Department of Mathematics and Statistics
Metric Geometry
Exercise 1
1.2.2006

Return by **Wednesday, February 1**.

1. Let (X, d) be a metric space.
   (a) Prove that $(X, d^\alpha),\ 0 < \alpha < 1$, is a metric space.
   (b) Prove that (X, d_0), where
   $$d_0(x, y) = \frac{d(x, y)}{1 + d(x, y)},$$
   is a metric space.
   (c) Study whether the topologies $\mathcal{T}_d,\ \mathcal{T}_{d^\alpha}$, and $\mathcal{T}_{d_0}$ are the same.

2. [Kuratowski] Prove that every metric space X can be isometrically embedded into the Banach-space $\ell^\infty(X)$.

3. [Fréchet] Prove that every separable metric space can be isometrically embedded into the Banach-space $\ell^\infty = \ell^\infty(\mathbb{N})$.

4. Prove that a metric space X is complete if and only if it has the following property: If (X_n) is a sequence of non-empty, closed subsets of X such that $X_{n+1} \subset X_n$ for every n and $\operatorname{diam}(X_n) \to 0$, then the sets X_n have a common point (i.e. $\cap_n X_n \neq \emptyset$).

5. (a) Let $f : X \to Y$ be a bi-Lipschitz homeomorphism. Prove that X is complete if and only if Y is complete.
   (b) Give an example of homeomorphic metric spaces X and Y such that X is complete while Y is not.

6. Let $X = (\mathbb{R}^2, d_\infty)$, where d_∞ is the metric given by the norm $\|\cdot\|_\infty$, and let $Y = \mathbb{R}^2$ with the standard metric. Let
   $$A = \{(-1, 1), (1, -1), (1, 1)\}$$
   and $f : A \to Y$,
   $$f(-1, 1) = (-1, 0), \quad f(1, -1) = (1, 0), \quad f(1, 1) = (0, \sqrt{3}).$$
   Show that f is 1-Lipschitz but it has no 1-Lipschitz extension to $A \cup \{(0, 0)\}$.

[Hints overleaf.]

2. Fix $x_0 \in X$ and study functions $f_y \colon X \to \mathbb{R}$, $f_y(x) = |x - y| - |x - x_0|$.
3. Fix a dense set $\{x_0, x_1, \dots\} \subset X$ and consider the mapping

$$x \mapsto \big(|x - x_1| - |x_1 - x_0|, |x - x_2| - |x_2 - x_0|, \dots\big).$$