set of all pairs (f,x), where $f\colon U \to Y$ and U is a neighborhood of x. We say that pairs (f,x) and (f',x') are equivalent if and only if $x = x'$ and $f = f'$ in some neighborhood of x. The equivalence class of (f,x) is called the *germ of f at x*.

Theorem 2.17. *Let X be a metric space, and let γ_1, γ_2, and γ_3 be three geodesics in X emanating from the same point $p \in X$. Then*

$$\angle_p(\gamma_1, \gamma_2) \leq \angle_p(\gamma_1, \gamma_3) + \angle_p(\gamma_3, \gamma_2). \tag{2.18}$$

Proof. We may assume that $\gamma_1, \gamma_2, \gamma_3$ are defined on $[0, a]$ for some $a > 0$ and $\gamma_i(0) = p,\ i = 1, 2, 3$. Suppose on the contrary that (2.18) does not hold. Then

$$\angle_p(\gamma_1, \gamma_2) > \angle_p(\gamma_1, \gamma_3) + \angle_p(\gamma_3, \gamma_2) + 3\delta \tag{2.19}$$

for some $\delta > 0$. Furthermore, by definition (of lim sup) there exists $\varepsilon > 0$ such that

(1) $\bar{\angle}_p(\gamma_1(t), \gamma_3(s)) < \angle_p(\gamma_1, \gamma_3) + \delta$ for all $s, t \in [0, \varepsilon]$,

(2) $\bar{\angle}_p(\gamma_3(s), \gamma_2(r)) < \angle_p(\gamma_3, \gamma_2) + \delta$ for all $r, s \in [0, \varepsilon]$, and

(3) $\bar{\angle}_p(\gamma_1(t), \gamma_2(r)) > \angle_p(\gamma_1, \gamma_2) - \delta$ for some $r, t \in [0, \varepsilon]$.

Fix $r, t \in [0, \varepsilon]$ such that (3) holds and choose a triangle in $\mathbb{R}^2$ with vertices $0, x_1, x_2$ such that

$$|x_1 - 0| = t = d(\gamma_1(t), p),$$
$$|x_2 - 0| = r = d(\gamma_2(r), p),$$

and that the angle α at 0 satisfies

$$\bar{\angle}_p(\gamma_1(t), \gamma_2(r)) > \alpha > \angle_p(\gamma_1, \gamma_2) - \delta. \tag{2.20}$$

In particular, $0 < \alpha < \pi$, and hence the triangle is non-degenerate. The left-hand inequality in (2.20) implies that

$$|x_1 - x_2| < d\big(\gamma_1(t), \gamma_2(r)\big). \tag{2.21}$$

The right-hand inequality in (2.20) and (2.19) imply that

$$\alpha > \angle_p(\gamma_1, \gamma_3) + \angle_p(\gamma_3, \gamma_2) + 2\delta.$$

Hence there exists a point $x \in [x_1, x_2]$ such that

$$\alpha_1 := \angle_0([0, x_1], [0, x]) > \angle_p(\gamma_1, \gamma_3) + \delta,$$
$$\alpha_2 := \angle_0([0, x], [0, x_2]) > \angle_p(\gamma_3, \gamma_2) + \delta.$$

Let $s = |x - 0|$. Since $s \leq \max\{r, t\}$, we may apply (1) to obtain

$$\bar{\angle}_p(\gamma_1(t), \gamma_3(s)) < \angle_p(\gamma_1, \gamma_3) + \delta < \alpha_1.$$

Hence

$$d\big(\gamma_1(t), \gamma_3(s)\big) < |x - x_1|.$$

Similarly,

$$d\big(\gamma_2(r), \gamma_3(s)\big) < |x - x_2|.$$

By (2.21), we have

$$d\big(\gamma_1(t), \gamma_2(r)\big) > |x_1 - x_2| = |x_1 - x| + |x - x_2| > d\big(\gamma_1(t), \gamma_3(s)\big) + d\big(\gamma_3(s), \gamma_2(r)\big)$$

which is a contradiction with the triangle inequality in X. $\square$

Theorem 2.22. *The spherical (resp. hyperbolic) angle between geodesic segments* $[p, x]$ *and* $[p, y]$ *in* $\mathbb{S}^n$ *(resp.* $\mathbb{H}^n$*) is equal to the Alexandrov angle between them.*

Proof. We present the proof in the hyperbolic case; the spherical case is similar. Let $a = d(p, x)$, $b = d(p, y)$, and let γ be the hyperbolic angle between $[p, x]$ and $[p, y]$. For $0 < t \leq a$ and $0 < s \leq b$, let $x_s \in [p, x]$ and $y_t \in [p, y]$ be the unique points such that $d(p, x_s) = s$ and $d(p, y_t) = t$. Let $c_{s,t} = d(x_s, y_t)$ and let $\gamma_{s,t}$ be the vertex angle at $\bar{p}$ in the comparison triangle $\bar{\Delta}(p, x_s, y_t) \subset \mathbb{R}^2$. We will show that $\gamma_{s,t} \to \gamma$ as $s, t \to 0$. By the usual cosine rule and the hyperbolic law of cosines we have

$$\cos \gamma_{s,t} = \frac{s^2 + t^2 - c_{s,t}^2}{2st}$$

and

$$\cosh c_{s,t} = \cosh s \cosh t - \sinh s \sinh t \cos \gamma. \tag{2.23}$$

We define a smooth function $h\colon \mathbb{R} \to \mathbb{R}$ by

$$h(r) = \sum_{i=1}^{\infty} \frac{r^i}{(2i)!}.$$

Since $h(0) = 0$ and $h'(0) = 1/2 \neq 0$, the restriction $h|(-\varepsilon, \varepsilon)$ has an inverse (for some $\varepsilon > 0$) which can be written as

$$h^{-1}(r) = 2r + \sum_{i=2}^{\infty} a_i r^i. \tag{2.24}$$

Since

$$h(r^2) = \cosh r - 1,$$

we obtain from (2.23) that

$$\begin{aligned} h(c_{s,t}^2) &= \cosh s \cosh t - \sinh s \sinh t \cos \gamma - 1 \\ &= (\cosh s - 1) \cosh t + \cosh t - 1 - \sinh s \sinh t \cos \gamma \\ &= h(s^2) \cosh t + h(t^2) - \sinh s \sinh t \cos \gamma. \end{aligned}$$

We define a smooth function $g\colon \mathbb{R}^2 \to \mathbb{R}$,

$$g(s, t) = h(s^2) \cosh t + h(t^2) - \sinh s \sinh t \cos \gamma \quad \big(= h(c_{s,t}^2)\big).$$

Then

$$\begin{aligned} g(0, 0) &= 0, \\ g(s, 0) &= h(s^2), \\ g(0, t) &= h(t^2). \end{aligned}$$

The function g can be expressed as the power series

$$g(s,t) = \left(\sum_{i=1}^{\infty} \frac{s^{2i}}{(2i)!}\right) \left(\sum_{i=0}^{\infty} \frac{t^{2i}}{(2i)!}\right) + \left(\sum_{i=1}^{\infty} \frac{t^{2i}}{(2i)!}\right) - \left(\sum_{i=0}^{\infty} \frac{s^{2i+1}}{(2i+1)!}\right) \left(\sum_{i=0}^{\infty} \frac{t^{2i+1}}{(2i+1)!}\right) \cos \gamma,$$

where the coefficient of st is equal to $-\cos\gamma$. Since $g(0,0)=0$, the function $f = h^{-1}\circ g$ is defined in a neighborhood of $(0,0)\in\mathbb{R}^2$. Furthermore, $f(0,0)=0$ and

$$f(s,t) = h^{-1}\big(h(c_{s,t}^2)\big) = c_{s,t}^2$$

for small $s,t>0$. We can write f as an absolutely convergent power series

$$f(s,t) = \sum_{i=1}^{\infty} f_{i,0}s^i + \sum_{j=1}^{\infty} f_{0,j}t^j + st\left(\sum_{i,j=1}^{\infty} f_{i,j}s^{i-1}t^{j-1}\right).$$

Here the coefficient of st is equal to $f_{1,1}$. Since $g(s,0)=h(s^2)$ and $g(0,t)=h(t^2)$, we have

$$s^2 = h^{-1}\big(\underbrace{g(s,0)}_{=h(s^2)}\big) = f(s,0) = \sum_{i=1}^{\infty} f_{i,0}s^i,$$

and similarly

$$t^2 = \sum_{j=1}^{\infty} f_{0,j}t^j.$$

Hence for small $s,t>0$

$$c_{s,t}^2 = f(s,t) = s^2+t^2+st\left(\sum_{i,j=1}^{\infty} f_{i,j}s^{i-1}t^{j-1}\right),$$

and so

$$\sum_{i,j=1}^{\infty} f_{i,j}s^{i-1}t^{j-1} = -\frac{s^2+t^2-c_{s,t}^2}{st}.$$

On the other hand,

$$f(s,t) = h^{-1}\big(g(s,t)\big) = 2g(s,t) + \sum_{i=2}^{\infty} a_i\big(g(s,t)\big)^i$$

by (2.24). Since the coefficient of st is equal to $-\cos\gamma$ in the power series expression of g, we obtain

$$f_{1,1} = -2\cos\gamma.$$

Hence

$$\begin{aligned}\cos\gamma_{s,t} &= \frac{s^2+t^2-c_{s,t}^2}{2st}\\ &= \frac{-st\left(\sum_{i,j=1}^{\infty} f_{i,j}s^{i-1}t^{j-1}\right)}{2st}\\ &= \cos\gamma - \tfrac{1}{2}\sum_{i+j\geq 3} f_{i,j}s^{i-1}t^{j-1}\\ &\to \cos\gamma\end{aligned}$$

as $s,t\to 0$. $\square$

2.25 Definitions of Alexandrov spaces

For $\kappa \in \mathbb{R}$ we denote by D_κ the diameter of the model space M^n_κ. Thus $D_\kappa = \pi/\sqrt{\kappa}$ for $\kappa > 0$ and $D_\kappa = \infty$ for $\kappa \le 0$.

Let X be a metric space. We say that $\Delta \subset X$ is a *geodesic triangle with vertices* $p, q, r \in X$ if

$$\Delta = \gamma_1[0, d(p,q)] \cup \gamma_2[0, d(p,r)] \cup \gamma_3[0, d(q,r)],$$

where γ_1, γ_2, and γ_3 are geodesics joining pairs p, q, p, r, and q, r, respectively. We denote by $\Delta(p,q,r)$ any geodesic triangle with vertices p, q, r. The number $d(p,q) + d(q,r) + d(r,p)$ is called the *perimeter* of Δ. We denote $[p,q] = \gamma_1[0, d(p,q)]$, $[p,r] = \gamma_2[0, d(p,r)]$, and $[q,r] = \gamma_3[0, d(q,r)]$ and call them the *sides* of Δ.

Theorem 2.26 (The law of cosines in M^n_κ)**.** *Let Δ be a geodesic triangle in M^n_κ with vertices A, B, C. Let $a = d(B,C), b = d(A,C), c = d(A,B)$, and let γ be the vertex angle of Δ at C. Then*

(a)

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

if $\kappa = 0$,

(b)

$$\cosh(\sqrt{-\kappa}c) = \cosh(\sqrt{-\kappa}a)\cosh(\sqrt{-\kappa}b) - \sinh(\sqrt{-\kappa}a)\sinh(\sqrt{-\kappa}b)\cos\gamma$$

if $\kappa < 0$, and

(c)

$$\cos(\sqrt{\kappa}c) = \cos(\sqrt{\kappa}a)\cos(\sqrt{\kappa}b) + \sin(\sqrt{\kappa}a)\sin(\sqrt{\kappa}b)\cos\gamma$$

if $\kappa > 0$.

Proof. The claims (for $\kappa \neq 0$) follow from Theorems 2.5 and 2.11 by rescaling the metric. Note that the vertex angle in M^n_κ for $\kappa > 0$ (resp. $\kappa < 0$) is defined exactly as in $\mathbb{S}^n$ (resp. $\mathbb{H}^n$). $\square$

Observe that, for a fixed a, b, and κ, c increases (strictly) from $|a-b|$ to $a+b$ as γ increases from 0 to π.

Definition 2.27. Let $\kappa \in \mathbb{R}$ and let p, q, r be distinct points in a metric space X such that $d(p,q) + d(q,r) + d(r,p) < 2D_\kappa$.

1. A *(κ-)comparison triangle* for the triple (p,q,r) is a geodesic triangle $\bar{\Delta}_\kappa(p,q,r) \subset M^2_\kappa$ consisting of vertices $\bar{p}, \bar{q}, \bar{r} \in M^2_\kappa$ and geodesic segments $[\bar{p},\bar{q}], [\bar{p},\bar{r}], [\bar{q},\bar{r}] \subset M^2_\kappa$ such that $d(\bar{p},\bar{q}) = d(p,q)$, $d(\bar{q},\bar{r}) = d(q,r)$, and $d(\bar{r},\bar{p}) = d(r,p)$.

2. If $\Delta \subset X$ is a geodesic triangle in X with vertices p, q, r, then $\bar{\Delta}_\kappa(p,q,r)$ is also called a (κ-)comparison triangle for Δ.

3. The *κ-comparison angle* between q and r at p, denoted by

$$\angle_p^{(\kappa)}(q,r),$$

is the vertex angle at $\bar{p}$ in a comparison triangle $\bar{\Delta}_\kappa(p,q,r) \subset M^2_\kappa$.

4. We say that $\bar{x} \in [\bar{q},\bar{r}]$ is a *comparison point* of $x \in [q,r]$ if $d(\bar{x},\bar{q}) = d(x,q)$. Comparison points on $[\bar{p},\bar{q}]$ and $[\bar{p},\bar{r}]$ are defined similarly.

Lemma 2.28 (Existence of comparison triangles)**.** *Given $\kappa \in \mathbb{R}$ and three distinct points p, q, r in a metric space X such that $d(p,q) + d(q,r) + d(r,p) < 2D_\kappa$, there exists a κ-comparison triangle $\bar{\Delta}(p,q,r) \subset M^2_\kappa$. It is unique up to an isometry of M^2_κ.*

Proof. Denote $a = d(p,q)$, $b = d(p,r)$, and $c = d(q,r)$. We may assume that $a \le b \le c$. By the triangle inequality, $c \le a + b$. Thus $c \le \pi/\sqrt{\kappa}$ if $\kappa > 0$. Hence we can solve $\gamma \in [0,\pi]$ uniquely from the law of cosines. Fix points $\bar{p}, \bar{q} \in M^2_\kappa$ with $d(\bar{p},\bar{q}) = a$. Let α be a geodesic starting from $\bar{p}$, with $\angle_{\bar{p}}(\alpha, [\bar{p},\bar{q}]) = \pi$. Let $\bar{r}$ be the (unique) point on α such that $d(\bar{p},\bar{r}) = b$. Then $d(\bar{q},\bar{r}) = c$ by the law of cosines. We omit the proof of the claim on uniqueness (cf. Exercises 6). □

Definition 2.29. 1. A metric space X is called k-geodesic, with $k > 0$, if all points $x, y \in X$ within distance $d(x,y) < k$ can be joined by a geodesic.

2. A set $C \subset X$ is called *convex* if all points $x, y \in C$ can be joined by a geodesic and all such geodesics lie in C.

Example 2.30. If $\kappa \le 0$, then all balls in M^n_κ are convex. If $\kappa > 0$, then all closed (open) balls of radius $< \pi/(2\sqrt{\kappa})$ (resp. $\le \pi/(2\sqrt{\kappa})$) are convex. To give an idea how to prove these statements, let us consider open balls in $\mathbb{H}^n$. Closed balls can be treated similarly and the case $\kappa < 0$ follows from these by scaling the metric. The proof for $\kappa > 0$ is similar and is left as an exercise.

Fix a ball $B(p,r) \subset \mathbb{H}^n$ and points $x, y \in B(p,r)$. We know that there exists a unique geodesic segment $[x,y] \subset \mathbb{H}^n$ joining x and y. It is obtained as the intersection of $\mathbb{H}^n$ and the 2-dimensional cone

$$\{s\big(tx + (1-t)y\big) \in \mathbb{R}^{n+1} \colon 0 \le t \le 1,\ s \ge 0\}$$

spanned by $0, x, y$. In the intersection (i.e. on $[x,y]$) we always have $s \le 1$. Thus all points of $[x,y]$ are of the form $z = \lambda x + \mu y$, with $\lambda + \mu \le 1$, $\lambda, \mu \ge 0$. It follows that $z \in B(p,r)$ since

$$\begin{aligned} \cosh d(p,z) &= -\langle p, z\rangle_{n,1} = -\lambda\langle p, x\rangle_{n,1} - \mu\langle p, y\rangle_{n,1} \\ &= \lambda \underbrace{\cosh d(p,x)}_{<\cosh r} + \mu \underbrace{\cosh d(p,y)}_{<\cosh r} \\ &< (\lambda + \mu)\cosh r \le \cosh r. \end{aligned}$$

Hence $[x,y] \subset B(p,r)$.

