Department of Mathematics and Statistics
Metric Geometry
Exercise 7
29.3.2006

Return by **Wednesday, March 29**.

Choose any 5 problems from the following (you can get bonus by solving all six problems).

1. Prove that the metric of a CAT(0)-space X is convex, that is, each pair of geodesics $\alpha\colon [0,a] \to X$ and $\beta\colon [0,b] \to X$, with $\alpha(0) = \beta(0)$, satisfy the inequality
$$d\big(\alpha(ta), \beta(tb)\big) \le t\, d\big(\alpha(a), \beta(b)\big)$$
for all $t \in [0,1]$.

2. Let X be a CAT(κ)-space and let $p, x, y \in X$ be such that $d(p,x) + d(p,y) < D_\kappa$. Prove that the geodesic segment $[x,y]$ is the union of $[x,p]$ and $[p,y]$ if and only if
$$\angle_p\big([p,x],[p,y]\big) = \pi.$$

3. Let X be a proper geodesic space. Suppose that there exists a unique geodesic segment $[x,y]$ joining points $x, y \in X$. Prove that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\operatorname{dist}(z,[x,y]) < \varepsilon$ whenever
$$d(x,z) + d(z,y) < d(x,y) + \delta.$$

4. Prove the following result: For every $\kappa \in \mathbb{R}$, $\ell < D_\kappa$, and $\varepsilon > 0$, there exists a constant δ (depending on $\kappa, \ell, \varepsilon$) such that for all $x, y \in M^2_\kappa$, with $d(x,y) \le \ell$, and for all $m' \in M^2_\kappa$, with
$$\max\{d(x,m'), d(y,m')\} < \tfrac{1}{2} d(x,y) + \delta,$$
we have
$$d(m,m') < \varepsilon,$$
where $m \in [x,y]$ is the midpoint of $[x,y]$ (i.e. $d(x,m) = d(y,m)$).

5. Prove that a geodesic space X is a CAT(κ)-space if and only if for every geodesic triangle $\Delta(p,q,r)$ of perimeter $< 2D_\kappa$ the midpoint $m \in [q,r]$ ($d(q,m) = d(m,r)$) and its comparison point $\bar{m} \in [\bar{q},\bar{r}] \subset \bar{\Delta}(p,q,r) \subset M^2_\kappa$ satisfy the inequality
$$d(p,m) \le d(\bar{p},\bar{m}).$$

6. Prove that a geodesic space X is a CAT(0)-space if and only if for all $p,q,r \in X$ and for all $m \in X$, with
$$d(q,m) = d(m,r) = \tfrac{1}{2} d(q,r),$$
we have
$$d(p,q)^2 + d(p,r)^2 \ge 2d(m,p)^2 + \tfrac{1}{2} d(q,r)^2.$$