Department of Mathematics and Statistics
Metric Geometry
Exercise 9
26.4.2006

Return by **Wednesday, April 26**.

1. Let X be a CAT(κ)-space. Suppose that $\alpha\colon [0,a] \to X$ and $\beta\colon [0,b] \to X$ are geodesics such that $\alpha(0) = p = \beta(0)$.
   (a) Show that the κ-comparison angle
   $$\angle_p^{(\kappa)}\big(\alpha(s), \beta(t)\big)$$
   is increasing in both $s > 0$ and $t > 0$.
   (b) Show that the Alexandrov angle satisfies
   $$\begin{aligned}\angle_p(\alpha,\beta) &= \lim_{s,t\to 0} \angle_p^{(\kappa)}\big(\alpha(s), \beta(t)\big)\\ &= \lim_{t\to 0} \angle_p^{(\kappa)}\big(\alpha(t), \beta(t)\big)\\ &= \lim_{t\to 0} 2\arcsin \tfrac{1}{2t} d\big(\alpha(t), \beta(t)\big).\end{aligned}$$

2. Let X be a CAT(κ)-space and let $x, y \in X\backslash\{p\}$ with $\max\{d(p,x), d(p,y)\} < D_\kappa$. Prove that
   (a) $(p,x,y) \mapsto \angle_p\big([p,x],[p,y]\big)$ is upper semicontinuous, and
   (b) for fixed p, $(x,y) \mapsto \angle_p\big([p,x],[p,y]\big)$ is continuous.

3. Prove that the κ-cone $X = C_\kappa Y$ over a metric space Y is complete if and only if Y is complete.

4. Suppose that the κ-cone $X = C_\kappa Y$ over a metric space Y is a CAT(κ)-space. Prove that for each pair of points $y_1, y_2 \in Y$, with $d(y_1,y_2) < \pi$, there exists a unique geodesic segment in Y joining y_1 and y_2.

5. Let (X,d) be a metric space of curvature $\leq \kappa$. For each $n \in \mathbb{N}$, we define a metric d_n by setting
   $$d_n(x,y) = n\, d(x,y).$$
   Prove that the tangent cone $C_0 S_p(X)$ at $p \in X$ (and its completion) is a 4-point limit of the sequence (X, d_n).