Given two points $p, q \in M^2_\kappa$, with $d(p,q) < D_\kappa$, there exists a unique (up to a reparameterization) local geodesic, called the *line* pq, $\mathbb{R} \to M^2_\kappa$ passing through p and q. It divides M^2_κ into two components. We say that points $x, y \in M^2_\kappa$ lie on opposite sides of a line if they are in different components of the complement of the line.

Lemma 2.31 (Alexandrov's lemma)**.** *Let $\kappa \in \mathbb{R}$ and consider distinct points $A, B, B', C \in M^2_\kappa$ (if $\kappa > 0$, we assume that $d(C,B) + d(C,B') + d(A,B) + d(A,B') < 2D_\kappa$). Suppose that B and B' lie on opposite sides of the line AC. (Note that the triangle inequality and the assumption above imply that $d(B,B') < D_\kappa$.)*

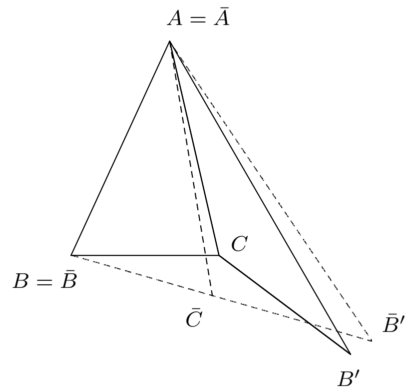
Consider geodesic triangles $\Delta = \Delta(A,B,C)$ and $\Delta' = \Delta(A,B',C)$. Let α, β, γ (resp. α', β', γ') be the vertex angles of Δ (resp. Δ') at A, B, C (resp. A, B', C). Suppose that $\gamma + \gamma' \ge \pi$. Then

$$d(B,C) + d(B',C) \le d(B,A) + d(B',A). \tag{2.32}$$

Let $\overline{\Delta} \subset M^2_\kappa$ be a geodesic triangle with vertices $\bar{A}, \bar{B}, \bar{B}'$ such that $d(\bar{A},\bar{B}) = d(A,B)$, $d(\bar{A},\bar{B}') = d(A,B')$, and $d(\bar{B},\bar{B}') = d(B,C) + d(C,B') < D_\kappa$. Let $\bar{C}$ be the point in $[\bar{B},\bar{B}']$ with $d(\bar{B},\bar{C}) = d(B,C)$. Let $\bar{\alpha}, \bar{\beta}, \bar{\beta}'$ be the vertex angles of $\overline{\Delta}$ at vertices $\bar{A}, \bar{B}, \bar{B}'$. Then

$$\bar{\alpha} \ge \alpha + \alpha',\ \ \bar{\beta} \ge \beta,\ \ \bar{\beta}' \ge \beta',\ \ \text{and } d(\bar{A},\bar{C}) \ge d(A,C). \tag{2.33}$$

Moreover, an equality in any of these implies the equality in the others, and occurs if and only if $\gamma + \gamma' = \pi$.



Proof. (The inequalities in (2.32) and in (2.33) are quite obvious in the special case $\kappa = 0$ as can be seen from a picture like above.)

Let $\tilde{B} \in M_\kappa^2$ be the unique point such that $d(C, \tilde{B}) = d(C, B')$ and $C \in [B, \tilde{B}]$. Then

$$\angle_C\big([C,A],[C,\tilde{B}]\big) \le \gamma' = \angle_C\big([C,A],[C,B']\big)$$

since $\gamma + \gamma' \ge \pi$. Hence

$$d(A, \tilde{B}) \le d(A, B') \tag{2.34}$$

by the law of cosines, with an equality, if and only if $\gamma + \gamma' = \pi$. Consequently,

$$\begin{aligned} d(B,A) + d(B',A) \ge d(B,A) + d(A,\tilde{B}) &\ge d(B,\tilde{B}) \\ &= d(B,C) + \underbrace{d(C,\tilde{B})}_{=d(C,B')}. \end{aligned}$$

Thus (2.32) holds.

Since $d(\bar{A}, \bar{B}') = d(A, B')$, we have

$$d(\bar{A}, \bar{B}') \ge d(A, \tilde{B}) \tag{2.35}$$

by (2.34). Furthermore,

$$d(\bar{B}, \bar{B}') = d(B,C) + d(C,B') \ge d(B,B'). \tag{2.36}$$

Applying the law of cosines to triangles $\overline{\Delta}$ and $\Delta(A, B, B')$ with the inequality (2.36) yields

$$\bar{\alpha} \ge \alpha + \alpha'.$$

This holds as an equality if and only if there is an equality in (2.36), i.e. $\gamma + \gamma' = \pi$. Similarly, the law of cosines, with (2.35) and the equality $d(\bar{B}, \bar{B}') = d(B, \tilde{B})$, implies that

$$\bar{\beta} \ge \beta.$$

Exchanging the roles of B and B' above yields

$$\bar{\beta}' \ge \beta',$$

Again these last two estimates hold as equalities if and only if $\gamma + \gamma' = \pi$. Since $d(\bar{A}, \bar{B}') = d(A, B')$, $d(\bar{C}, \bar{B}') = d(C, B')$, and $\bar{\beta}' \geq \beta'$, we have

$$d(\bar{A}, \bar{C}) \geq d(A, C)$$

again by the law of cosines. Here, too, the equality holds if and only if $\gamma + \gamma' = \pi$. □

Definition 2.37. (1) Let X be a metric space, $\kappa \in \mathbb{R}$, and let $\Delta = [p, q] \cup [p, r] \cup [q, r] \subset X$ be a geodesic triangle with perimeter $< 2D_\kappa$. Let $\bar{\Delta}_\kappa \subset M_\kappa^2$ be a comparison triangle for Δ. We say that Δ satisfies the CAT(κ) *inequality* if, for all $x \in [q, r]$,

$$d(p, x) \leq d(\bar{p}, \bar{x}),$$

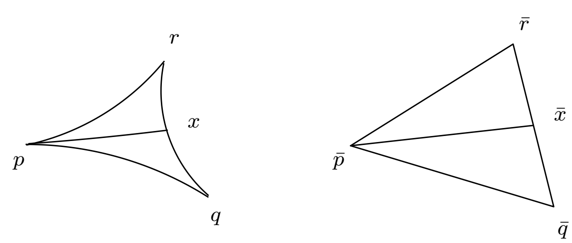
where $\bar{x} \in [\bar{q}, \bar{r}]$ is the comparison point of x.

(2) If $\kappa \leq 0$, a metric space X is called a CAT(κ)*-space* if X is geodesic and all geodesic triangles of X satisfies the CAT(κ)-inequality.

(3) If $\kappa > 0$, a metric space X is called a CAT(κ)*-space* if X is D_κ-geodesic and all geodesic triangles of X with perimeter $< 2D_\kappa$ satisfies the CAT(κ)-inequality.

A complete CAT(0)-space is called a *Hadamard-space*.

The name CAT comes from initials of Cartan, Alexandrov, and Toponogov.



Definition 2.38. 1. A length space X is said to be of *curvature* $\leq \kappa$ if it is locally a CAT(κ)-space. That is, every point $x \in X$ has a neighborhood U which is a CAT(κ)-space when equipped with the induced metric.

2. We say that X is *non-positively curved* if it is of curvature ≤ 0.

3. A metric space X is said to be of *curvature* $\geq \kappa$ if each point of X has a neighborhood U which is geodesic (with respect to the induced metric) and an inequality

$$d(p, x) \geq d(\bar{p}, \bar{x})$$

holds for all geodesic triangles $\Delta = [p, q] \cup [p, r] \cup [q, r] \subset U$ of perimeter $< 2D_\kappa$ and for every $x \in [q, r]$ and its comparison point $\bar{x} \in [\bar{q}, \bar{r}]$.

4. We say that X is *non-negatively curved* if it is of curvature ≥ 0.

In general, metric spaces with curvature bounded from below or from above are called Alexandrov spaces.

3 CAT(κ)-spaces and spaces of curvature bounded from above

3.1 Characterizations and basic properties of CAT(κ)-spaces

First we present some characterizations of CAT(κ)-spaces.

Theorem 3.2. *Let $\kappa \in \mathbb{R}$ and suppose that X is D_κ-geodesic. Then the following are equivalent (if $\kappa > 0$, all geodesic triangles below are assumed to have perimeter $< 2D_\kappa$):*

(1) X is a CAT(κ)*-space.*

(2) For every geodesic triangle $\Delta \subset X$ and for all $x, y \in \Delta$,

$$d(x,y) \le d(\bar{x},\bar{y}),$$

where $\bar{x}, \bar{y} \in \bar{\Delta}_\kappa \subset M_\kappa^2$ are the comparison points of x and y.

(3) For every geodesic triangle $\Delta \subset X$ with vertices p, q, r, and for all $x \in [p,q]$, $y \in [p,r]$, with $x \ne p \ne y$, we have

$$\angle_p^{(\kappa)}(x,y) \le \angle_p^{(\kappa)}(q,r).$$

(4) For every geodesic triangle $\Delta \subset X$, with distinct vertices p, q, r, the Alexandrov angle between $[p,q]$ and $[p,r]$ at p is at most the κ-comparison angle between q and r at p, i.e.

$$\angle_p\big([p,q],[p,r]\big) \le \angle_p^{(\kappa)}(q,r).$$

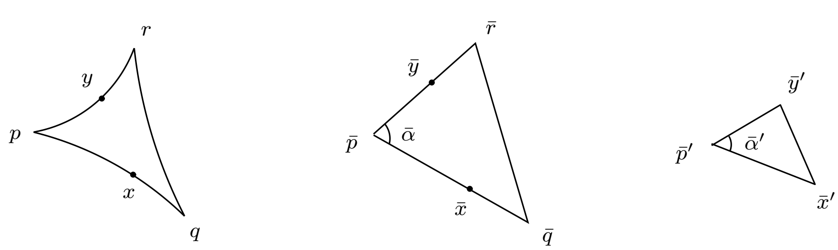
(5) For every geodesic triangle $\Delta \subset X$, with distinct vertices p, q, r and with the Alexandrov angle $\gamma = \angle_p([p,q],[p,r])$ between $[p,q]$ and $[p,r]$ at p, if $\Delta(\hat{p},\hat{q},\hat{r}) \subset M_\kappa^2$ is a geodesic triangle such that $d(\hat{p},\hat{q}) = d(p,q)$, $d(\hat{p},\hat{r}) = d(p,q)$, and $\gamma = \angle_{\hat{p}}(\hat{q},\hat{r})$ (= the vertex angle between $[\hat{p},\hat{q}]$ and $[\hat{p},\hat{r}]$), then

$$d(q,r) \ge d(\hat{q},\hat{r}).$$

Proof. First we note that (2) implies (1) trivially. Also it is easily seen, by using the law of cosines, that (4) and (5) are equivalent. Furthermore, it follows from Theorem 2.22 that one could use κ-comparison angles instead of Euclidean comparison angles in the definition of an Alexandrov angle. Hence (3) implies (4).

Let p, q, r, x, and y be as in (3). Let $\bar{\Delta} = \bar{\Delta}_\kappa(p,q,r)$ and $\bar{\Delta}' = \bar{\Delta}_\kappa(p,x,y)$ be κ-comparison triangles of $\Delta(p,q,r)$ and $\Delta(p,x,y)$ with vertices $\bar{p},\bar{q},\bar{r}$ and $\bar{p}',\bar{x}',\bar{y}'$, respectively. Denote by $\bar{x} \in \bar{\Delta}$ and $\bar{y} \in \bar{\Delta}$ the comparison points of x and y. Let

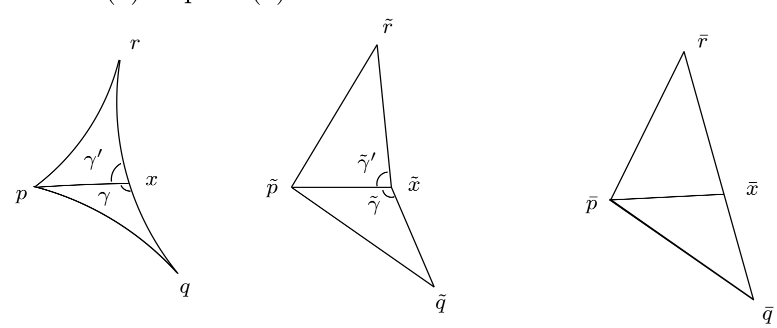
$$\bar{\alpha} = \angle_p^{(\kappa)}(q,r) \quad \text{and} \quad \bar{\alpha}' = \angle_p^{(\kappa)}(x,y).$$



By the law of cosines,
$$d(\bar{x},\bar{y}) \geq \underbrace{d(\bar{x}',\bar{y}')}_{=d(x,y)} \iff \bar{\alpha} \geq \bar{\alpha}'.$$
Hence (2) and (3) are equivalent.

Next we prove that (1) implies (3). Let p,q,r,x, and y be as in (3) and let $\bar{\Delta}$, $\bar{\Delta}'$, $\bar{\alpha}$, and $\bar{\alpha}'$ be as above. Furthermore, let $\bar{\Delta}'' = \bar{\Delta}_\kappa(p,x,r)$ be a κ-comparison triangle of $\Delta(p,x,r)$ with vertices $\bar{p}'',\bar{x}'',\bar{r}''$ and denote
$$\bar{\alpha}'' = \angle_p^{(\kappa)}(x,r).$$
By the assumption (1),
$$d(x,y) \leq d(\bar{x}'',\bar{y}''),$$
where $\bar{y}'' \in [\bar{p}'',\bar{r}'']$ is the comparison point of $y \in [p,r]$. Since $d(\bar{x}',\bar{y}') = d(x,y)$, we get
$$\bar{\alpha}' \leq \bar{\alpha}''$$
from the law of cosines. Similarly, by (1),
$$d(\bar{x},\bar{r}) \geq d(x,r) = d(\bar{x}'',\bar{r}''),$$
and so
$$\bar{\alpha} \geq \bar{\alpha}''$$
again by the law of cosines. Hence
$$\bar{\alpha}' \leq \bar{\alpha}$$
and (3) follows.

Finally, we prove that (4) implies (1).



Let $\Delta \subset X$ be a geodesic triangle with vertices p,q,r and let $x \in [q,r], p \neq x \neq q$. Let $\bar{\Delta} = \bar{\Delta}_\kappa(p,q,r)$ be a comparison triangle with vertices $\bar{p},\bar{q},\bar{r}$. Choose comparison triangles $\bar{\Delta}' = \bar{\Delta}_\kappa(p,x,q)$ and $\bar{\Delta}'' = \bar{\Delta}_\kappa(p,x,r)$ with vertices $\tilde{p},\tilde{x},\tilde{q}$ and $\tilde{p},\tilde{x},\tilde{r}$, respectively, such that they have a common side $[\tilde{p},\tilde{x}]$ and that $\tilde{q}$ and $\tilde{r}$ lie on opposite sides of the line $\tilde{p}\tilde{x}$. Let
$$\gamma = \angle_x([x,p],[x,q]) \quad \text{and} \quad \gamma' = \angle_x([x,p],[x,r])$$
be Alexandrov angles and let
$$\tilde{\gamma} = \angle_{\tilde{x}}(\tilde{p},\tilde{q}) = \angle_x^{(\kappa)}(p,q) \quad \text{and} \quad \tilde{\gamma}' = \angle_{\tilde{x}}(\tilde{p},\tilde{r}) = \angle_x^{(\kappa)}(p,r)$$
be vertex angles at $\tilde{x}$ in M_κ^2. The triangle inequality for Alexandrov angles (Theorem 2.17) and Remark 2.16.2 imply that $\gamma + \gamma' \geq \pi$. By the assumption (4),
$$\tilde{\gamma} \geq \gamma \quad \text{and} \quad \tilde{\gamma}' \geq \gamma'.$$

Hence $\tilde{\gamma} + \tilde{\gamma}' \geq \pi$. By Alexandrov's lemma 2.31,

$$d(\bar{p}, \bar{x}) \geq d(\tilde{p}, \tilde{x}) = d(p, x).$$

Hence X is a CAT(κ) space, i.e. (1) holds. $\square$

Theorem 3.3. *For any $\kappa \in \mathbb{R}$, M^2_κ is a* CAT($\tilde{\kappa}$)*-space if and only if $\tilde{\kappa} \geq \kappa$.*

Proof. We will give two proofs for the result. The first one uses the criterion 3.2(4) and the law of cosines. The second one that appears in Remark 3.6 involves features from Riemannian geometry.

