Department of Mathematics and Statistics
Metric Geometry
Exercise 6
22.3.2006

Return by **Wednesday, March 22**.

1. (a) Prove that any closed ball $\bar{B}(p,r) \subset \mathbb{S}^n$ of radius $r < \pi/2$ is convex. That is, if $x, y \in \mathbb{S}^n$ and $[x,y] \subset \mathbb{S}^n$ is the geodesic segment joining x and y, then $[x,y] \subset \bar{B}(p,r)$.
   (b) Prove that all balls (open or closed) in $\mathbb{H}^n$ are convex.

2. *Hyperplanes in* $\mathbb{H}^n$. A hyperplane H in $\mathbb{H}^n$ is a non-empty intersection of $\mathbb{H}^n$ with an n-dimensional vector subspace V of $\mathbb{R}^{n+1}$. The reflection through H is the mapping $r_H \colon \mathbb{H}^n \to \mathbb{H}^n$,
$$r_H(x) = x - 2\langle x, u\rangle_{n,1} u,$$
where u is a unit vector orthogonal (w.r.t. $\langle \cdot, \cdot \rangle_{n,1}$) to V (u is unique up to a sign). Given distinct points $x, y \in \mathbb{H}^n$, the set
$$H_{xy} = \{z \in \mathbb{H}^n \colon d(z,x) = d(z,y)\}$$
is a hyperplane, called the hyperplane bisector of x, y.
   (a) Show that
$$H_{xy} = \mathbb{H}^n \cap (x-y)^{\perp}.$$
   (b) Let H be a hyperplane in $\mathbb{H}^n$. Prove that the reflection r_H through H is an isometry and that $r_H(x) = x \iff x \in H$.
   (c) Let H be a hyperplane in $\mathbb{H}^n$ and $x \in \mathbb{H}^n \setminus H$. Show that H is the hyperplane bisector of x and $r_H(x)$.
   (d) Prove that $r_H(x) = y$ if H is the hyperplane bisector of x and y.

3. Prove (by induction): Given $k \in \mathbb{N}$ and points $x_1, \ldots, x_k \in \mathbb{H}^n$, $y_1, \ldots, y_k \in \mathbb{H}^n$ such that $d(x_i, x_j) = d(y_i, y_j)$ for all $i, j \in \{1, \ldots, k\}$, there exists an isometry $f \colon \mathbb{H}^n \to \mathbb{H}^n$ such that $f(x_i) = y_i$. Moreover, such an isometry can be obtained by composing k or fewer reflections through hyperplanes.

4. Prove the following theorem: Let $f \colon \mathbb{H}^n \to \mathbb{H}^n$ be an isometry.
   (a) If f is not the identity, there exists a hyperplane in $\mathbb{H}^n$ containing all fixed points of f.
   (b) If H is a hyperplane in $\mathbb{H}^n$ such that $f|H = \mathrm{id}$, then f is either the identity or the reflection r_H through H.
   (c) The isometry f can be written as a composition of $n+1$ or fewer reflections through hyperplanes.

5. Study the notions and results of Exercises 2–4 with $\mathbb{H}^n$ replaced by $\mathbb{S}^n$, $\mathbb{R}^n$, and M^n_κ, $\kappa \in \mathbb{R}$. Write down your observations and thoughts.