Fix $\kappa \in \mathbb{R}$ and $\tilde{\kappa} > \kappa$. Clearly M^2_κ is a CAT(κ)-space. We will use the criterion 3.2(4) to show that M^2_κ is a CAT($\tilde{\kappa}$)-space, but not a CAT(κ')-space for any $\kappa' < \kappa$. Fix $p \in M^2_\kappa$ and $\tilde{p} \in M^2_{\tilde{\kappa}}$. Consider geodesic triangles $\Delta_t \subset M^2_\kappa$ and $\tilde{\Delta}_t \subset M^2_{\tilde{\kappa}}$ with vertices $p, q, r_t \in M^2_\kappa$ and $\tilde{p}, \tilde{q}, \tilde{r}_t \in M^2_{\tilde{\kappa}}$ such that

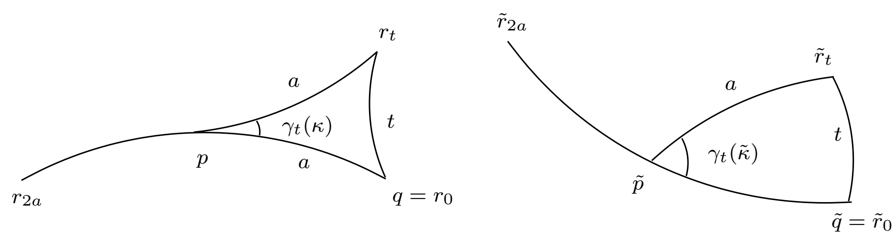
$$d(p, q) = d(p, r_t) = d(\tilde{p}, \tilde{q}) = d(\tilde{p}, \tilde{r}_t) = a \in (0, D_{\tilde{\kappa}}/2)$$

and that

$$d(q, r_t) = d(\tilde{q}, \tilde{r}_t) = t \in (0, 2a).$$

It suffices to show that

$$\gamma_t(\kappa) := \angle_p(q, r_t) < \angle_{\tilde{p}}(\tilde{q}, \tilde{r}_t) =: \gamma_t(\tilde{\kappa}). \tag{3.4}$$



By the law of cosines,

$$\cos \gamma_t(\kappa) = \begin{cases} \dfrac{\cosh^2(\sqrt{-\kappa}a) - \cosh(\sqrt{-\kappa}t)}{\sinh^2(\sqrt{-\kappa}a)}, & \kappa < 0; \\[2ex] 1 - \dfrac{t^2}{2a^2}, & \kappa = 0; \\[2ex] \dfrac{\cos(\sqrt{\kappa}t) - \cos^2(\sqrt{\kappa}a)}{\sin^2(\sqrt{\kappa}a)}, & \kappa > 0. \end{cases}$$

Hence (3.4) follows once we show that, for fixed a and t, the function $\kappa \mapsto \cos \gamma_t(\kappa)$ is strictly decreasing on the interval $(-\infty, \pi^2/a^2)$. We omit the verification of this. $\square$

Theorem 3.5. *(1) If X is a* CAT(κ')*-space for all $\kappa' > \kappa$, then it is also a* CAT(κ)*-space.*

(2) A CAT(κ)*-space X is a* CAT(κ')*-space for all $\kappa' > \kappa$.*

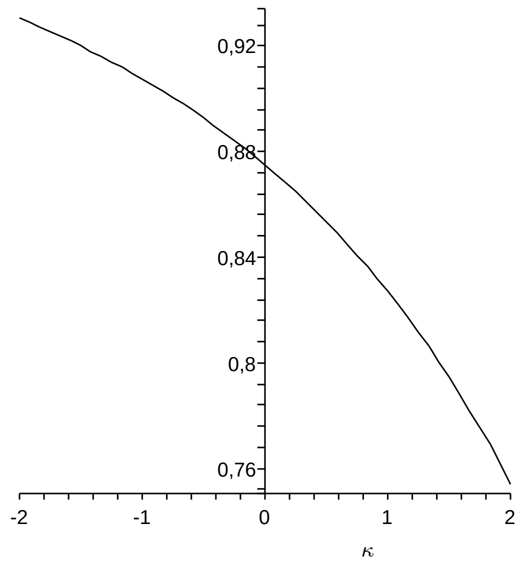


Figure 1: Graph of the function $\kappa \mapsto \cos\gamma_t(\kappa)$ with $a = 1$ and $t = 1/2$.

Proof. Suppose that X is a CAT(κ')-space for all $\kappa' > \kappa$. If $x, y \in X$ with $d(x, y) < D_\kappa$, then $d(x, y) < D_{\kappa'}$ for all $\kappa' > \kappa$ sufficiently close to κ. Since X is a CAT(κ')-space, it is, in particular, $D_{\kappa'}$-geodesic. Hence there exists a geodesic joining x and y. It follows that X is D_κ-geodesic. Let $\Delta = \Delta(p, q, r) \subset X$ be a geodesic triangle of perimeter $< 2D_\kappa$. Consider sufficiently small $\kappa' > \kappa$ so that the perimeter of Δ is less that $2D_{\kappa'}$. Write $a = d(p, q),\ b = d(p, r),\ c = d(q, r)$ and let $\gamma = \angle_p\big([p, q], [p, r]\big)$ be the Alexandrov angle at p.

We will use the characterization 3.2(5) of the CAT(κ')-property of X. For $\kappa \geq 0$, we have

$$\cos(\sqrt{\kappa'}a)\cos(\sqrt{\kappa'}b) + \sin(\sqrt{\kappa'}a)\sin(\sqrt{\kappa'}b)\cos\gamma = \cos\big(\sqrt{\kappa'}\,\underbrace{d(\hat{q}, \hat{r})}_{\leq c}\big) \geq \cos(\sqrt{\kappa'}c),$$

where $\hat{q}, \hat{r}$ are as in 3.2(5). By letting $\kappa' \to \kappa$, we obtain, in the case $\kappa > 0$, the same inequality with κ' replaced by κ. If $\kappa = 0$, we get the inequality

$$c^2 \geq a^2 + b^2 - 2ab\cos\gamma.$$

Thus in both cases, 3.2(5) implies that X is a CAT(κ)-space. If $\kappa < 0$, applying 3.2(5) with $\kappa' \in (\kappa, 0)$ yields

$$\cosh(\sqrt{-\kappa'}c) \geq \cosh(\sqrt{-\kappa'}a)\cosh(\sqrt{-\kappa'}b) - \sinh(\sqrt{-\kappa'}a)\sinh(\sqrt{-\kappa'}b)\cos\gamma.$$

Letting $\kappa' \to \kappa$, we obtain the same inequality with κ' replaced by κ. Hence X is a CAT(κ)-space by 3.2(5). We have proved (1).

We may use Theorem 3.3 to prove (2). Suppose that X is a CAT(κ)-space and $\kappa' > \kappa$. Let $\Delta \subset X$ be a geodesic triangle with vertices p, q, r and let $x \in [q, r]$. Let $\bar{\Delta} = \bar{\Delta}_\kappa(p, q, r) \subset M^2_\kappa$ and $\bar{\Delta}' = \bar{\Delta}_{\kappa'}(p, q, r) \subset M^2_{\kappa'}$ be comparison triangles of Δ with vertices $\bar{p}, \bar{q}, \bar{r}$ and $\bar{p}', \bar{q}', \bar{r}'$, respectively. Let $\bar{x} \in \bar{\Delta}$ and $\bar{x}' \in \bar{\Delta}'$ be the comparison points of x. Observe that $\bar{\Delta}'$ is a κ'-comparison triangle of $\bar{\Delta}$. Since X is a CAT(κ)-space and M^2_κ is a CAT(κ')-space, we have

$$d(p, x) \leq d(\bar{p}, \bar{x}) \leq d(\bar{p}', \bar{x}').$$

Hence X is a CAT(κ')-space. $\square$

Remark 3.6. Here we present another proof of Theorem 3.3. For that purpose we introduce polar coordinates in M^2_κ. Suppose first that $\kappa = -1$. Let $p = (0,0,1) \in M^2_{-1} \subset \mathbb{R}^3$ and consider geodesic rays starting at p. They are intersections of M^2_{-1} and 2-planes containing the x_3-axis and they are parameterized by $\alpha\colon [0,\infty) \to M^2_{-1} \subset \mathbb{R}^3$,

$$\alpha(r) = (\cosh r)\underbrace{(0,0,1)}_{=p} + (\sinh r)\underbrace{(\cos\vartheta, \sin\vartheta, 0)}_{=u}, \tag{3.7}$$

where $u \in p^\perp = \{(x,y,0)\colon (x,y) \in \mathbb{R}^2\}$ and $\langle u,u\rangle_{2,1} = 1$. Note that $\langle\cdot,\cdot\rangle_{2,1}|p^\perp$ coincides with the usual inner product of $\mathbb{R}^2$. Since every point $x \in M^2_{-1} \setminus \{p\}$ can be joined to p by a unique geodesic, the formula (3.7) defines polar coordinates $(r,\vartheta) \in (0,\infty) \times \mathbb{S}^1$ for points in $M^2_{-1} \setminus \{p\}$. It is convenient to identify the angle ϑ with the point $(\cos\vartheta, \sin\vartheta) \in \mathbb{S}^1$.

Since M^2_κ, for $\kappa < 0$, is obtained from M^2_{-1} by scaling the metric, we have polar coordinates (r,ϑ) also for points in $M^2_\kappa \setminus \{p\}$. (Here r is the distance to the fixed point p with respect to the metric in M^2_κ.) Similarly, we obtain polar coordinates for points $x \in M^2_\kappa$, $0 < d(p,x) < D_\kappa$ if $\kappa > 0$ and $p \in M^2_\kappa$ is fixed.

What is the length of the circle $S_\kappa(p,r) = \{x \in M^2_\kappa\colon d(x,p) = r\}$? Let us again consider the case $\kappa = -1$, $p = (0,0,1)$ and denote $S(r) = S_{-1}(p,r)$. Then

$$\begin{aligned} S(r) &= \{x \in \mathbb{R}^3\colon \langle x,x\rangle_{2,1} = -1, \cosh r = -\langle x,(0,0,1)\rangle_{2,1}\} \\ &= \{(x_1,x_2,x_3) \in \mathbb{R}^3\colon x_3 = \cosh r, x_1^2 + x_2^2 = \sinh^2 r\}. \end{aligned}$$

Thus $S(r)$ is a circle of Euclidean radius $\sinh r$ on the affine plane

$$\{(x_1,x_2,x_3) \in \mathbb{R}^3\colon x_3 = \cosh r\}.$$

It can be parameterized by $\gamma\colon [0,2\pi] \to S(r)$,

$$\gamma(\vartheta) = (\cos\vartheta\sinh r, \sin\vartheta\sinh r, \cosh r).$$

This can be obtained also directly from (3.7).

By the law of cosines,

$$\cosh d\big(\gamma(\vartheta+t),\gamma(\vartheta)\big) = \cosh^2 r - \sinh^2 r\cos t,$$

and hence we obtain the equality

$$|\dot\gamma|(\vartheta) = \lim_{t\to 0}\frac{d\big(\gamma(\vartheta+t),\gamma(\vartheta)\big)}{|t|} = \sinh r$$

for the metric derivative of γ. Thus

$$\ell(\gamma) = \int_0^{2\pi} |\dot\gamma|(\vartheta)\,d\vartheta = 2\pi\sinh r.$$

It is worth noting that the derivative of γ at ϑ is the vector

$$\gamma'(\vartheta) = (-\sin\vartheta\,\sinh r, \cos\vartheta\,\sinh r, 0) \in \mathbb{R}^3$$

and hence

$$\langle \gamma'(\vartheta), \gamma'(\vartheta)\rangle_{2,1}^{1/2} = \sinh r.$$

The other values of κ can be treated similarly and we have

$$|\dot{\gamma}|(\vartheta) = \begin{cases} \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa} r), & \kappa < 0; \\ r, & \kappa = 0; \\ \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa} r), & \kappa > 0. \end{cases} \tag{3.8}$$

We denote by $f(\kappa, r)$ the function defined by the right-hand side of (3.8). It is easy to see that, for a fixed r, the function $\kappa \mapsto f(\kappa, r)$ is strictly decreasing

Since any point of M^2_κ can be mapped to $p = (0,0,1)$ by an isometry of M^2_κ (cf. Exercises 6), we may place the "origin" of polar coordinates to any point of M^2_κ. Suppose that $\tilde{\kappa} > \kappa$. Fix $p \in M^2_\kappa$ and a geodesic ray M^2_κ starting at p. Similarly, we fix $\tilde{p} \in M^2_{\tilde{\kappa}}$ and a geodesic ray starting at $\tilde{p}$. Then we have polar coordinates $(r, \vartheta)_\kappa$ in M^2_κ and $(r, \vartheta)_{\tilde{\kappa}}$ in $M^2_{\tilde{\kappa}}$, where r is the distance to p (resp. $\tilde{p}$) and the angle ϑ is measured from the fixed geodesic rays. Using these polar coordinates we define a mapping

$$h\colon \underbrace{B(\tilde{p}, D_{\tilde{\kappa}})}_{\subset M^2_{\tilde{\kappa}}} \to \underbrace{B(p, D_{\tilde{\kappa}})}_{\subset M^2_\kappa},$$
$$h\big((r,\vartheta)_{\tilde{\kappa}}\big) = (r,\vartheta)_\kappa, \quad h(\tilde{p}) = p.$$

Then h preserves the distance from $\tilde{p}$, that is,

$$d\big(h(x), h(\tilde{p})\big) = d(x, \tilde{p}) \quad \forall x \in B(\tilde{p}, D_{\tilde{\kappa}}).$$

We claim that

$$d\big(h(x), h(y)\big) \ge d(x, y), \tag{3.9}$$

with an equality if and only if $\tilde{p}$, x and y lie on a same geodesic. If $\tilde{p}$, x, and y lie on a same geodesic, there are three possible cases:

$$\begin{aligned} d(x,y) &= d(x,\tilde{p}) + d(\tilde{p},y) \quad \text{or} \\ d(\tilde{p},x) &= d(\tilde{p},y) + d(y,x) \quad \text{or} \\ d(\tilde{p},y) &= d(\tilde{p},x) + d(x,y). \end{aligned}$$

There is an equality in (3.9) in all these cases. In order to prove the rest of the claim above, let us study how the length of a (smooth) path changes under h. Let $I \subset \mathbb{R}$ be an open interval and let $\alpha\colon I \to M^2_{-1}$ be a smooth path, i.e. α is a smooth mapping into $\mathbb{R}^3$ and $\alpha(t) \in M^2_{-1}$ for all t. We write $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, where $\alpha_i\colon I \to \mathbb{R},\ i = 1,2,3$. For all $t \in I$, we have

$$\alpha'(t) = \underbrace{\big(\alpha_1'(t), \alpha_2'(t), \alpha_3'(t)\big)}_{\in \mathbb{R}^3} \in \alpha(t)^\perp$$

since

$$\langle \alpha'(t), \alpha(t)\rangle_{2,1} = \tfrac{1}{2}\tfrac{d}{dt} \underbrace{\langle \alpha(t), \alpha(t)\rangle_{2,1}}_{\equiv -1} \equiv 0.$$

Next we express α in the polar coordinates as

$$\alpha(t) = \big(\alpha_r(t), \alpha_\vartheta(t)\big) \in [0,\infty) \times \mathbb{S}^1.$$

Then

$$\begin{aligned}
\alpha_1(t) &= \sinh \alpha_r(t) \cos \alpha_\vartheta(t),\\
\alpha_2(t) &= \sinh \alpha_r(t) \sin \alpha_\vartheta(t),\\
\alpha_3(t) &= \cosh \alpha_r(t),
\end{aligned}$$

and

$$\begin{aligned}
\alpha_1'(t) &= \cosh \alpha_r(t) \cos \alpha_\vartheta(t)\alpha_r'(t) - \sinh \alpha_r(t) \sin \alpha_\vartheta(t)\alpha_\vartheta'(t),\\
\alpha_2'(t) &= \cosh \alpha_r(t) \sin \alpha_\vartheta(t)\alpha_r'(t) + \sinh \alpha_r(t) \cos \alpha_\vartheta(t)\alpha_\vartheta'(t),\\
\alpha_3'(t) &= \sinh \alpha_r(t)\alpha_r'(t).
\end{aligned}$$

We claim that

$$|\dot\alpha|(t) = \sqrt{\langle \alpha'(t), \alpha'(t)\rangle_{2,1}} = \sqrt{\alpha_r'(t)^2 + \sinh^2 \alpha_r(t)\alpha_\vartheta'(t)^2} \tag{3.10}$$

for all t. The equation on the right-hand side of (3.10) follows from the equations above since, by definition,

$$\langle \alpha'(t), \alpha'(t)\rangle_{2,1} = -\alpha_3'(t)^2 + \alpha_1'(t)^2 + \alpha_2'(t)^2.$$

To prove the equation on the left-hand side of (3.10), we first observe that

$$\frac{-2\big\langle \alpha(t), \alpha(t+s)\big\rangle_{2,1} - 2}{s^2} = \frac{-\big(\alpha_3(t+s) - \alpha_3(t)\big)^2 + \big(\alpha_1(t+s) - \alpha_1(t)\big)^2 + \big(\alpha_2(t+s) - \alpha_2(t)\big)^2}{s^2}$$

$$\to -\alpha_3'(t)^2 + \alpha_1'(t)^2 + \alpha_2'(t)^2 = \langle \alpha'(t), \alpha'(t)\rangle_{2,1}$$

as $s \to 0$. Hence there exists, for every $t \in I$, a constant $L_t > 0$ such that

$$\cosh d\big(\alpha(t+s), \alpha(t)\big) = -\big\langle \alpha(t+s), \alpha(t)\big\rangle_{2,1} \le 1 + \frac{L_t^2}{2} s^2$$

whenever $|s|$ is small enough. Since

$$1 + \frac{L_t^2}{2} s^2 \le \cosh(L_t|s|),$$

we obtain

$$d\big(\alpha(t), \alpha(t+s)\big) \le L_t|s|$$

for small $|s|$. Therefore

$$\begin{aligned}
-\langle \alpha(t), \alpha(t+s)\rangle_{2,1} &= \cosh d\big(\alpha(t), \alpha(t+s)\big)\\
&= 1 + \tfrac{1}{2} d^2\big(\alpha(t), \alpha(t+s)\big) + O(s^4),
\end{aligned}$$

and so

$$\begin{aligned}
\frac{d^2\big(\alpha(t), \alpha(t+s)\big)}{s^2} &= \frac{-2\langle \alpha(t), \alpha(t+s)\rangle_{2,1} - 2}{s^2} + O(s^2)\\
&\to \langle \alpha'(t), \alpha'(t)\rangle_{2,1}
\end{aligned}$$

as $s \to 0$. Hence (3.10) holds. Similarly, for a smooth path $\alpha\colon I \to M^2_\kappa,\ \kappa \in \mathbb{R}$, we have

$$|\dot\alpha|(t) = \sqrt{\alpha_r'(t)^2 + f^2\big(\kappa, \alpha_r(t)\big)\alpha_\vartheta'(t)^2}. \tag{3.11}$$

We can now easily prove the claim (3.9) for the mapping $h\colon B(\tilde p, D_{\tilde\kappa}) \to B(p, D_{\tilde\kappa})$ between the balls $B(\tilde p, D_{\tilde\kappa}) \subset M^2_{\tilde\kappa}$ and $B(p, D_{\tilde\kappa}) \subset M^2_\kappa$. Suppose that x and y are points in $B(\tilde p, D_{\tilde\kappa}) \subset M^2_{\tilde\kappa}$ such that $\tilde p, x, y$ do not lie on a same geodesic. Denote $d = d\big(h(x), h(y)\big)$ and let $\alpha\colon [0,d] \to M^2_\kappa$ be a geodesic from $h(x)$ to $h(y)$. Suppose that $d\big(p, \alpha(t)\big) < D_{\tilde\kappa}$ for all $t \in [0,d]$ (the other case is left as an exercise). Then $\beta = h^{-1} \circ \alpha$ is a path from x to y, and hence

$$d(x,y) \le \ell(\beta) = \int_0^d |\dot\beta|(t)\,dt.$$

By (3.11),

$$|\dot\beta|(t) = \sqrt{\beta_r'(t)^2 + f^2\big(\tilde\kappa, \beta_r(t)\big)\beta_\vartheta'(t)^2}.$$

Here $\alpha_r'(t) \equiv \beta_r'(t)$ and $\alpha_\vartheta'(t) \equiv \beta_\vartheta'(t) \not\equiv 0$ since $\beta_r = \alpha_r$ and $\beta_\vartheta = \alpha_\vartheta$ and $\tilde p, x, y$ do not lie on a same geodesic. Then

$$0 < f\big(\tilde\kappa, \beta_r(t)\big) < f\big(\kappa, \alpha_r(t)\big),$$

and we obtain

$$d(x,y) < \ell(\alpha) = d\big(h(x), h(y)\big).$$

Finally, (3.9) and the criterion 3.2(5) imply that M^2_κ is a CAT($\tilde\kappa$)-space if and only if $\tilde\kappa \ge \kappa$.

Theorem 3.12. *A* CAT(κ)*-space X has the following properties:*

(1) For each $x, y \in X$, with $d(x,y) < D_\kappa$, there exists a unique geodesic segment from x to y. This geodesic segment varies continuously with its endpoints. That is, if $x_n \to x,\ y_n \to y$, with $d(x_n, y_n) < D_\kappa$, and if $\alpha_n\colon [0,1] \to X$ and $\alpha\colon [0,1] \to X$ are constant speed geodesics such that $\alpha_n(0) = x_n,\ \alpha(0) = x,\ \alpha_n(1) = y_n$, and $\alpha(1) = y$, then $\alpha_n \to \alpha$ uniformly.

(2) Local geodesics in X of length $\le D_\kappa$ are geodesics.

(3) Balls in X of radius $< D_\kappa/2$ are convex. That is, any two points in a ball of radius $< D_\kappa/2$ can be joined by a unique geodesic segment and this geodesic segment is contained in the ball.

(4) Balls in X of radius $< D_\kappa$ are contractible.[5]

(5) For every $\lambda < D_\kappa$ and $\varepsilon > 0$ there exists $\delta = \delta(\kappa, \lambda, \varepsilon)$ such that if m is the midpoint of a geodesic segment $[x,y] \subset X$, with $d(x,y) \le \lambda$, and if

$$\max\{d(x,m'), d(y,m')\} \le \tfrac12 d(x,y) + \delta,$$

then $d(m,m') < \varepsilon$.

[5]Recall that a topological space Y is *contractible* if there exists a point $x_0 \in X$ and a homotopy $h\colon X \times [0,1] \to X$ such that

$$h(x,0) = x \quad\text{and}\quad h(x,1) = x_0\ \forall x \in X.$$

Proof. (1) Let $p, q \in X$ with $d(p,q) < D_\kappa$. Since X is D_κ-geodesic by definition, there exists a geodesic from x to y. Suppose that $[p,q]$ and $[p,q]'$ are geodesic segments. Let $r \in [p,q]$, $p \neq r \neq q$, and $r' \in [p,q]'$, $p \neq r' \neq q$, be such that $d(p,r) = d(p,r')$. Consider the geodesic triangle

$$\Delta = [p,r] \cup [r,q] \cup [p,q]',$$

where $[p,r], [r,q] \subset [p,q]$. Then any κ-comparison triangle of Δ is degenerate, and therefore the comparison points $\bar{r}$ and $\bar{r}'$ (of r and r') are the same. By the criterion 3.2(2),

$$d(r,r') \leq d(\bar{r}, \bar{r}') = 0.$$

Since $r \in [p,q]$ is arbitrary, we have $[p,q] = [p,q]'$.

To prove the second statement in (1), let x_n, y_x, α_n, and α be as in the claim. We may assume that

$$d(x,y), d(x_n,y_n), d(x,y_n) \leq L < D_\kappa. \tag{3.13}$$

Let $\alpha_n' \colon [0,1] \to X$ be the (unique) constant speed geodesic from x to y_n. If $\kappa \leq 0$, we obtain (cf. Exercise 7/1)

$$d\big(\alpha_n(t), \alpha_n'(t)\big) \leq (1-t) d(x_n, x) \leq d(x_n, x)$$

and

$$d\big(\alpha_n'(t), \alpha(t)\big) \leq t\, d(y_n, y) \leq d(y_n, y),$$

and hence $d\big(\alpha_n(t), \alpha(t)\big) \to 0$ uniformly in t. Suppose then that $\kappa > 0$. Since $y_n \to y$, the perimeter of $\Delta(x, y_n, y)$ is less that $2D_\kappa$ for large n, and we have by the criterion 3.2(3) that

$$\angle_x^{(\kappa)}\big(\alpha_n'(t), \alpha(t)\big) \leq \angle_x^{(\kappa)}(y_n, y)$$

for every $t \in [0,1]$. Furthermore, $\angle_x^{(\kappa)}(y_n, y) \to 0$ by the law of cosines and (3.13). Let $\bar{x}, \bar{\alpha}(t)$, and $\bar{\alpha}_n'(t)$ be the vertices of a κ-comparison triangle of $\Delta(x, \alpha(t), \alpha_n'(t))$. Furthermore, let $\tilde{x}, \tilde{y}, \tilde{y}_n, \tilde{\alpha}_n'(t)$, and $\tilde{\alpha}(t)$ be points in $\mathbb{R}^2$ such that

$$\begin{aligned}
\tilde{\alpha}_n'(t) \in [\tilde{x}, \tilde{y}_n], &\quad \tilde{\alpha}(t) \in [\tilde{x}, \tilde{y}], \\
|\tilde{x} - \tilde{\alpha}_n'(t)| &= d\big(x, \alpha_n'(t)\big) = d\big(\bar{x}, \bar{\alpha}_n'(t)\big), \\
|\tilde{x} - \tilde{y}_n| &= d(x, y_n), \\
|\tilde{x} - \tilde{\alpha}(t)| &= d\big(x, \alpha(t)\big) = d\big(\bar{x}, \bar{\alpha}(t)\big), \\
|\tilde{x} - \tilde{y}| &= d(x,y), \quad \text{and} \\
\angle_x^{(0)}(y_n, y) &= \angle_x^{(\kappa)}(y_n, y).
\end{aligned}$$

Since $\mathbb{R}^2$ is a CAT(κ)-space for $\kappa \geq 0$, we obtain by the criterion 3.2(5) that

$$d\big(\alpha(t), \alpha_n'(t)\big) = d\big(\bar{\alpha}(t), \bar{\alpha}_n'(t)\big) \leq |\tilde{\alpha}(t) - \tilde{\alpha}_n'(t)| \leq |\tilde{y} - \tilde{y}_n|.$$

By the (usual) law of cosines, $|\tilde{y} - \tilde{y}_n| \to 0$ as $n \to \infty$. Hence

$$d\big(\alpha(t), \alpha_n'(t)\big) \to 0$$

uniformly in t. Similarly,

$$d\big(\alpha_n(t), \alpha_n'(t)\big) \to 0$$

uniformly in t, and therefore $\alpha_n \to \alpha$ uniformly.

(2) Let $\gamma\colon [0,L] \to X$ be a local geodesic of length $L \le D_\kappa$. Let

$$T = \{t \in [0,L] \colon \gamma|[0,t] \text{ is a geodesic}\}.$$

Then clearly T is closed and non-empty. Thus we obtain $T = [0,L]$ if we show that T is also open. To prove this, let $t_0 \in T,\ 0 < t_0 < L$. Since γ is a local geodesic, there exists $0 < \varepsilon < \min\{L - t_0, t_0\}$ such that $\gamma|[t_0 - \varepsilon, t_0 + \varepsilon]$ is a geodesic. Consider a geodesic triangle

$$\Delta = \gamma[0,t_0] \cup \gamma[t_0, t_0 + \varepsilon] \cup [\gamma(0), \gamma(t_0 + \varepsilon)].$$

Then the Alexandrov angle between segments $\gamma[t_0 - \varepsilon, t_0]$ and $\gamma[t_0, t_0 + \varepsilon]$ at $\gamma(t_0)$ is equal to π, and therefore, by 3.2(4), the same is true for the κ-comparison angle. It follows that any κ-comparison triangle of Δ is degenerate, and consequently

$$\ell(\gamma|[0, t_0 + \varepsilon]) = d\big(\gamma(0), \gamma(t_0 + \varepsilon)\big).$$

Hence $[0, t_0 + \varepsilon] \subset T$, and so $T = [0,L]$.

(3) Let $x, y \in B(p,r)$, where $r < D_\kappa/2$. Then $d(x,y) < D_\kappa$, and hence there exists a unique geodesic segment $[x,y]$. Since X is a CAT(κ)-space and $B(\bar{p}, r) \subset M^2_\kappa$ is convex, we have

$$d(p,z) \le d(\bar{p}, \bar{z}) < r$$

for all $z \in [x,y]$. Hence $[x,y] \subset B(p,r)$, and so $B(p,r)$ is convex.

(4) Let $B = \bar{B}(x,r),\ r < D_\kappa$, and let $h\colon B \times [0,1] \to X$ be the mapping such that $h(y,t)$ is the unique point z on the unique geodesic segment $[x,y]$, with $d(z,y) = t\, d(x,y)$. By (1), the segment $[x,y]$ varies continuously with y and hence h is continuous. Clearly, $h(y,0) = y$ and $h(y,1) = x$ for every $y \in B$. Thus B is contractible.

(5) Let x, y, m', and m be as in the claim. Consider a κ-comparison triangle $\bar{\Delta}_\kappa(x,y,m')$. Then

$$d(m,m') \le d(\bar{m}, \bar{m}')$$

and the claim follows from Exercise 7/4. □

Corollary 3.14. *A* CAT(κ)*-space X, with $\kappa \le 0$, is contractible. In particular, X is simply connected.*

3.15 CAT(κ) 4-point condition and 4-point limits of CAT(κ)-spaces

In this section we formulate a condition for a CAT(κ)-space by using quadrilaterals.

Definition 3.16. Let X be a metric space, $x_1, y_1, x_2, y_2 \in X$, and $\kappa \in \mathbb{R}$. We say that a 4-tuple $(\bar{x}_1, \bar{y}_1, \bar{x}_2, \bar{y}_2)$ of points in M^2_κ is a *subembedding in M^2_κ* of (x_1, y_1, x_2, y_2) if

$$\begin{aligned} d(\bar{x}_i, \bar{y}_j) &= d(x_i, y_j) \quad \text{for } i,j \in \{1,2\}, \\ d(x_1, x_2) &\le d(\bar{x}_1, \bar{x}_2), \quad \text{and} \\ d(y_1, y_2) &\le d(\bar{y}_1, \bar{y}_2). \end{aligned}$$

Definition 3.17. A metric space X satisfies the CAT(κ) *4-point condition* if every 4-tuple (x_1, y_1, x_2, y_2) with (perimeter) $d(x_1,y_1) + d(y_1,x_2) + d(x_2,y_2) + d(y_2,x_1) < 2D_\kappa$ has a subembedding in M^2_κ.

We say that a pair of points $x, y \in X$ has *approximate midpoints* (cf. Theorem 1.64) if, for every $\varepsilon > 0$ there exists $m' \in X$ such that

$$\max\{d(x,m'), d(y,m')\} \le \tfrac{1}{2} d(x,y) + \varepsilon.$$

Theorem 3.18. *For a complete metric space X the following two conditions are equivalent:*

(1) X is a $\mathrm{CAT}(\kappa)$*-space.*

(2) X satisfies the $\mathrm{CAT}(\kappa)$ *4-point condition and each pair of points $x, y \in X$, with $d(x,y) < D_\kappa$, has approximate midpoints.*

Proof. (1) $\Rightarrow$ (2): Since X is D_κ-geodesic, there are approximate midpoints for all x, y, with $d(x,y) < D_\kappa$. Let (x_1, y_1, x_2, y_2) be a 4-tuple with $d(x_1,y_1)+d(y_1,x_2)+d(x_2,y_2)+d(y_2,x_1) < 2D_\kappa$. Choose κ-comparison triangles

$$\Delta(\bar{x}_1, \bar{x}_2, \bar{y}_1) = \bar{\Delta}_\kappa(x_1, x_2, y_1) \quad \text{and} \quad \Delta(\bar{x}_1, \bar{x}_2, \bar{y}_2) = \bar{\Delta}_\kappa(x_1, x_2, y_2)$$

such that they have a common side $[\bar{x}_1, \bar{x}_2]$ and that $\bar{y}_1$ and $\bar{y}_2$ lie on opposite sides of the line $\bar{x}_1\bar{x}_2$. There are two cases: either the segments (diagonals) $[\bar{x}_1, \bar{x}_2], [\bar{y}_1, \bar{y}_2]$ intersect at some point $\bar{z}$ or they do not intersect. In the first case, let $z \in [x_1, x_2]$ be such that $d(x_1, z) = d(\bar{x}_1, \bar{z})$. Then

$$\begin{aligned} d(y_1, y_2) &\le d(y_1, z) + d(z, y_2) \\ &\le d(\bar{y}_1, \bar{z}) + d(\bar{z}, \bar{y}_2) \\ &= d(\bar{y}_1, \bar{y}_2) \end{aligned}$$

by the triangle and $\mathrm{CAT}(\kappa)$ inequalities. Note that $d(x_1, x_2) = d(\bar{x}_1, \bar{x}_2)$. Hence (2) holds. In the second case (i.e. $[\bar{x}_1, \bar{x}_2] \cap [\bar{y}_1, \bar{y}_2] = \emptyset$), there exists a geodesic triangle in M_κ^2 with vertices $\tilde{x}_k, \tilde{y}_1, \tilde{y}_2$, where $k = 1$ or $k = 2$, and $\tilde{x}_n \in [\tilde{y}_1, \tilde{y}_2]$, $n \in \{1,2\} \setminus \{k\}$, such that

$$\begin{aligned} d(\tilde{x}_i, \tilde{y}_j) &= d(x_i, y_j), \quad i, j \in \{1,2\} \quad \text{and} \\ d(\tilde{y}_1, \tilde{y}_2) &= d(\tilde{y}_1, \tilde{x}_n) + d(\tilde{x}_n, \tilde{y}_2) \\ &= d(y_1, x_n) + d(x_n, y_2) \\ &\ge d(y_1, y_2). \end{aligned}$$

By Alexandrov's lemma,

$$d(\tilde{x}_k, \tilde{x}_n) \ge d(x_1, x_2).$$

Hence $(\tilde{x}_k, \tilde{y}_1, \tilde{x}_n, \tilde{y}_2)$ is a subembedding of (x_1, y_1, x_2, y_2).

(2) $\Rightarrow$ (1): Let $\Delta = \Delta(p,q,r) \subset X$ be a geodesic triangle of perimeter $< D_\kappa$ and let $x \in [q, r]$. Let $(\bar{p}, \bar{q}, \bar{x}, \bar{r})$ be a subembedding in M_κ^2 of (p, q, x, r). Since

$$d(q,r) \le d(\bar{q}, \bar{r}) \le d(\bar{q}, \bar{x}) + d(\bar{x}, \bar{r}) = d(q,x) + d(x,r) = d(q,r),$$

the triangle $\Delta(\bar{p}, \bar{q}, \bar{r}) \subset M_\kappa^2$ is a κ-comparison triangle of Δ and $\bar{x}$ is the comparison point of x. By the definition of a subembedding,

$$d(p,x) \le d(\bar{p}, \bar{x}),$$

and hence Δ satisfies the $\mathrm{CAT}(\kappa)$-inequality.

It remains to prove that X is D_κ-geodesic. Since X is assumed to be complete, it suffices to show that each pair of points $x, y \in X$, with $d(x,y) < D_\kappa$ has the midpoint. Let (m_i) be a sequence of approximate midpoints of x and y such that

$$\max\{d(x,m_i), d(y,m_i)\} \le \tfrac{1}{2}d(x,y) + 1/i.$$

We claim that (m_i) is a Cauchy-sequence. If this is the case, its limit m_0 will be the midpoint of x, y. Fix $\varepsilon > 0$ and $d(x,y) < \ell < D_\kappa$. Recall from Exercise 7/4 that there exists $\delta = \delta(\kappa, \ell, \varepsilon)$ such that, if $p, q \in M_\kappa^2$, with $d(p,q) \le \ell$ and if

$$\max\{d(p,m'), d(q,m')\} < \tfrac{1}{2}d(p,q) + \delta,$$

then $d(m,m') < \varepsilon$, where m is the midpoint of $[p,q]$. For each i, j, let $(\bar{x}, \bar{m}_i, \bar{y}, \bar{m}_j)$ be a subembedding in M_κ^2 of (x, m_i, y, m_j). Then, by definition,

$$d(m_i, m_j) \le d(\bar{m}_i, \bar{m}_j)$$

and

$$\begin{aligned} d(\bar{x}, \bar{y}) &\le d(\bar{x}, \bar{m}_i) + d(\bar{m}_i, \bar{y}) \\ &= d(x, m_i) + d(m_i, y) \\ &\le d(x,y) + 2/i. \end{aligned}$$

Thus $d(\bar{x}, \bar{y}) \le \ell$ and $\max\{1/i, 1/j\} < \delta$ for all sufficiently large i, j. For such i, j,

$$d(\bar{m}_i, \bar{m}) < \varepsilon \quad \text{and} \quad d(\bar{m}_j, \bar{m}) < \varepsilon,$$

where $\bar{m}$ is the midpoint of $[\bar{x}, \bar{y}]$. It follows that

$$d(m_i, m_j) \le d(\bar{m}_i, \bar{m}_j) < 2\varepsilon$$

for all sufficiently large i, j. Thus (m_i) is a Cauchy-sequence. $\square$

Remark 3.19. The assumption that X be complete was not used in the proof of (1) $\Rightarrow$ (2). Thus every $\mathrm{CAT}(\kappa)$-space satisfies the $\mathrm{CAT}(\kappa)$ 4-point condition.

Definition 3.20. A metric space (X, d) is called a *4-point limit* of a sequence of metric spaces (X_n, d_n) if, for all 4-tuple (x_1, x_2, x_3, x_4) of points in X and all $\varepsilon > 0$, there exist infinitely many $n \in \mathbb{N}$ such that there are 4-tuples $\big(x_1(n), x_2(n), x_3(n), x_4(n)\big)$ in X_n with $|d(x_i, x_j) - d_k\big(x_i(n), x_j(n)\big)| < \varepsilon$ for $1 \le i, j \le 4$.

Theorem 3.21. *Let (X_n, d_n) be a sequence of $\mathrm{CAT}(\kappa_n)$-spaces, with $\kappa = \lim_{n\to\infty} \kappa_n$. Let (X, d) be a complete metric space such that each pair of points $x, y \in X$, with $d(x,y) < D_\kappa$, has approximate midpoints. If (X, d) is a 4-point limit of the sequence (X_n, d_n), then X is a $\mathrm{CAT}(\kappa)$-space.*

Proof. We will show that X is a $\mathrm{CAT}(\kappa')$-space for all $\kappa' > \kappa$. By 3.5(1), this implies that X is a $\mathrm{CAT}(\kappa)$-space. By 3.18, it suffices to show that X satisfies the $\mathrm{CAT}(\kappa')$ 4-point condition for all $\kappa' > \kappa$. Fix $\kappa' > \kappa$. Then, for all sufficiently large n, $\kappa_n < \kappa'$ and hence X_n is a $\mathrm{CAT}(\kappa')$-space. Since X is a 4-point limit of X_n's, there exist a sequence of integers $n_i \to \infty$ and 4-tuples $\big(x_1(n_i), y_1(n_i), x_2(n_i), y_2(n_i)\big)$ of points of X_{n_i} such that

$$\begin{aligned} &d_{n_i}\big(x_j(n_i), x_k(n_i)\big) \to d(x_j, x_k), \quad d_{n_i}\big(y_j(n_i), y_k(n_i)\big) \to d(y_j, y_k), \quad \text{and} \\ &d_{n_i}\big(x_j(n_i), y_k(n_i)\big) \to d(x_j, y_k) \end{aligned}$$

for $j,k \in \{1,2\}$ as $n_i \to \infty$. Since X_{n_i} is a $\mathrm{CAT}(\kappa')$-space, the 4-tuple $\big(x_1(n_i), y_1(n_i), x_2(n_i), y_2(n_i)\big)$ has a subembedding $\big(\bar{x}_1(n_i), \bar{y}_1(n_i), \bar{x}_2(n_i), \bar{y}_2(n_i)\big)$ in $M^2_{\kappa'}$. We may assume that $\bar{x}_1(n_i) = \bar{x}_1$ for all n_i. Then all the points $\bar{x}_2(n_i)$, $\bar{y}_1(n_i)$, $\bar{y}_2(n_i)$ belong to a compacts set. By passing to a subsequence, we may assume that

$$\bar{x}_2(n_i) \to \bar{x}_2, \quad \bar{y}_1(n_i) \to \bar{y}_1, \quad \text{and} \quad \bar{y}_2(n_i) \to \bar{y}_2.$$

Clearly $(\bar{x}_1, \bar{y}_1, \bar{x}_2, \bar{y}_2)$ is a subembedding of (x_1, y_1, x_2, y_2) in $M^2_{\kappa'}$. Hence X satisfies the $\mathrm{CAT}(\kappa')$ 4-point condition. ☐

Corollary 3.22. *If (X,d) is a $\mathrm{CAT}(\kappa)$-space, then its completion $(\tilde{X}, \tilde{d})$ is a $\mathrm{CAT}(\kappa)$-space.*

Proof. Clearly $(\tilde{X}, \tilde{d})$ is a 4-point limit of the constant sequence $(X_n, d_n) = (X, d)$ and it has approximate midpoints. Thus $\tilde{X}$ is a $\mathrm{CAT}(\kappa)$-space. ☐

3.23 Cones

Let (Y,d) be a metric space and $\kappa \in \mathbb{R}$. The *κ-cone over Y*, denoted by

$$X = C_\kappa Y,$$

is the following metric space. For $\kappa \le 0$, X (as a set) is the quotient space

$$X = [0,\infty) \times Y/\!\sim,$$

where $\sim$ is the equivalence relation

$$(t,y) \sim (t',y') \iff t = t' = 0 \text{ or } (t,y) = (t'y').$$

If $\kappa > 0$, then

$$X = [0, D_\kappa/2] \times Y/\!\sim,$$

with the same equivalence relation as above. We denote points of X (i.e. equivalence classes) by $ty = [(t,y)]$ and $0 = [(0,y)]$ and call 0 the vertex of $C_\kappa Y$.

Next we define the metric on $C_\kappa Y$. Let $d_\pi(y,y') = \min\{\pi, d(y,y')\}$ and $x = ty,\ x' = t'y'$. If $x' = 0$, we set $d(x,x') = t$. If $t, t' > 0$, we define $d(x,x')$ so that

$$\angle_0^{(\kappa)}(x,x') = d_\pi(y,y').$$

Thus

$$d(x,x')^2 = t^2 + t'^2 - 2tt'\cos(d_\pi(y,y'))$$

if $\kappa = 0$,

$$\cosh\big(\sqrt{-\kappa}\,d(x,x')\big) = \cosh(\sqrt{-\kappa}t)\cosh(\sqrt{-\kappa}t') - \sinh(\sqrt{-\kappa}t)\sinh(\sqrt{-\kappa}t')\cos(d_\pi(y,y'))$$

if $\kappa < 0$, and

$$\cos\big(\sqrt{\kappa}\,d(x,x')\big) = \cos(\sqrt{\kappa}t)\cos(\sqrt{\kappa}t') + \sin(\sqrt{\kappa}t)\sin(\sqrt{\kappa}t')\cos(d_\pi(y,y')) \quad \text{and} \quad d(x,x') \le D_\kappa$$

if $\kappa > 0$.

Remark 3.24. If $Y = \mathbb{S}^{n-1}$, then $C_\kappa Y$ is isometric to M^n_κ if $\kappa \le 0$, or to a closed ball in M^n_κ of radius $D_\kappa/2$ if $\kappa > 0$. [This can be seen by using polar coordinates $(t,\vartheta) \in [0, D_\kappa] \times \mathbb{S}^{n-1}$ in M^n_κ.]

Theorem 3.25. *(1) $d(x, x')$ defines a metric in $X = C_\kappa Y$.*

(2) Y is complete $\iff$ $C_\kappa Y$ is complete.

Proof. We will prove only (1), the proof of (2) is left as an exercise. It suffices to verify the triangle inequality. Let $x_i = t_i y_i \in X, i = 1, 2, 3$. We want to show that

$$d(x_1, x_3) \le d(x_1, x_2) + d(x_2, x_3). \tag{3.26}$$

If $t_i = 0$ for some $i = 1, 2, 3$, then (3.26) follows easily from the triangle inequality in M^2_κ. Suppose that $t_i > 0$ for $i = 1, 2, 3$. There are two cases:

(i) $d(y_1, y_2) + d(y_2, y_3) < \pi$. The triangle inequality in Y implies that $d(y_1, y_3) < \pi$. Choose $\bar{y}_1, \bar{y}_2, \bar{y}_3 \in \mathbb{S}^2$ such that $d(\bar{y}_i, \bar{y}_j) = d(y_1, y_j)$ for $i, j \in \{1, 2, 3\}$. It follows from the definition of d that the subcone $C_\kappa\{y_1, y_2, y_3\} \subset X$ is isometric to a subcone $C_\kappa\{\bar{y}_1, \bar{y}_2, \bar{y}_3\} \subset M^3_\kappa$ [use polar coordinates in M^3_κ]. The inequality (3.26) follows then from the triangle inequality in M^3_κ.

(ii) $d(y_1, y_2) + d(y_2, y_3) \ge \pi$. Fix three points $\bar{y}_1, \bar{y}_2, \bar{y}_3 \in \mathbb{S}^1$ (occurring in that order) such that

$$d(\bar{y}_1, \bar{y}_2) = d_\pi(y_1, y_2) \quad \text{and} \quad d(\bar{y}_2, \bar{y}_3) = d_\pi(y_2, y_3).$$

Identify $C_\kappa\mathbb{S}^1$ with M^2_κ if $\kappa \le 0$ or with a closed hemisphere in M^2_κ if $\kappa > 0$. Let $\bar{x}_i = t\bar{y}_i,\ i = 1, 2, 3$. Then

$$d(x_1, x_2) = d(\bar{x}_1, \bar{x}_2), \quad d(x_2, x_3) = d(\bar{x}_2, \bar{x}_3),$$

and

$$d(x_1, x_3) \le d(x_1, 0) + d(0, x_3) = t_1 + t_3.$$

Since $d(\bar{y}_1, \bar{y}_2) + d(\bar{y}_2, \bar{y}_3) \ge \pi$, we have

$$t_1 + t_3 \le d(\bar{x}_1, \bar{x}_2) + d(\bar{x}_2, \bar{x}_3)$$

by Alexandrov's lemma. Hence

$$d(x_1, x_3) \le t_1 + t_3 \le d(x_1, x_2) + d(x_2, x_3).$$

$\square$

Theorem 3.27. *The κ-cone $X = C_\kappa Y$ over a metric space Y is a* CAT(κ)*-space if and only if Y is a* CAT(1)*-space.*

Proof. Suppose that Y is a CAT(1)-space. First we claim that every pair of points $x_1 = t_1 y_1, x_2 = t_2 y_2 \in X$ can be joined by a geodesic. This is clear if $t_i = 0$ for some $i = 1, 2$. Therefore, suppose that $t_1, t_2 > 0$. If $d(y_1, y_2) \ge \pi$, then $d(x_1, x_2) = t_1 + t_2$ and the claim follows. If $d(y_1, y_2) < \pi$ (and $t_1, t_2 > 0$), then the subcone $C_\kappa[y_1, y_2] \subset X$ is isometric to a sector (subcone) $C_\kappa[\bar{y}_1, \bar{y}_2] \subset M^2_\kappa$, which is convex. (Here $\bar{y}_1, \bar{y}_2 \in \mathbb{S}^1$, with $d(\bar{y}_1, \bar{y}_2) = d(y_1, y_2)$.) Hence there is a geodesic segment joining x_1 and x_2.

Next we verify the CAT(κ)-inequality for a geodesic triangle $\Delta \subset X$ with vertices $x_1 = t_i y_i, i = 1, 2, 3$, and perimeter $< 2D_\kappa$. If $t_i = 0$ for some $i = 1, 2, 3$, then the triangle Δ is isometric to its comparison triangle in M^2_κ and hence satisfies the CAT(κ)-inequality.

Thus we may assume that $t_i > 0, i = 1, 2, 3$. Then there are three cases:

(i) $d(y_1, y_2) + d(y_2, y_3) + d(y_3, y_1) < 2\pi$;

(ii) $d(y_1, y_2) + d(y_2, y_3) + d(y_3, y_1) \ge 2\pi$ but $d(y_i, y_j) < \pi$ for all $i, j = 1, 2, 3$;

(iii) $d(y_i, y_j) \geq \pi$ for some $i, j = 1, 2, 3$.

(i): Denote $\Delta_Y = [y_1, y_2] \cup [y_2, y_3] \cup [y_3, y_1] \subset Y$. Fix a comparison triangle $\bar{\Delta}_Y \subset M_1^2 = \mathbb{S}^2$ with vertices $\bar{y}_1, \bar{y}_2, \bar{y}_3$. The (comparison) map $\bar{\Delta}_Y \to \Delta_Y$, $\bar{y} \mapsto y$, extends to a bijection

$$\underbrace{C_\kappa \bar{\Delta}_Y}_{\subset M_\kappa^3} \to C_\kappa \Delta_Y \subset X, \quad t\bar{y} \mapsto ty,$$

where $\bar{y} \in \bar{\Delta}_Y$ is the comparison point of $y \in \Delta_Y$. Fix an arbitrary point $x = ty \in [x_2, x_3]$ and let $\bar{y} \in [\bar{y}_2, \bar{y}_3]$ be the comparison point of $y \in [y_2, y_3]$. The triangle $\Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3) \subset M_\kappa^3$, with $\bar{x}_i = t_i \bar{y}_i$, can be interpreted as a κ-comparison triangle of $\Delta = \Delta(x_1, x_2, x_3)$, with $\bar{x} = t\bar{y}$ as the comparison point of $x \in [x_2, x_3]$. Since Y is a CAT(1)-space, $d(y, y_1) \leq d(\bar{y}, \bar{y}_1)$, and it follows from the definition of the metric d on X, that

$$d(x, x_1) \leq d(\bar{x}, \bar{x}_1).$$

Hence Δ satisfies the CAT(κ)-inequality.

(ii): Choose κ-comparison triangles $\Delta(\tilde{0}, \tilde{x}_1, \tilde{x}_2) \subset M_\kappa^2$ and $\Delta(\tilde{0}, \tilde{x}_1, \tilde{x}_3) \subset M_\kappa^2$ of $\Delta(0, x_1, x_2) \subset X$ and $\Delta(0, x_1, x_3) \subset X$, respectively, such that $\tilde{x}_2$ and $\tilde{x}_3$ lie on opposite sides of the line $\tilde{0}\tilde{x}_1$. By the definition of the metric d on X, we have

$$\begin{aligned}
&\angle_{\tilde{0}}(\tilde{x}_1, \tilde{x}_2) = d(y_1, y_2), \\
&\angle_{\tilde{0}}(\tilde{x}_1, \tilde{x}_3) = d(y_1, y_3), \\
&\angle_{\tilde{x}_1}(\tilde{0}, \tilde{x}_2) = \angle_{x_1}\big([x_1, 0], [x_1, x_2]\big), \quad \text{and} \\
&\angle_{\tilde{x}_1}(\tilde{0}, \tilde{x}_3) = \angle_{x_1}\big([x_1, 0], [x_1, x_3]\big).
\end{aligned}$$

The last two equalities hold since $\Delta(0, x_1, x_2)$ is isometric to $\Delta(\tilde{0}, \tilde{x}_1, \tilde{x}_2)$ and $\Delta(0, x_1, x_3)$ is isometric to $\Delta(\tilde{0}, \tilde{x}_1, \tilde{x}_3)$. By the assumption (ii),

$$d(y_1, y_2) + d(y_1, y_3) > \pi,$$

and therefore

$$\angle_{\tilde{0}}(\tilde{x}_2, \tilde{x}_3) = 2\pi - \underbrace{\angle_{\tilde{0}}(\tilde{x}_2, \tilde{x}_1)}_{=d(y_1, y_2)} - \underbrace{\angle_{\tilde{0}}(\tilde{x}_3, \tilde{x}_1)}_{=d(y_1, y_3)} \leq d(y_2, y_3) = \angle_0^{(\kappa)}(x_2, x_3),$$

and thus

$$d(\tilde{x}_2, \tilde{x}_3) \leq d(x_2, x_3).$$

Hence we have, for the comparison triangle $\bar{\Delta} = \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ of $\Delta = \Delta(x_1, x_2, x_3)$,

$$\begin{aligned}
\angle_{\bar{x}_1}(\bar{x}_2, \bar{x}_3) &\geq \angle_{\tilde{x}_1}(\tilde{x}_2, \tilde{x}_3) \\
&= \angle_{\tilde{x}_1}(\tilde{x}_2, \tilde{0}) + \angle_{\tilde{x}_1}(\tilde{x}_3, \tilde{0}) \\
&= \angle_{x_1}\big([x_1, 0], [x_1, x_2]\big) + \angle_{x_1}\big([x_1, 0], [x_1, x_3]\big) \\
&\geq \angle_{x_1}\big([x_1, x_2], [x_1, x_3]\big),
\end{aligned}$$

that is, the condition 3.2(4) holds.

(iii): Suppose that $d(y_1, y_3) \geq \pi$. Then

$$[x_1, x_3] = [x_1, 0] \cup [0, x_3].$$

Choose κ-comparison triangles $\bar{\Delta}_1 = \Delta(\bar{0}, \bar{x}_1, \bar{x}_2)$ and $\bar{\Delta}_3 = \Delta(\bar{0}, \bar{x}_3, \bar{x}_2)$ of $\Delta_1 = \Delta(0, x_1, x_2)$ and $\Delta_3 = \Delta(0, x_3, x_2)$, respectively, such that $\bar{x}_1$ and $\bar{x}_3$ lie on opposite sides of the line $\bar{0}\bar{x}_2$. Note that $\bar{\Delta}_1$ is isometric to $\Delta(0, x_1, x_2)$ and $\bar{\Delta}_3$ is isometric to $\Delta(0, x_3, x_2)$. Hence we can estimate the Alexandrov's angles of $\Delta(x_1, x_2, x_3)$

$$\begin{aligned}
\angle_{x_1}\big([x_1, x_2], [x_1, x_3]\big) &= \angle_{x_1}\big([x_1, x_2], [x_1, 0]\big) = \angle_{\bar{x}_1}(\bar{x}_2, \bar{0}),\\
\angle_{x_3}\big([x_3, x_2], [x_3, x_1]\big) &= \angle_{x_3}\big([x_3, x_2], [x_3, 0]\big) = \angle_{\bar{x}_3}(\bar{x}_2, \bar{0}), \quad \text{and}\\
\angle_{x_2}\big([x_2, x_1], [x_2, x_3]\big) &\le \angle_{x_2}\big([x_2, x_1], [x_2, 0]\big) + \angle_{x_2}\big([x_2, 0], [x_2, x_3]\big)\\
&= \angle_{\bar{x}_2}(\bar{x}_1, \bar{0}) + \angle_{\bar{x}_2}(\bar{0}, \bar{x}_3).
\end{aligned}$$

Since

$$\pi \le d(y_1, y_3) \le d(y_1, y_2) + d(y_2, y_3),$$

we have

$$d_\pi(y_1, y_2) + d_\pi(y_2, y_3) \ge \pi.$$

Hence

$$\angle_{\bar{0}}(\bar{x}_1, \bar{x}_2) + \angle_{\bar{0}}(\bar{x}_2, \bar{x}_3) = d_\pi(y_1, y_2) + d_\pi(y_2, y_3) \ge \pi.$$

By Alexandrov's lemma, the vertex angles of a κ-comparison triangle $\bar{\Delta}(x_1, x_2, x_3)$ are greater than or equal to the corresponding Alexandrov's angles of $\Delta(x_1, x_2, x_3)$, i.e. the condition 3.2(4) holds.

Suppose then that X is a CAT(κ)-space. We leave it as an exercise to show that Y is π-geodesic. Let $\Delta \subset Y$ be a geodesic triangle with vertices y_1, y_2, y_3 and perimeter $< 2\pi$. Let $\bar{\Delta} = \Delta(\bar{y}_1, \bar{y}_2, \bar{y}_3) \subset M_1^2 = \mathbb{S}^2$ be its comparison triangle. For $y \in [y_2, y_3]$, let $\bar{y} \in [\bar{y}_2, \bar{y}_3]$ denote its comparison point. Let $x_i = \varepsilon y_i, i = 1, 2, 3$, be points of the subcone $C_\kappa \Delta \subset X$, where $\varepsilon > 0$ is so small that the perimeter of $\Delta' = \Delta(x_1, x_2, x_3)$ is $< 2D_\kappa$. Now $C_\kappa \bar{\Delta} \subset C_\kappa \mathbb{S}^2 \subset M_\kappa^3$, and the points $\bar{x}_i = \varepsilon \bar{y}_1, i = 1, 2, 3$, are the vertices of a κ-comparison triangle $\bar{\Delta}'$ of Δ'. If $x = ty \in [x_2, x_3] \subset \Delta'$, then $\bar{x} = t\bar{y}$ is its comparison point. Since X is a CAT(κ)-space, we have

$$d(x, x_1) \le d(\bar{x}, \bar{x}_1).$$

Hence

$$d(y, y_1) \le d(\bar{y}, \bar{y}_1)$$

by the definition of the metric on $X = C_\kappa Y$. □

Corollary 3.28. *The following conditions are equivalent for a metric space Y:*

(1) $C_\kappa Y$ is a CAT(κ)*-space.*

(2) $C_\kappa Y$ is of curvature $\le \kappa$.

(3) A neighborhood of the vertex $0 \in C_\kappa Y$ is a CAT(κ)*-space.*

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) hold trivially and (3) $\Rightarrow$ (1) follows from the last part of the proof above. □

3.29 Space of directions and tangent cone

Definition 3.30. Let X be a metric space and $p \in X$. We say that two geodesics $\alpha\colon [0,a] \to X$ and $\beta\colon [0,b] \to X$, with $\alpha(0) = p = \beta(0)$, define the *same direction* at p if the Alexandrov angle $\angle_p(\alpha,\beta) = 0$. The triangle inequality for Alexandrov angles (Theorem 2.17) implies that

$$\alpha \sim \beta \iff \angle_p(\alpha,\beta) = 0$$

is an equivalence relation in the set of geodesics emanating from p. Furthermore, $\angle_p(\cdot,\cdot)$ defines a metric in the set of equivalence classes. The resulting metric space is denoted by $S_p(X)$ and called the *space of directions at* p. The 0-cone (Euclidean cone) over $S_p(X)$, $C_0S_p(X)$, is called the *tangent cone at* p.

Theorem 3.31. *Let X be a metric space of curvature $\le \kappa$ for some $\kappa \in \mathbb{R}$. Then the completion of $S_p(X)$ is a* CAT(1)*-space and the completion of $C_0S_p(X)$ is a* CAT(0)*-space for every $p \in X$.*

Proof. By Theorems 3.25 and 3.27, it suffices to prove that the completion of $C_0S_p(X)$ is a CAT(0)-space. Furthermore, by Theorem 3.18 and Corollary 3.28, it is enough to show that a neighborhood of the vertex $0 \in C_\kappa(S_p(X))$ satisfies the CAT(0) 4-point condition and has approximate midpoints. Since $S_p(X)$ depends only on a neighborhood of p, we may assume that X is a CAT(κ)-space of diameter $< D_\kappa/2$. Then there exists a unique geodesic segment $[p,x]$ for every $x \in X \setminus \{p\}$. We denote by $\vec{x} \in S_p(X)$ the equivalence class of $[p,x]$. Let $j\colon X \to C_0S_p(X)$ be the mapping

$$j(x) = \begin{cases} 0, & x = p; \\ d(x,p)\vec{x}, & x \ne p. \end{cases}$$

For each $t \in [0,1]$, we denote by tx the unique point in $[p,x]$ such that $d(p,tx) = td(p,x)$. If $\varepsilon \in (0,1]$, we define a pseudometric d_ε by setting

$$d_\varepsilon(x,y) = \tfrac{1}{\varepsilon} d(\varepsilon x, \varepsilon y).$$

Then

$$d_\varepsilon(p,x) = \tfrac{1}{\varepsilon} d(\varepsilon p, \varepsilon x) = \tfrac{1}{\varepsilon} d(p, \varepsilon x) = d(p,x).$$

Note that (X, d_ε) satisfies the CAT($\varepsilon^2\kappa$) 4-point. Fix $x, y \in X \setminus \{p\}$, and let

$$\gamma_\varepsilon = \bar{\angle}_p(\varepsilon x, \varepsilon y) \quad \bigl(= \angle_p^{(0)}(\varepsilon x, \varepsilon y)\bigr).$$

Then

$$\begin{aligned} d(\varepsilon x, \varepsilon y)^2 &= d(p,\varepsilon x)^2 + d(p,\varepsilon y)^2 - 2d(p,\varepsilon x)d(p,\varepsilon y)\cos\gamma_\varepsilon \\ &= \varepsilon^2 d(p,x)^2 + \varepsilon^2 d(p,y)^2 - \varepsilon^2 2d(p,x)d(p,y)\cos\gamma_\varepsilon, \end{aligned}$$

and so

$$d_\varepsilon(x,y)^2 = d(p,x)^2 + d(p,y)^2 - 2d(p,x)d(p,y)\cos\gamma_\varepsilon. \tag{3.32}$$

Since X is a CAT(κ)-space, the limit

$$\angle_p\bigl([p,x],[p,y]\bigr) = \lim_{\varepsilon\to 0}\gamma_\varepsilon$$

exists (in strong sense). Hence by (3.32) the limit

$$d_0(x,y) := \lim_{\varepsilon\to 0} d_\varepsilon(x,y)$$

exists for all $x, y \in X$. Note that

$$\angle_p\big([p,x],[p,y]\big) = \angle_p(\vec{x},\vec{y}) = d(\vec{x},\vec{y}),$$

the distance between points $\vec{x},\vec{y} \in S_p(X)$. By the definition of the metric in $C_0S_p(X)$, we have

$$d\big(j(x),j(y)\big)^2 = d(p,x)^2 + d(p,y)^2 - 2d(p,x)d(p,y)\cos d(\vec{x},\vec{y}),$$

and hence the mapping

$$j\colon (X,d_0) \to C_0S_p(X)$$

satisfies

$$d\big(j(x),j(y)\big) = d_0(x,y).$$

Moreover, the pseudometric space (X,d_0) satisfies the CAT(0) 4-point condition since it satisfies the CAT$(\varepsilon^2\kappa)$ 4-point condition for every $\varepsilon \in (0,1]$. The image $jX \subset C_0S_p(X)$ contains a neighborhood of the vertex 0, and hence a neighborhood of 0 satisfies the CAT(0) 4-point condition. It remains to prove that each pair $j(x), j(y) \in C_0S_p(X)$ has approximate midpoints.

Suppose first that $\kappa \le 0$. Let m_ε be the midpoint of $[\varepsilon x, \varepsilon y]$. We claim that the points $\frac{1}{\varepsilon}j(m_\varepsilon)$ are approximate midpoints of $j(x)$ and $j(y)$ for small ε. By Exercise 7/1, the metric of X is convex, and thus

$$d_0(x,y) = \lim_{\varepsilon\to 0} d_\varepsilon(x,y) = \lim_{\varepsilon\to 0} \tfrac{1}{\varepsilon}\underbrace{d(\varepsilon x,\varepsilon y)}_{\le \varepsilon d(x,y)} \le d(x,y)$$

for all $x, y \in X$. Hence

$$\begin{aligned} d\big(j(x),\tfrac{1}{\varepsilon}j(m_\varepsilon)\big) &= \tfrac{1}{\varepsilon}d\big(\varepsilon j(x), j(m_\varepsilon)\big) = \tfrac{1}{\varepsilon}d_0(\varepsilon x, m_\varepsilon) \le \tfrac{1}{\varepsilon}d(\varepsilon x, m_\varepsilon) \\ &= \tfrac{1}{2\varepsilon}d(\varepsilon x,\varepsilon y) = \tfrac{1}{2}d_\varepsilon(x,y). \end{aligned}$$

Similarly,

$$d\big(j(y),\tfrac{1}{\varepsilon}j(m_\varepsilon)\big) \le \tfrac{1}{2}d_\varepsilon(x,y).$$

Since for every $\delta > 0$ there exists $\varepsilon \in (0,1]$ such that

$$\tfrac{1}{2}d_\varepsilon(x,y) \le \tfrac{1}{2}d_0(x,y) + \delta = \tfrac{1}{2}d\big(j(x),j(y)\big) + \delta,$$

we see that the points $\frac{1}{\varepsilon}j(m_\varepsilon)$ are approximate midpoints of $j(x)$ and $j(y)$ for small ε.

Suppose then that $\kappa > 0$. The proof is similar to the case of $\kappa \le 0$ except that we replace inequalities $d_0(\varepsilon x, m_\varepsilon) \le d(x, m_\varepsilon)$ and $d_0(\varepsilon y, m_\varepsilon) \le d(y, m_\varepsilon)$ above (which hold for $\kappa \le 0$) by estimates

$$d_0(\varepsilon x, m_\varepsilon) = \lim_{\bar\varepsilon\to 0} d_{\bar\varepsilon}(\varepsilon x, m_\varepsilon) = \lim_{\bar\varepsilon\to 0} \tfrac{1}{\bar\varepsilon}\underbrace{d(\bar\varepsilon\varepsilon x, \bar\varepsilon m_\varepsilon)}_{\le \bar\varepsilon C(\varepsilon)d(\varepsilon x, m_\varepsilon)} \le C(\varepsilon)d(\varepsilon x, m_\varepsilon)$$

and

$$d_0(\varepsilon y, m_\varepsilon) \le C(\varepsilon)d(\varepsilon y, m_\varepsilon),$$

where $C(\varepsilon) \to 1$ as $\varepsilon \to 0$. These estimates follow from the CAT(κ) criterion 3.2(2) and from Lemma 3.33 below. $\square$

Lemma 3.33. *For all $\kappa \in \mathbb{R}$ there exists a function $C\colon [0, D_\kappa/2) \to \mathbb{R}$ such that $\lim_{R\to 0} C(R) = 1$ and for all $p \in M^2_\kappa$ and all $x, y \in B(p,R)$, we have*

$$d(\varepsilon x, \varepsilon y) \le \varepsilon\, C(R) d(x,y).$$

Proof. The claim for $\kappa \le 0$ holds with $C(r) \equiv 1$ by the convexity of the metric in CAT(0)-spaces. Thus we may assume that $\kappa > 0$. Let $x, y \in B(p,R)$ and let $\alpha\colon [0, d(x,y)] \to [x,y] \subset B(p,R)$ be the geodesic joining x and y. We write α in polar coordinates (origin at p) as

$$\alpha(t) = \big(\alpha_r(t), \alpha_\vartheta(t)\big)_\kappa.$$

Then the path $\beta\colon [0, d(x,y)] \to B(p,R)$,

$$\beta(t) = \big(\varepsilon\alpha_r(t), \alpha_\vartheta(t)\big)_\kappa,$$

joins εx and εy, and hence by (3.11)

$$\begin{aligned} d(\varepsilon x, \varepsilon y) \le \ell(\beta) &= \int_0^{d(x,y)} |\dot\beta|(t)\, dt \\ &= \int_0^{d(x,y)} \sqrt{\varepsilon^2 \alpha_r'(t)^2 + \tfrac{1}{\kappa}\sin^2\big(\sqrt{\kappa}\varepsilon\alpha_r(t)\big)\alpha_\vartheta'(t)^2} \\ &= \varepsilon \int_0^{d(x,y)} \sqrt{\alpha_r'(t)^2 + \tfrac{1}{\kappa\varepsilon^2}\sin^2\big(\sqrt{\kappa}\varepsilon\alpha_r(t)\big)\alpha_\vartheta'(t)^2}. \end{aligned}$$

The claim follows since

$$\frac{\sin(t\gamma)}{t\sin\gamma} \to 1$$

as $\gamma \to 0$. □

4 The Cartan-Hadamard theorem

We start with some definitions. Let (X,d) be a metric space. We say that the *metric* d on X is *convex* if

(a) X is a geodesic space and

(b) all geodesics $\alpha\colon [0,a] \to X$ and $\beta\colon [0,b] \to X$, with $\alpha(0) = \beta(0)$, satisfy the inequality

$$d\big(\alpha(ta), \beta(tb)\big) \le t\, d\big(\alpha(a), \beta(b)\big)$$

for all $t \in [0,1]$.

The metric d is said to be *locally convex* if every point has a neighborhood where the induced metric is convex. It follows immediately from the definition that X is (locally) uniquely geodesic if d is (locally) convex.

If Y is a topological space and $\tilde Y$ is a simply connected covering space of Y, then $\tilde Y$ (or the pair $(\tilde Y, \pi)$, where $\pi\colon \tilde Y \to Y$ is a covering map) is called a *universal covering space* of Y. It is a covering space of any other covering space of Y, and hence unique up to a homeomorphism. It is known that a connected, locally path-connected, and semi-locally simply connected space Y has a universal covering space. In particular, if the metric d on X is locally convex, then X is (locally) contractible by Exercise 8/2, and hence X has the universal covering space $\tilde{X}$. It follows from Theorem 1.107 that there exists a unique length metric $\tilde{d}$ in $\tilde{X}$ such that $\pi\colon \tilde{X} \to X$ is a local isometry. It is defined by

$$\tilde{d}(\tilde{x}, \tilde{y}) = \inf_{\tilde{\gamma}} \ell(\pi \circ \gamma),$$

where the infimum is taken over all paths $\gamma\colon I \to \tilde{X}$ joining $\tilde{x}$ and $\tilde{y}$, and it is called the *induced length metric*.

Theorem 4.1. *Let X be a connected complete metric space.*

(1) If the metric of X is locally convex, then the induced length metric on the universal covering space $\tilde{X}$ is convex. In particular, $\tilde{X}$ is uniquely geodesic and geodesics in $\tilde{X}$ vary continuously with their endpoints.

(2) If X is of curvature $\leq \kappa \leq 0$, then $\tilde{X}$ is a $\mathrm{CAT}(\kappa)$-space.

For the proof of the Cartan-Hadamard theorem we need some lemmas and definitions. We say that a metric space (X, d) is *locally complete* if, for each point $x \in X$, there is $r_x > 0$ such that $\big(\bar{B}(x, r_x), d\big)$ is a complete metric space.

Lemma 4.2. *Let X be a locally complete metric space whose metric d is locally convex. Let $\gamma\colon [0,1] \to X$ be a local constant speed geodesic from x to y. Let $\varepsilon > 0$ be so small that the induced metric in $\bar{B}\big(\gamma(t), 2\varepsilon\big)$ is complete and convex for all $t \in [0,1]$. Then*

(1) for all $\bar{x}, \bar{y} \in X$, with $d(x, \bar{x}) < \varepsilon$ and $d(y, \bar{y}) < \varepsilon$, there exists a unique local constant speed geodesic $\bar{\gamma}\colon [0,1] \to X$ from $\bar{x}$ to $\bar{y}$ such that

$$t \mapsto d\big(\gamma(t), \bar{\gamma}(t)\big)$$

is a convex function, and

(2)

$$\ell(\bar{\gamma}) \leq \ell(\gamma) + d(x, \bar{x}) + d(y, \bar{y}).$$

Proof. First we note that such an $\varepsilon > 0$ exists by the compactness of $\gamma[0,1]$.

We prove first the uniqueness of $\bar{\gamma}$ and that (1) implies (2). Observe that if $\bar{\gamma}$ exists, then the convexity of

$$t \mapsto d\big(\gamma(t), \bar{\gamma}(t)\big)$$

implies that $d\big(\gamma(t), \bar{\gamma}(t)\big) < \varepsilon$ for all $t \in [0,1]$.

Suppose that $\alpha, \beta\colon [0,1] \to X$ are local constant speed geodesics such that

$$d\big(\gamma(t), \alpha(t)\big) < \varepsilon \quad \text{and} \quad d\big(\gamma(t), \beta(t)\big) < \varepsilon$$

for all $t \in [0,1]$. Since the metric is convex in each ball $B\big(\gamma(t), 2\varepsilon\big)$, the function

$$t \mapsto d\big(\alpha(t), \beta(t)\big)$$

is locally convex, hence convex. In particular, if $\alpha(0) = \beta(0)$, then

$$d\big(\alpha(t), \beta(t)\big) \leq t\, d\big(\alpha(1), \beta(1)\big) \tag{4.3}$$

for all $t \in [0,1]$. Furthermore,

$$d\big(\alpha(0),\alpha(t)\big) = \ell(\alpha|[0,t]) = t\,\ell(\alpha) \quad \text{and}$$
$$d\big(\beta(0),\beta(t)\big) = \ell(\beta|[0,t]) = t\,\ell(\beta)$$

for small $t > 0$ since α and β are local constant speed geodesics. Hence

$$\begin{aligned} t\,\ell(\beta) &= d\big(\beta(0),\beta(t)\big) \\ &= d\big(\alpha(0),\beta(t)\big) \\ &\le d\big(\alpha(0),\alpha(t)\big) + d\big(\alpha(t),\beta(t)\big) \\ &\le t\,\ell(\alpha) + t\,d\big(\alpha(1),\beta(1)\big), \end{aligned}$$

and so

$$\ell(\beta) \le \ell(\alpha) + d\big(\alpha(1),\beta(1)\big). \tag{4.4}$$

Suppose then that (1) holds. Let $\tilde\gamma$ be the unique local constant speed geodesic from $\bar x$ to y given by (1) for the pair $\bar x, y$. We apply (4.4) with $\alpha = \tilde\gamma$ and $\beta = \bar\gamma$ to obtain

$$\ell(\bar\gamma) \le \ell(\tilde\gamma) + d(y,\bar y).$$

Similarly, applying (1) with $\alpha(t) = \gamma(1-t)$ and $\beta(t) = \tilde\gamma(1-t)$ yields

$$\ell(\tilde\gamma) \le \ell(\gamma) + d(x,\bar x).$$

Thus

$$\ell(\bar\gamma) \le \ell(\gamma) + d(y,\bar y) + d(x,\bar x)$$

and hence (1) implies (2). On the other hand, if also $\alpha(1) = \beta(1)$, then $\alpha = \beta$ by (4.3). This shows the uniqueness of $\bar\gamma$ (provided $\bar\gamma$ exists).

It remains to prove the existence of $\bar\gamma$. For $L > 0$ consider the following property:

$P(L)$ For all $a, b \in [0,1]$, with $0 < b - a \le L$, and for all $\bar p \in B(\gamma(a),\varepsilon)$ and $\bar q \in B(\gamma(b),\varepsilon)$ there exists a local constant speed geodesic $\bar\gamma\colon [a,b] \to X$ such that $\bar\gamma(a) = \bar p$, $\bar\gamma(b) = \bar q$, and $d\big(\gamma(t),\bar\gamma(t)\big) < \varepsilon$ for all $t \in [a,b]$.

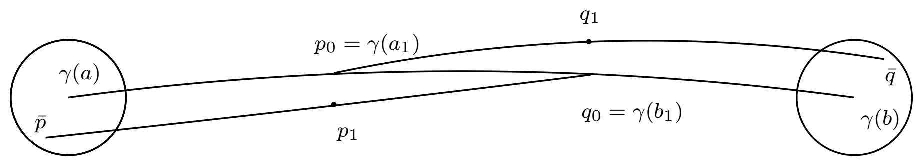
If $L < \varepsilon/\ell(\gamma)$, the property $P(L)$ holds. Hence it is sufficient to prove that

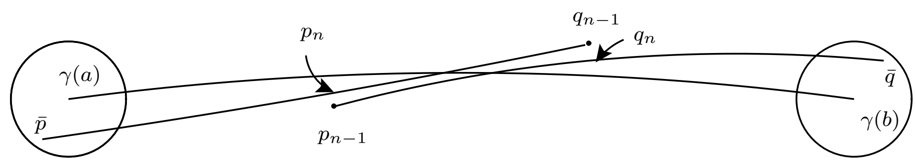
$$P(L) \;\Rightarrow\; P(\tfrac{3}{2}L).$$

Suppose that $P(L)$ holds for $L > 0$ and fix $a, b \in [0,1]$ such that $0 < b - a \le \frac{3}{2}L$. Divide $[a,b]$ into three intervals $[a,a_1]$, $[a_1,b_1]$, and $[b_1,b]$ of equal length. Let $\bar p \in B(\gamma(a),\varepsilon)$ and $\bar q \in B(\gamma(b),\varepsilon)$. First we construct Cauchy-sequences (p_n) and (q_n) in $\bar B(\gamma(a_1),\varepsilon)$ and $\bar B(\gamma(b_1),\varepsilon)$, respectively, as follows. Let $p_0 = \gamma(a_1)$ and $q_0 = \gamma(b_1)$ and assume that p_{n-1} and q_{n-1} are already defined. By the property $P(L)$ there exist local constant speed geodesics $\gamma_n\colon [a,b_1] \to X$ joining $\bar p$ to q_{n-1} and $\gamma_n'\colon [a_1,b] \to X$ joining p_{n-1} to $\bar q$, respectively, such that

$$d\big(\gamma(t),\gamma_n(t)\big) < \varepsilon \quad \text{for all } t \in [a,b_1] \quad \text{and}$$
$$d\big(\gamma(t),\gamma_n'(t)\big) < \varepsilon \quad \text{for all } t \in [a_1,b].$$

Define $p_n = \gamma_n(a_1)$ and $q_n = \gamma_n'(b_1)$.





By convexity of d in balls $B(\gamma(t), \varepsilon)$, we have that

$$d(p_0, p_1) \le \tfrac{1}{2} d(\gamma(a), \bar{p}) < \varepsilon/2$$

and that functions

$$t \mapsto d\big(\gamma_n(t), \gamma_{n+1}(t)\big)$$

are locally convex on $[a, b_1]$, hence convex. Thus

$$d(p_n, p_{n+1}) \le d(q_{n-1}, q_n)/2.$$

Similarly,

$$d(q_0, q_1) < \varepsilon/2 \quad \text{and} \quad d(q_n, q_{n+1}) \le d(p_{n-1}, p_n).$$

Hence

$$d(p_n, p_{n+1}) < \varepsilon/2^{n+1} \quad \text{and} \quad d(q_n, q_{n+1}) < \varepsilon/2^{n+1}$$

for all $n \in \mathbb{N}$, and therefore (p_n) and (q_n) are Cauchy-sequences in $\bar{B}(p_0, \varepsilon)$ and $\bar{B}(q_0, \varepsilon)$, respectively. Since the function

$$t \mapsto d\big(\gamma_n(t), \gamma_{n+1}(t)\big)$$

is convex and bounded by $d(q_{n-1}, q_n) < \varepsilon/2^n$, the sequence $\big(\gamma_n(t)\big)$ is Cauchy in $\bar{B}(\gamma(t), \varepsilon)$ for every $t \in [a, b_1]$. Similarly, $\big(\gamma'_n(t)\big)$ is a Cauchy-sequence in $\bar{B}(\gamma(t), \varepsilon)$ for every $t \in [a_1, b]$. Thus the local constant speed geodesics γ_n and γ'_n converge uniformly to local constant speed geodesics whose restrictions to $[a_1, b_1]$ coincide. The union of these local constant speed geodesics gives a local constant speed geodesic $\bar{\gamma}[a, b] \to X$ satisfying $P(\frac{3}{2}L)$. $\square$

Definition 4.5. Let X be a metric space and $p \in X$. Denote by $\tilde{X}_p$ the set consisting of the constant path $\tilde{p}\colon [0,1] \to X$, $\tilde{p}(t) \equiv p$, and of all local constant speed geodesics $\gamma\colon [0,1] \to X$ with $\gamma(0) = p$. The *exponential map at p* is the mapping $\exp_p\colon \tilde{X}_p \to X$,

$$\exp_p(\gamma) = \gamma(1).$$

We equip $\tilde{X}_p$ with the metric

$$d(\alpha, \beta) = \max\{|\alpha(t) - \beta(t)|\colon t \in [0,1]\}, \quad \alpha, \beta \in \tilde{X}_p.$$

Lemma 4.6. *Let X be a locally complete metric space whose metric is locally convex. Then*

(a) $\tilde{X}_p$ is contractible (in particular, simply connected) for every $p \in X$,

(b) $\exp_p\colon \tilde{X}_p \to X$ is a local isometry, and

(c) for each $\gamma \in \tilde{X}_p$ there exists a unique local constant speed geodesic in $\tilde{X}_p$ from $\tilde{p}$ to γ.

Proof. (a) For each $\gamma \in \tilde{X}_p$ and $s \in [0,1]$, let $h_s(\gamma)\colon [0,1] \to X$ be a local constant speed geodesic defined by

$$h_s(\gamma)(t) = \gamma(st).$$

Then we define a mapping $H\colon \tilde{X}_p \times [0,1] \to \tilde{X}_p$ by

$$H(\gamma, s) = h_s(\gamma).$$

Fix $\gamma \in \tilde{X}_p$ and $s \in [0,1]$. It follows from (4.3) that

$$d\big(H(\gamma,s), H(\alpha,s')\big) = \max\{|\gamma(st) - \alpha(s't)| \colon t \in [0,1]\} = |\gamma(s) - \alpha(s')|$$

if $d(\gamma,\alpha)$ and $|s - s'|$ are small enough. Hence

$$d\big(H(\gamma,s), H(\alpha,s')\big) \le |\gamma(s) - \alpha(s)| + |\alpha(s) - \alpha(s')| \le d(\gamma,\alpha) + |\alpha(s) - \alpha(s')|$$

for small $d(\gamma,\alpha)$ and $|s - s'|$, and therefore H is continuous. Thus H is a homotopy from $\tilde{p}$ to the identity map of $\tilde{X}_p$.

(b) By the proof of Lemma 4.2(1) (more precisely, (4.3)), for every $\gamma \in \tilde{X}_p$ there exists $\varepsilon > 0$ such that $\exp_p |B(\gamma,\varepsilon)$ is an isometry onto $B(\gamma(1),\varepsilon)$. Indeed, given $\gamma \in \tilde{X}_p$

$$|\exp_p(\alpha) - \exp_p(\beta)| = |\alpha(1) - \beta(1)| \overset{(4.3)}{=} \max\{|\alpha(t) - \beta(t)| \colon t \in [0,1]\} = d(\alpha,\beta)$$

for all $\alpha, \beta \in B(\gamma,\varepsilon)$, where $\varepsilon > 0$ is given by Lemma4.2(1).

(c) For each $\gamma \in \tilde{X}_p$, the path $s \mapsto H(\gamma,s)$ is a local constant speed geodesic from $\tilde{p}$ to γ since

$$d\big(H(\gamma,s), H(\gamma,s')\big) = |\gamma(s) - \gamma(s')| = \ell(\gamma)|s - s'|$$

whenever $|s - s'|$ is sufficiently small. On the other hand, since $\exp_p$ is a local isometry, a path $\tilde{\gamma}$ in $\tilde{X}_p$ is a local constant speed geodesic if and only if $\exp_p \circ \tilde{\gamma}$ is a local constant speed geodesic in X. In particular, the mapping $\tilde{\gamma} \mapsto \exp_p \circ \tilde{\gamma}$ is a bijection from the set of all local constant speed geodesics $\tilde{\gamma}$ in $\tilde{X}_p$ starting at $\tilde{p}$ to the set of all local constant speed geodesics in X starting at p. Thus for each $\gamma \in \tilde{X}_p$, the path $H(\gamma,\cdot)$ is the unique local constant speed geodesic from $\tilde{p}$ to γ. (Note that $\exp_p \circ H(\gamma,\cdot) = \gamma$.) Indeed, if $\tilde{\gamma}' \neq H(\gamma,\cdot)$ is another local constant speed geodesic in $\tilde{X}_p$ starting at $\tilde{p}$, then

$$\gamma' := \exp_p \circ \tilde{\gamma}'$$

is a local constant speed geodesic in X starting at p and $\gamma' \neq \gamma$ since $\exp_p$ is a local isometry. Hence $H(\gamma,\cdot)$ is the only local constant speed geodesic from $\tilde{p}$ to γ because $\tilde{\gamma}'$ ends at $\gamma' \neq \gamma$. □

Lemma 4.7. *Suppose that X is a complete metric space, $p \in X$, and that the metric is locally convex. Then $\tilde{X}_p$ is complete.*

Proof. Let (γ_n) be a Cauchy-sequence in $\tilde{X}_p$. Since X is complete, the Cauchy-sequence $\big(\gamma_n(t)\big)$ converges for every $t \in [0,1]$. Denote the pointwise limit by $\gamma(t)$. We may assume that $\gamma(t) \neq p$ for some $t \in [0,1]$ and that $\gamma_n \neq \tilde{p}$ for any n. Fix $t_0 \in [0,1]$ and choose $\varepsilon > 0$ such that the metric in $B(\gamma(t_0), 4\varepsilon)$ is convex. (In particular, $B(\gamma(t_0), 4\varepsilon)$ is geodesic.) Let n_ε be an integer such that $d(\gamma_n, \gamma_m) < \varepsilon$ for all $n, m \geq n_\varepsilon$. Let $[t_1, t_2] \subset [0,1]$ be the maximal interval such that

$$\gamma_{n_\varepsilon}[t_1, t_2] \subset \bar{B}(\gamma(t_0), \varepsilon).$$

Since

$$|\gamma_{n_\varepsilon}(t_0) - \gamma(t_0)| = \lim_{n\to\infty} |\gamma_{n_\varepsilon}(t_0) - \gamma_n(t_0)| \leq \varepsilon$$

and $\gamma_{n_\varepsilon} \in \tilde{X}_p \setminus \{\tilde{p}\}$, we have $t_0 \in [t_1, t_2]$ and $t_1 < t_2$. Furthermore, for all $n \geq n_\varepsilon$ and $t \in [t_1, t_2]$

$$|\gamma_n(t) - \gamma(t_0)| \leq |\gamma_n(t) - \gamma_{n_\varepsilon}(t)| + |\gamma_{n_\varepsilon}(t) - \gamma(t_0)| < 2\varepsilon.$$

Hence $\gamma_n(t_1)$ and $\gamma_n(t_2)$ can be joined by a constant speed geodesic $\alpha_n \colon [t_1, t_2] \to B(\gamma(t_0), 4\varepsilon)$. By (4.3), $\gamma_n|[t_1, t_2] = \alpha_n$, and hence $\gamma_n|[t_1, t_2]$ is a constant speed geodesic for all $n \geq n_\varepsilon$. It follows that for all $t, s \in [t_1, t_2]$

$$\begin{aligned} |\gamma(t) - \gamma(s)| &= \lim_{n\to\infty} |\gamma_n(t) - \gamma_n(s)| = \lim_{n\to\infty} \frac{|\gamma_n(t_2) - \gamma_n(t_1)|}{t_2 - t_1} |t - s| \\ &= \frac{|\gamma(t_2) - \gamma(t_1)|}{t_2 - t_1} |t - s|. \end{aligned}$$

Hence γ is a local constant speed geodesic, and $\tilde{X}_p$ is complete. □

Theorem 4.8. *Suppose that X is a connected complete metric space, $p \in X$, and that the metric is locally convex. Then*

(1) $(\tilde{X}_p, \exp_p)$ is a universal covering space of X (i.e. $\tilde{X}_p$ is simply connected and $\exp_p \colon \tilde{X}_p \to X$ is a covering map) and

(2) there exists a unique local constant speed geodesic between each pair of points in $\tilde{X}_p$.

Proof. By 4.6 and 4.7, $\tilde{X}_p$ is a complete simply connected metric space and $\exp_p$ is a local isometry. Furthermore, since the metric is locally convex, each point in X has a neighborhood which is uniquely geodesic and these geodesics vary continuously with their endpoints. Thus we can apply Theorem 1.108 to obtain the claim (1).

To prove the claim (2), we first show that every path $\alpha \colon [0,1] \to X$ is homotopic to a unique local constant speed geodesic. Let $x = \alpha(0)$ and let $\tilde{\alpha} \colon [0,1] \to \tilde{X}_x$ be the maximal lift of α (under $\exp_x$) starting at $\tilde{x}$. Denote by A the set of all paths in $\tilde{X}_x$ from $\tilde{x}$ to $\tilde{\alpha}(1)$. Since $(\tilde{X}_x, \exp_x)$ is a universal covering, the set A is bijective to the set of paths in X that are homotopic to α. By Lemma 4.6(3), the set A contains a unique local constant speed geodesic $\tilde{\gamma}$. Then $\exp_x \circ \tilde{\gamma}$ is the unique local constant speed geodesic that is homotopic to α.

Let then $\tilde{\alpha}, \tilde{\beta} \in \tilde{X}_p$. Since $\tilde{X}_p$ is simply connected, there exists exactly one homotopy class of paths in $\tilde{X}_p$ joining $\tilde{\alpha}$ and $\tilde{\beta}$. Since $(\tilde{X}_p, \exp_p)$ is a universal covering, the exponential map $\exp_p$ maps this class bijectively onto a single homotopy class of paths in X. By the argument above, the latter class contains a unique local constant speed geodesic. The lift (under $\exp_p$) of this path is then the unique constant speed geodesic in $\tilde{X}_p$ joining $\tilde{\alpha}$ and $\tilde{\beta}$. □

The proof of the Cartan-Hadamard theorem 4.1(1)

Suppose that X is as in Theorem 4.1(1), that is, a connected complete metric space whose metric is locally convex. Fix $p \in X$ and let d be the metric in $\tilde{X}_p$ defined in 4.5. By Theorem 4.8, $(\tilde{X}_p, \exp_p)$ is a universal covering space of X for every $p \in X$. Let $\tilde{d}$ be the induced length metric (cf. Definition 1.103). By Theorem 1.107 and Lemma 4.6(b), the identity map $\mathrm{id}\colon (\tilde{X}_p, \tilde{d}) \to (\tilde{X}_p, d)$ is a local isometry. In particular, a path in $\tilde{X}_p$ is a local constant speed geodesic simultaneously with respect to $\tilde{d}$ and d. By Lemma 4.7, $(\tilde{X}_p, \tilde{d})$ is locally complete, and the metric $\tilde{d}$ is locally convex since $\exp_p$ is a local isometry. By Theorem 4.8(2), there exists a unique local constant speed geodesic between any pair of points in $\tilde{X}_p$ and these local constant speed geodesics vary continuously with their endpoints by Lemma 4.2(1). Theorem 4.1(1) follows by applying the following lemma with $Y = \tilde{X}_p$.

Lemma 4.9. *Let Y be a simply connected locally complete length space whose metric is locally convex. Suppose that for each $x, y \in Y$ there exists a unique local constant speed geodesic $\gamma_{x,y}\colon [0,1] \to Y$ from x to y and that these local constant speed geodesics vary continuously with their endpoints. Then*

(1) each $\gamma_{x,y}$ is a constant speed geodesic and

(2) the metric in Y is convex.

Proof. To prove (1) it suffices to show that

$$\ell\big(\gamma_{\alpha(0),\alpha(t)}\big) \le \ell\big(\alpha|[0,t]\big) \tag{4.10}$$

for every rectifiable path $\alpha\colon [0,1] \to Y$ and for every $t \in [0,1]$. Fix a rectifiable path $\alpha\colon [0,1] \to Y$ and let $T \subset [0,1]$ be the set of all $t' \in [0,1]$ such that (4.10) holds for all $t \le t'$. For sufficiently small $t > 0$, (the unique local constant speed geodesic) $\gamma_{\alpha(0),\alpha(t)}$ is a constant speed geodesic since the metric is locally convex. Hence T is non-empty. Obviously, T is closed. We claim that T is also open, and hence $T = [0,1]$. If $t_0 \in T$, then Lemma 4.2(2) implies that

$$\begin{aligned}
\ell\big(\gamma_{\alpha(0),\alpha(t_0+\delta)}\big) &\le \ell\big(\gamma_{\alpha(0),\alpha(t_0)}\big) + \ell\big(\alpha|[t_0, t_0+\delta]\big) \\
&\le \ell\big(\alpha|[0,t_0]\big) + \ell\big(\alpha|[t_0, t_0+\delta]\big) \\
&\le \ell\big(\alpha|[0, t_0+\delta]\big)
\end{aligned}$$

for sufficiently small $\delta > 0$ as desired.

By assumptions and (1), Y is a locally complete uniquely geodesic space whose geodesics vary continuously with their endpoints. In order to prove (2), it is enough to show that

$$d\big(\gamma_{p,q_0}(1/2), \gamma_{p,q_1}(1/2)\big) \le \tfrac{1}{2}\, d(q_0, q_1) \tag{4.11}$$

for each pair of constant speed geodesics $\gamma_{p,q_0}, \gamma_{p,q_1}\colon [0,1] \to Y$. Indeed, the convexity of the metric follows from (4.11) by iteration. Let $\alpha\colon [0,1] \to Y$ be the constant speed geodesic from q_0 to q_1 and denote $q_s = \alpha(s)$. By Lemma 4.2(1),

$$d\big(\gamma_{p,q_s}(1/2), \gamma_{p,q_t}(1/2)\big) \le \tfrac{1}{2} d(q_s, q_t) \tag{4.12}$$

whenever $|t-s|$ is sufficiently small. Choose $0 = s_0 < s_1 < \cdots < s_k = 1$ such that (4.12) holds with $s = s_i, t = s_{i+1},\ i = 0, \ldots, k-1$. We obtain (4.11) from inequalities (4.12) by the triangle inequality. $\square$

The proof of the Cartan-Hadamard theorem 4.1(2)

The second claim in Theorem 4.1 follows from the first by the following theorem.

Theorem 4.13 (Alexandrov's Patchwork). *Let Y be a metric space of curvature $\le \kappa$. Suppose that there exists a unique geodesic joining each pair of points $x, y \in Y$, with $d(x, y) < D_\kappa$. If these geodesics vary continuously with their endpoints, then Y is a* $\mathrm{CAT}(\kappa)$*-space.*

Indeed, if X is of curvature $\le \kappa \le 0$, the metric of X is locally convex. By the first part of the Cartan-Hadamard theorem 4.1(1), the universal covering space $(\tilde{X}_p, \tilde{d})$ is uniquely geodesics and the geodesics in $\tilde{X}_p$ vary continuously with their endpoints. Furthermore, $\tilde{X}_p$ is of curvature $\le \kappa$ since the exponential map $\exp_p$ is a local isometry. By Theorem 4.13, $\tilde{X}_p$ is a $\mathrm{CAT}(\kappa)$-space.

Theorem 4.13 is a consequence of the characterization 3.2(4) of $\mathrm{CAT}(\kappa)$-spaces and the following two lemmas.

Lemma 4.14. *Suppose that $\kappa \in \mathbb{R}$ and that Y is a D_κ-geodesic metric space. Let*

$$\Delta = \Delta\big([p, q_1], [p, q_2], [q_1, q_2]\big)$$

be a geodesic triangle with distinct vertices and perimeter $< 2D_\kappa$. Let $r \in [q_1, q_2] \setminus \{q_1, q_2\}$ and let $[p, r]$ be a geodesic segment from p to r. Let $\bar{\Delta}_i$ be a κ-comparison triangle of

$$\Delta_i = \Delta\big([p, q_i], [p, r], [q_i, r]\big), \quad i = 1, 2.$$

If the Alexandrov angles of Δ_i are at most the corresponding vertex angles of $\bar{\Delta}_i$, $i = 1, 2$, then the Alexandrov angles of Δ are at most the corresponding vertex angles of any κ-comparison triangle of Δ.

Proof. Choose κ-comparison triangles $\bar{\Delta}_i = \bar{\Delta}_\kappa(p, q_i, r)$ with vertices $\bar{p}, \bar{q}_i, \bar{r}$, $i = 1, 2$, such that they have a common side $[\bar{p}, \bar{r}]$ and that $\bar{q}_1$ and $\bar{q}_2$ lie on opposite sides of the line $\bar{p}\bar{r}$. By the triangle inequality (for Alexandrov angles),

$$\angle_r([r, p], [r, q_1]) + \angle_r([r, p], [r, q_2]) \ge \angle_r([r, q_1], [r, q_2]) = \pi.$$

Hence

$$\angle_r^{(\kappa)}(p, q_1) + \angle_r^{(\kappa)}(p, q_2) \ge \pi$$

by the assumption. The claim then follows from Alexandrov's lemma 2.31. □

Lemma 4.15. *Let Y be a metric space of curvature $\le \kappa$. Let $\gamma\colon [0, 1] \to Y$ be a constant speed geodesic from $q_0 = \gamma(0)$ to $q_1 = \gamma(1)$, $q_0 \ne q_1$ and let $p \in Y \setminus \gamma[0, 1]$. Suppose that for each $s \in [0, 1]$ there exists a constant speed geodesic $\alpha_s\colon [0, 1] \to Y$ from p to $q_s = \gamma(s)$ and that the mapping $s \mapsto \alpha_s$ is continuous (with respect to the metric defined in 4.5). Let Δ be the geodesic triangle with sides $\gamma[0, 1]$, $\alpha_0[0, 1]$, and $\alpha_1[0, 1]$. Then the Alexandrov angles at p, q_0, and q_1 between the sides of Δ are at most the corresponding vertex angles in any κ-comparison triangle $\bar{\Delta} \subset M_\kappa^2$. (If $\kappa > 0$, we assume that the perimeter of Δ is less than $2D_\kappa$.)*

Proof. By the assumption, the mapping $\alpha\colon [0, 1] \times [0, 1] \to Y$,

$$\alpha(s, t) = \alpha_s(t),$$

is continuous and each point in Y has a neighborhood which is a $\mathrm{CAT}(\kappa)$-space. Hence there are partitions

$$0 = s_0 < s_1 < \cdots < s_k = 1 \quad \text{and} \quad 0 = t_0 < t_1 < \cdots < t_k = 1$$

such that there exists an open set $U_{i,j}$ of diameter $< D_\kappa/2$ which is a CAT(κ)-space and which contains

$$\alpha\big([s_{i-1}, s_i] \times [t_{j-1}, t_j]\big).$$

By repeated use of Lemma 4.14, it suffices to prove the claim for geodesic triangles

$$\Delta_i = \gamma[s_{i-1}, s_i] \cup \alpha_{s_{i-1}} \cup \alpha_{s_i}$$

and their κ-comparison triangles $\bar{\Delta}_i$. For each i, let

$$\Delta_i^1, \Delta_i^2, \tilde{\Delta}_i^2, \ldots, \Delta_i^k, \tilde{\Delta}_i^k$$

be adjoining geodesic triangles, where

$$\Delta_i^1 \subset U_{i,1} \quad \text{and} \quad \Delta_i^j, \tilde{\Delta}_i^j \subset U_{i,j}$$

are as in Figure 2. In each of these triangles the Alexandrov angles at the vertices are at most the

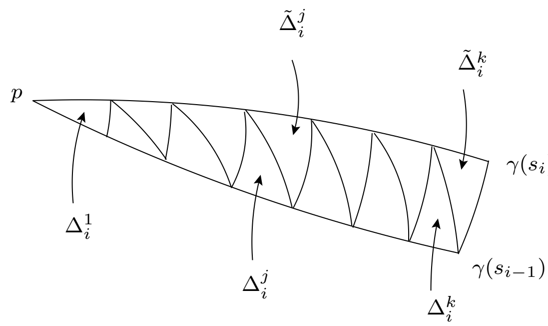


Figure 2: Adjoining geodesic triangles.

corresponding vertex angles in their κ-comparison triangles since the sets $U_{i,j}$ are CAT(κ)-spaces. By repeated use of Lemma 4.14 (starting with triangles Δ_i^1 and Δ_i^2) we obtain the claim for Δ_i as desired. $\square$

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